HOMOTOPY THEORY OF SMALL DIAGRAMS OVER LARGE CATEGORIES

BORIS CHORNY AND WILLIAM G. DWYER

ABSTRACT. Let $\mathcal{D}$ be a large category which is cocomplete. We construct a model structure (in the sense of Quillen) on the category of small functors from $\mathcal{D}$ to simplicial sets. As an application we construct homotopy localization functors on the category of simplicial sets which satisfy a stronger universal property than the customary homotopy localization functors do.

1. Introduction

Let $\mathcal{S}$ be the category of simplicial sets. In this paper we introduce axiomatic homotopy theory into the study of functors from a large category $\mathcal{D}$ into $\mathcal{S}$, in other words, into the study of diagrams in $\mathcal{S}$ indexed by $\mathcal{D}$. Such diagrams arise naturally (for instance in the treatment of Goodwillie calculus, which in one form deals with functors from $\mathcal{S}$ itself into $\mathcal{S}$) but in the past they have been dealt with by *ad hoc* techniques. The novelty of our approach is the introduction of a model category structure, which allows for the use of standard tools from axiomatic homotopy theory.

There is an obvious set–theoretic difficulty in dealing with the index categories we wish to consider: if $\mathcal{D}$ is large, the totality of natural transformations between two functors $\mathcal{S} \to \mathcal{S}$ does not necessarily form a set, and so the collection of all such functors is not even a category in the usual sense, much less a model category. We overcome this difficulty by restricting our attention to the category $\mathcal{S}^{\mathcal{D}}_{\mathrm{sm}}$ of *small* (2.1) functors $\mathcal{D} \to \mathcal{S}$. This category is always cocomplete. If $\mathcal{D}$ itself is cocomplete, then $\mathcal{S}^{\mathcal{D}}_{\mathrm{sm}}$ is also complete, and it is in this situation we can construct a model category on $\mathcal{S}^{\mathcal{D}}_{\mathrm{sm}}$. This model structure reduces to the ordinary projective model category structure on the category of all functors $\mathcal{D} \to \mathcal{S}$ if $\mathcal{D}$ is small [3], therefore we call the model structure of Theorem 3.1 also *projective*; observe though that for technical reasons our model structure in general lacks functorial factorization.

We discuss in detail two examples, $\mathcal{D} = \mathcal{S}^{\mathrm{op}}$ and $\mathcal{D} = \mathcal{S}$. For $\mathcal{D} = \mathcal{S}^{\mathrm{op}}$, we generalize the arguments of [10] to show that our model structure on $\mathcal{S}^{\mathcal{D}}_{\mathrm{sm}}$ is Quillen equivalent to the equivariant model structure developed by Farjoun [7] on the category of maps of spaces. The model category $\mathcal{S}^{\mathcal{S}}_{\mathrm{sm}}$ does not seem to have an analogous interpretation. The category of pro-spaces may be viewed as dual to the subcategory of pro-representable functors in $\mathcal{S}^{\mathcal{S}}_{\mathrm{sm}}$, and its model structure [12] [16] is perhaps the closest relative to our model structure on $\mathcal{S}^{\mathcal{S}}_{\mathrm{sm}}$.

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An immediate application of the model structure on $\mathcal{S}^{\mathcal{S}}_{\rm sm}$ is a construction of homotopy localizations in this category. Although this construction itself involves factorizations and is thus non-functorial in $\mathcal{S}^{\mathcal{S}}_{\rm sm}$, an application of the constuction to the identity functor yields an object of $\mathcal{S}^{\mathcal{S}}_{\rm sm}$ (i.e., a functor $\mathcal{S} \to \mathcal{S}$) which is equivalent to the ordinary homotopy localization functor on simplicial sets but has a stronger universal property. We finish the paper by using these homotopy localization functors to construct natural A-Postnikov towers in $\mathcal{S}$. Another application of the model structure on $\mathcal{S}^{\mathcal{S}}_{\rm sm}$ is developed in [1].

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1.1. **Notation.** We continue to let $\mathcal{S}$ denote the category of simplicial sets, which we also refer to as the category of *spaces*. If $\mathcal{C}$ and $\mathcal{D}$ are categories, we simplify notation by using $\mathcal{C}^{\mathcal{D}}$ to denote the category $\mathcal{C}^{\mathcal{D}}_{\rm sm}$ of all small functors $\mathcal{D} \to \mathcal{C}$. If $\mathcal{D}$ itself is small, this is the category of all functors $\mathcal{D} \to \mathcal{C}$. A *simplicial category* is a category enriched over $\mathcal{S}$, such as $\mathcal{S}$ itself; functors between two such categories are assumed to respect the enrichments, in the sense that they provide simplicial maps between the respective function complexes.

2. Preliminaries on small functors

The object of study of this paper is homotopy theory of functors from a large simplicial category to $\mathcal{S}$. The totality of these functors does not form a category in the usual sense, since the natural transformations between two functors need not form a set in general, but rather a proper class. We are willing to be satisfied with a treatment of a reasonable subcollection of functors, a subcollection which does form a category. The purpose of this section is to describe such a subcollection.

Definition 2.1. Let $\mathcal{D}$ be a (not necessarily small) simplicial category. A functor $\underset{\sim}{X} : \mathcal{D} \to \mathcal{S}$ is *representable* if there is an object $D \in \mathcal{D}$ such that $\underset{\sim}{X}$ is naturally equivalent to R^D, where $R^D(D') = \hom_{\mathcal{D}}(D, D')$. A functor $\underset{\sim}{X} \colon \mathcal{C} \to \mathcal{S}$ is called *small* if $\underset{\sim}{X}$ is a small weighted colimit of representables.

Remark 2.2. Since the category of small functors is tensored over simplicial sets, the *small weighted colimit* above may be expressed as a coend of the form

$$R^F \otimes_{\mathcal{I}} G = \int^{I \in \mathcal{I}} R^{F(I)} \otimes G(I), \tag{1}$$

where $\mathcal{I}$ is a small category and $F \colon \mathcal{I} \to \mathcal{D}$, $G \colon \mathcal{I} \to \mathcal{S}$ are functors. Here $R^F : \mathcal{I}^{\rm op} \to \mathcal{S}^{\mathcal{D}}$ assigns to $I \in \mathcal{I}$ the representable functor $R^{F(I)} \colon \mathcal{D} \to \mathcal{S}$. For the general treatment of weighted limits and colimits see [17].

Since the simplicial tensor structure on the category of small fuctors $\mathcal{S}^{\mathcal{D}}$ is given by the objectwise direct product, we will use $\underset{\sim}{X} \times K$ to denote tensor product of $\underset{\sim}{X} \in \mathcal{S}$ with $K \in \mathcal{S}$.

The above coend is the (enriched) left Kan extension of the functor G over the functor F. Using the transitivity of left Kan extensions, it is easy to see that the following four conditions are equivalent [17, Prop. 4.83]:

- $\underset{\sim}{X} \colon \mathcal{D} \to \mathcal{S}$ is a small functor,

- there is a small simplicial category $\mathcal{I}$ and a functor $G : \mathcal{I} \to \mathcal{S}$, such that $\underset{\sim}{X}$ is isomorphic to the left Kan extension of G over some functor $\mathcal{I} \to \mathcal{D}$,
- there a small simplicial subcategory $i \colon \mathcal{D}' \to \mathcal{D}$ and a functor $G : \mathcal{D}' \to \mathcal{S}$, such that $\underset{\sim}{X}$ is isomorphic to the left Kan extension of G over i, and
- there is a small full simplicial subcategory $i \colon \mathcal{D}_{\underset{\sim}{X}} \to \mathcal{D}$ such that $\underset{\sim}{X}$ is isomorphic to the left Kan extension of $i^*(\underset{\sim}{X})$ over i.

If $D \in \mathcal{D}$ and $\underset{\sim}{Y}$ is a functor $\mathcal{D} \to \mathcal{S}$, then by Yoneda's lemma the simplicial class of natural transformations $R^D \to \underset{\sim}{Y}$ is $\underset{\sim}{Y}(D)$; in particular, this simplicial class is a simplicial set. It follows easily that if $\underset{\sim}{X}$ is a small functor $\mathcal{D} \to \mathcal{S}$, then the natural transformations $\underset{\sim}{X} \to \underset{\sim}{Y}$ also form a simplicial set (this also follows from 2.2 above and the adjointness property of the left Kan extension). In particular, the collection of all small functors is a simplicial category.

Remark 2.3. M.G. Kelly [17] calls small functors *accessible* and weighted colimits *indexed*. He proves that small functors form a simplicial category which is closed under small (weighted) colimits [17, Prop. 5.34].

In order to do homotopy theory we need to work in a category which is not only cocomplete, but also complete (at least under finite limits). Fortunately, there is a simple sufficient condition in the situation of small functors.

Theorem 2.4. *If $\mathcal{D}$ is cocomplete, then the category $\mathcal{S}^{\mathcal{D}}$ of small functors $\mathcal{D} \to \mathcal{S}$ is complete.*

Remark 2.5. There is a long story behind this theorem. P. Freyd [13] introduced the notion of *petty* and *lucid* set-valued functors. A set-valued functor is called *petty* if it is a quotient of a small sum of representable functors. Any small functor is clearly petty. A functor $F \colon \mathcal{A} \to \mathcal{S}ets$ is called *lucid* if it is petty and for any functor $G \colon \mathcal{A} \to \mathcal{S}ets$ and any pair of natural transformations $\alpha, \beta \colon G \rightrightarrows F$, the equalizer of α and β is petty. Freyd proved [13, 1.12] that the category of lucid functors from $\mathcal{A}^{\mathrm{op}}$ to $\mathcal{S}ets$ is complete if and only if $\mathcal{A}$ is *approximately* complete (that means that the category of cones over any small diagram in $\mathcal{A}$ has a weakly initial set). J. Rosický then proved [19, Lemma 1] that if the category $\mathcal{A}$ is approximately complete, a functor $F \colon \mathcal{A}^{\mathrm{op}} \to \mathcal{S}ets$ is small if and only if it is lucid. Finally, these results were partially generalized by B. Day and S. Lack [18] to the enriched setting. They show, in particular, that the category of small $\mathcal{V}$–enriched functors $\mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ is complete if $\mathcal{K}$ is complete and $\mathcal{V}$ is a symmetric monoidal closed category which is locally finitely presentable as a closed category. This last condition is certainly satisfied if $\mathcal{V} = \mathcal{S}$.

3. A model category on $\mathcal{S}^{\mathcal{D}}$

As usual, $\mathcal{S}^{\mathcal{D}}$ denotes the category of small functors $\mathcal{D} \to \mathcal{S}$.

Theorem 3.1. *Assume that $\mathcal{D}$ is cocomplete. Then the category $\mathcal{S}^{\mathcal{D}}$ has a model category structure in which weak equivalences and fibrations are defined objectwise and the cofibrations are the maps which have the left lifting property with respect to acyclic fibrations. (The factorizations provided by this model category structure are not necessarily functorial.)*

Remark 3.2. The use of "objectwise" above signifies that a map $F \to G$ is a weak equivalence (fibration) if and only if for each $X \in \mathcal{D}$ the induced map $F(X) \to G(X)$

is a weak equivalence (fibration) of simplicial sets. We are using the ordinary model category structure on simplicial sets, in which a map is a weak equivalence if its geometric realization is a weak equivalence of topological spaces, and a fibration if it is a Kan fibration (see, e.g., [15, Thm. 3.6.5]).

Recall from 2.1 the notion of representable functor, as well as the notation $R^D = \hom(D, –)$ for the functor represented by D. We first need a definition and some lemmas, which exhibit yet additional uses of the word *small*.

Definition 3.3. A collection $\mathcal{L}$ of objects in a category $\mathcal{B}$ is said to be *locally small* in $\mathcal{B}$ if for every object X of $\mathcal{B}$ there exists a *set* of objects $\mathcal{O}_X \subset \mathcal{L}$ such that any map $Y \to X$ with $Y \in \mathcal{L}$ can be factored as a composite $Y \to Y' \to X$ for some $Y' \in \mathcal{O}_X$.

Remark 3.4. More standardly, the statement that $\mathcal{L}$ is locally small is expressed by saying that $\mathcal{L}$ satisfies the *co–solution set* condition. The set $\mathcal{O}_X$ is called the co–solution set associated to X. Our terminology follows [7], since the idea of the proof of Theorem 3.1 also goes back to [7].

Lemma 3.5. *The collection of representable functors is locally small in $\mathcal{S}^{\mathcal{D}}$.*

Proof. Suppose that $\underset{\sim}{X}$ is in $\mathcal{S}^{\mathcal{D}}$, and write $\underset{\sim}{X}$ as a small weighted colimit as in (1). Given a representable functor R^D, consider the simplicial set $\hom(R^D, \underset{\sim}{X})$. From the generalized Yoneda lemma, and the fact that weighted colimits of diagrams are computed levelwise, we obtain:

$$\hom(R^D, \underset{\sim}{X}) = \underset{\sim}{X}(D) = R^F(D) \otimes_{\mathcal{I}} G = \int^{I \in \mathcal{I}} \hom(R^D, R^{F(I)}) \times G(I). \tag{2}$$

Comparing the sets of the zero simplices of the simplicial sets above, we conclude that every map $R^D \to \underset{\sim}{X}$ factors through a map $R^D \to R^{F(I)}$ for some $I \in \mathcal{I}$. □

Definition 3.6. The *category of maps in $\mathcal{S}^{\mathcal{D}}$*, denoted $\mathcal{M}ap(\mathcal{S}^{\mathcal{D}})$, is the category whose objects are the arrows $f \colon \underset{\sim}{X} \to \underset{\sim}{Y}$ in $\mathcal{S}^{\mathcal{D}}$. A morphism $f \to f'$ is a commutative diagram

$$\begin{array}{ccc} \underset{\sim}{X} & \longrightarrow & \underset{\sim}{X}' \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f'} \\ \underset{\sim}{Y} & \longrightarrow & \underset{\sim}{Y}' \end{array}$$

Lemma 3.7. *If $g : K \to L$ is a map of spaces, then the collection*

$$\mathcal{L}(g) = \{R^D \times K \to R^D \times L \mid D \in \mathcal{D}\}$$

is locally small in $\mathcal{M}ap(\mathcal{S}^{\mathcal{D}})$.

Remark 3.8. It follows immediately that if $\{g_\alpha\}_{\alpha \in A}$ is a set of maps between spaces, then the union $\cup_\alpha \mathcal{L}(g_\alpha)$ is also locally small in $\mathcal{M}ap(\mathcal{S}^{\mathcal{D}})$.

Proof of 3.7. Consider a morphism

$$\begin{array}{ccc} R^D \times K & \longrightarrow & \underset{\sim}{X} \\ \Big\downarrow{\scriptstyle g_D} & & \Big\downarrow{\scriptstyle f} \\ R^D \times L & \longrightarrow & \underset{\sim}{Y} \end{array}$$

in $\mathcal{M}ap(\mathcal{S}^{\mathcal{D}})$. This morphism gives rise, by adjunction, to the following commutative diagram:

$$\begin{tikzcd}
R^D \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \underset{\sim}{W} \arrow[r] \arrow[d] & \underset{\sim}{X}^K \arrow[d] \\
& \underset{\sim}{Y}^L \arrow[r] & \underset{\sim}{Y}^K,
\end{tikzcd}$$

where $\underset{\sim}{W}$ is defined so that the square is a pullback square. By Lemma 3.5 there exists a set of representable functors $\mathcal{O}_{\underset{\sim}{W}}$, such that any morphism from a representable functor to $\underset{\sim}{W}$ can be factored through an object in $\mathcal{O}_{\underset{\sim}{W}}$. Now take $\mathcal{O}_f = \{F \times K \to F \times L \mid F \in \mathcal{O}_{\underset{\sim}{W}}\}$, and observe that any map from g_D to f will factor, by adjunction, through one of the objects in $\mathcal{O}_f$. □

Let us now briefly remind the setup of the generalized small object argument, which applies for locally small collections of maps with small domains. The reader might want to consult [5] for a more extensive discussion. Suppose that $\mathcal{L}$ is a locally small collection in $\mathcal{M}ap(\mathcal{S}^{\mathcal{D}})$, that $f\colon \underset{\sim}{X} \to \underset{\sim}{Y}$ is an object in $\mathcal{M}ap(\mathcal{S}^{\mathcal{D}})$, and that $\mathcal{O}_f$ is the associated co–solution set for f (3.4). We define $\Gamma^1_{\mathcal{L}}(f)$ to be the natural map $\gamma^1_{\mathcal{L}}(f) \to Y$, where $\gamma^1_{\mathcal{L}}(f)$ is determined by the following pushout diagram:

$$\begin{tikzcd}
\coprod_\beta \underset{\sim}{U}_\beta \arrow[r] \arrow[d] & \underset{\sim}{X} \arrow[d] \\
\coprod_\beta \underset{\sim}{V}_\beta \arrow[r] & \gamma^1_{\mathcal{L}}(f).
\end{tikzcd}$$

Here β runs through the set of pairs (g_β, h_β), where $g_\beta\colon \underset{\sim}{U}_\beta \to \underset{\sim}{V}_\beta$ belongs to $\mathcal{O}_f$ and $h_\beta\colon g_\beta \to f$ is a morphism in $\mathcal{M}ap(\mathcal{S}^{\mathcal{D}})$. It is easy to see that the map $\underset{\sim}{X} \to \underset{\sim}{Y}$ extends to a map $\gamma^1_{\mathcal{L}}(f) \to \underset{\sim}{Y}$. For $n > 1$, we let $\gamma^n_{\mathcal{L}}(f) = \gamma^1_{\mathcal{L}}(\Gamma^{n-1}_{\mathcal{L}}(f))$, and $\Gamma^n_{\mathcal{L}}(f)\colon\ \gamma^n_{\mathcal{L}}(f) \to Y$ the induced natural map. Finally, $\gamma^\infty_{\mathcal{L}}(f)$ denotes $\operatorname{colim}_n \gamma^n_{\mathcal{L}}(f)$, and $\Gamma^\infty_{\mathcal{L}}(f)\colon\ \gamma^\infty_{\mathcal{L}}(f) \to Y$ is the evident natural map.

Recall that a simplicial set is said to be *finite* if it has a finite number of non-degenerate simplicies and a finite simplicial set K is $\aleph_0$–small in the category of simplicial sets, in the sense that $\hom(K, -)$ commutes with countable sequential colimits.

In order to conclude, by the generalized small object argument, that the induced map $\Gamma^\infty_{\mathcal{L}}(f) : \gamma^\infty(f) \to \underset{\sim}{Y}$ has the right lifting property with respect to all of the maps in $\mathcal{L}$, the class $\mathcal{L}$ must satisfy an additional condition (to local smallness) that all domains of maps in $\mathcal{L}$ are λ–small for some fixed cardinal λ.

Yoneda's lemma and smallness of finite simplicial sets imply the last condition if $\mathcal{L}$ is the collection $\cup_\alpha \mathcal{L}(g_\alpha)$ for a set $\{g_\alpha\colon K_\alpha \to L_\alpha\}_{\alpha \in A}$ of monomorphisms between finite simplicial sets. $\mathcal{L}$ is locally small by 3.8.

Remark 3.9. The construction of the map $\Gamma^\infty_{\mathcal{L}}(f) : \gamma^\infty(f) \to \underset{\sim}{Y}$ has a natural generalization to an arbitrary transfinite cardinal λ: $\Gamma^\lambda_{\mathcal{L}}(f) : \gamma^\lambda(f) \to \underset{\sim}{Y}$. We will not need the transfinite version until we prove Theorem 5.1, than we refer to an even more general approach of [5].

Remark 3.10. The construction of $\Gamma^1_{\mathcal{L}}(f)$ or $\Gamma^\infty_{\mathcal{L}}(f)$ is not functorial unless the co–solution sets $\mathcal{O}_f$ depend in some natural way on f. This would be the case, for instance, if $\mathcal{O}_f = \mathcal{L}$ for all f, but of course this would be allowed only if $\mathcal{L}$ itself is a set. Another example where $\mathcal{O}_f$ depends functorially on f occurs in the equivariant model category of [7]. See [4] for the construction of functorial factorizations in this model category. In general, there are two versions of the generalized small object argument: functorial and non-functorial [5]. We apply the non-functorial version in this work.

Proof of 3.1. There are several variations in the literature of the definition of a model category. We prove the axioms MC0–MC5 in the form of [11]. The required limits and colimits exist by the discussion in the previous section. The '2-out-of-3' axiom and the fact that weak equivalences and fibrations are closed under retracts follows from the corresponding properties of the category $\mathcal{S}$. By the definition of cofibration, every cofibration has the left lifting property with respect to any acyclic fibration. In particular, cofibrations are closed under retracts.

Although our model category is not cofibrantly generated [15, Section 2.1], it has a similar structure, namely, it is class-cofibrantly generated [5, Def. 1.3]. In order to verify the second lifting property and two factorization properties, let us define the classes of generating cofibrations and generating acyclic cofibrations to be

$$\begin{aligned} I &= \{R^D \times \partial\Delta^n \hookrightarrow R^D \times \Delta^n | D \in \mathcal{D}, n \geq 0\} \\ J &= \{R^D \times \Lambda^n_k \hookrightarrow R^D \times \Delta^n | D \in \mathcal{D}, n > 0, n \geq k \geq 0\} \end{aligned}$$

where as usual Δ^n is the n-simplex, $\partial\Delta^n$ its boundary, and Λ^n_k the space obtained by removing the k'th face of Δ^n from $\partial\Delta^n$. A map in $\mathcal{S}$ is an acyclic fibration if and only if it has the right lifting property with respect to the inclusions $\partial\Delta^n \to \Delta^n$, $n \geq 0$ and so it follows by adjunction that a map in $\mathcal{S}^{\mathcal{D}}$ is an acyclic fibration if and only if it has the right lifting property with respect to the maps in I. Similarly, a map in $\mathcal{S}^{\mathcal{D}}$ is a fibration if and only if it has the right lifting property with respect to the maps in J.

Suppose that $f\colon \underset{\sim}{X} \to \underset{\sim}{Y}$ is a map in $\mathcal{S}^{\mathcal{D}}$, and note that by Remark 3.8 above the classes I and J permit the generalized small object argument [5]. Therefore, the composite $\underset{\sim}{X} \to \gamma^\infty_I(f) \to \underset{\sim}{Y}$ is a factorization of f into the composite of a cofibration with an acyclic fibration, while $\underset{\sim}{X} \to \gamma^\infty_J(f) \to \underset{\sim}{Y}$ is a factorization into the composite of an acyclic cofibration and a fibration.

The second lifting property is achieved by a standard trick; see, e.g., [10]. Given an acyclic cofibration $f : \underset{\sim}{A} \to \underset{\sim}{B}$, let $\underset{\sim}{C} = \gamma^\infty_J(f)$ and factor f as a composite $\underset{\sim}{A} \to \underset{\sim}{C} \to \underset{\sim}{B}$. By construction the map $\underset{\sim}{A} \to \underset{\sim}{C}$ has the left lifting property with respect to any fibration. Since $\underset{\sim}{C} \to \underset{\sim}{B}$ is actually an acyclic fibration (by the '2-out-of-3' property), lifting in the diagram

$$\begin{tikzcd} \underset{\sim}{A} \arrow[r] \arrow[d] & \underset{\sim}{C} \arrow[d] \\ \underset{\sim}{B} \arrow[r] & \underset{\sim}{B} \end{tikzcd}$$

shows that $\underset{\sim}{A} \to \underset{\sim}{B}$ is a retract of $\underset{\sim}{A} \to \underset{\sim}{C}$ and thus also has the left lifting property with respect to any fibration. $\square$

4. Example: $\mathcal{D} = \mathcal{S}^{\mathrm{op}}$

In order to illustrate the concept of the projective model structure on the category of small functors $\mathcal{S}^{\mathcal{D}}$ by a familiar model category, we consider $\mathcal{D} = \mathcal{S}^{\mathrm{op}}$. In this case we construct a Quillen equivalence between $\mathcal{S}^{\mathcal{D}}$ and the category $\mathcal{M}ap(\mathcal{S})_{\mathrm{eq}}$; the subscript "eq" signifies that this is the category $\mathcal{M}ap(\mathcal{S})$ of maps in $\mathcal{S}$ (3.6) endowed with the "equivariant" or "fine" model structure of [7].

Recall from [9] that an *orbit* in $\mathcal{M}ap(\mathcal{S})$ is a diagram $A \to B$ in $\mathcal{S}$ whose colimit is isomorphic to the one–point space $* = \Delta^0$. Since the colimit of $A \to B$ is B, an orbit in $\mathcal{M}ap(\mathcal{S})$ is simply an object of the form $A \to *$, and so, via the functor which assigns to such a diagram the space A, the category $\mathcal{O}$ of orbits is equivalent to $\mathcal{S}$. We will let $O_A = (A \to *)$ be the orbit in $\mathcal{M}ap(\mathcal{S})$ corresponding the space A. The following definition was given in [7]:

Definition 4.1. The *equivariant model structure* or *fine model structure* on $\mathcal{M}ap(\mathcal{S})$ is the model category $\mathcal{M}ap(\mathcal{S})_{\mathrm{eq}}$ in which the underlying category is $\mathcal{M}ap(\mathcal{S})$, and in which a map $X \to Y$ between objects of $\mathcal{M}ap(\mathcal{S})$ is a weak equivalence (fibration) if and only for each $A \in \mathcal{S}$ it gives a weak equivalence (fibration) $\hom(O_A, X) \to \hom(O_A, Y)$ in $\mathcal{S}$.

The above definition suggests assigning to each object X of $\mathcal{M}ap(\mathcal{S})$ the diagram $X^{\mathcal{O}} \colon \mathcal{S}^{\mathrm{op}} \to \mathcal{S}$ sending A to $\hom(O_A, X)$; the functor $(-)^{\mathcal{O}} \colon \mathcal{M}ap(\mathcal{S}) \to \mathcal{S}^{\mathcal{S}^{\mathrm{op}}}$ both preserves and reflects weak equivalences.

Lemma 4.2. *For every $M \in \mathcal{M}ap(\mathcal{S})$, the functor $M^{\mathcal{O}} \colon \mathcal{S}^{\mathrm{op}} \to \mathcal{S}$ is a small functor; in particular, $M \mapsto M^{\mathcal{O}}$ gives a functor $(-)^{\mathcal{O}} \colon \mathcal{M}ap(\mathcal{S}) \to \mathcal{S}^{\mathcal{S}^{\mathrm{op}}}$.*

Proof. This was essentially shown in [6]. More precisely, Farjoun proved (in a more general context) that for any object $M \in \mathcal{M}ap(\mathcal{S})$ there exists a small full subcategory $i \colon \mathcal{O}_M \hookrightarrow \mathcal{O}$ such that $M^{\mathcal{O}}$ is the left Kan extension of $i^*(M^{\mathcal{O}})$ (cf. 2.2). □

Let us construct the left adjoint to the functor $(-)^{\mathcal{O}}$ by verifying the conditions of the adjoint functor theorem: the orbit-point functor obviously preserves limits, so it remains to verify the solution-set condition. This means for any small diagram $\underset{\sim}{X} \in \mathcal{S}^{\mathcal{S}^{\mathrm{op}}}$ we need to find a set of arrows $f_i \colon \underset{\sim}{X} \to \underset{\sim}{Y_i^{\mathcal{O}}}$ such that any arrow $\underset{\sim}{f} \colon \underset{\sim}{X} \to Z^{\mathcal{O}}$, for $Z \in \mathcal{M}ap(\mathcal{S})$ factorizes as $f = \underset{\sim}{(k)^{\mathcal{O}}} \circ f_i$ for some map $k \colon Y_i \to Z$.

Recall from [10] that for every full simplicial subcategory $\mathcal{I} \subset \mathcal{O} \cong \mathcal{S}$ there is a pair of adjoint functors

$$|{-}|_{\mathcal{I}} \colon \mathcal{S}^{\mathcal{I}^{\mathrm{op}}} \leftrightarrows \mathcal{M}ap(\mathcal{S}) : (-)^{\mathcal{I}}, \tag{3}$$

(which is a Quillen equivalence if one considers the model structure induced by the set $\mathcal{I}$ of orbits on $\mathcal{M}ap(\mathcal{S})$).

If $\underset{\sim}{X}$ is small, then it is a left Kan extension of $i^*\underset{\sim}{X} \colon \mathcal{I}^{\mathrm{op}} \to \mathcal{S}$ for a small simplicial full subcategory $\mathcal{I}$ of the orbit category $\mathcal{O} \cong \mathcal{S}$ and $i \colon \mathcal{I}^{\mathrm{op}} \to \mathcal{O}^{\mathrm{op}}$.

Given $f \colon \underset{\sim}{X} \to Z^{\mathcal{O}}$, consider $i^* f \colon i^*\underset{\sim}{X} \to i^* Z^{\mathcal{O}} = Z^{\mathcal{I}}$ and look at the adjoint map $(i^* f)^{\sharp} \colon |i^*\underset{\sim}{X}|_{\mathcal{I}} \to Z$. Let $Y = |i^*\underset{\sim}{X}|_{\mathcal{I}}$ and $k = (i^* f)^{\sharp}$. We obtain the following commutative triangle:

$$\begin{tikzcd} i^*\underset{\sim}{X} \arrow[rr] \arrow[dr] & & i^*(Y^{\mathcal{O}}) \arrow[dl] \\ & i^*(Z^{\mathcal{O}}). & \end{tikzcd}$$

The horizontal arrow is the unit of adjunction (3): $i^*\underset{\sim}{X} \to (|i^*\underset{\sim}{X}|_{\mathcal{I}})^{\mathcal{I}} = Y^{\mathcal{I}} = i^*(Y^{\mathcal{O}})$. Consider the adjoint triangle:

$$\begin{tikzcd} \mathrm{Lan}_i(i^*\underset{\sim}{X}) \cong \underset{\sim}{X} \arrow[rr] \arrow[dr] & & \mathrm{Lan}_i(i^*(Y^{\mathcal{O}})) \arrow[dl] \\ & Z^{\mathcal{O}}. & \end{tikzcd}$$

Since Y has orbit type $\mathcal{O}_Y = \mathcal{I}$ by construction, [6, 4.26] implies $\mathrm{Lan}_i i^*(Y^{\mathcal{O}}) \cong Y^{\mathcal{O}}$ and we obtain the required factorization $\underset{\sim}{X} \to Y^{\mathcal{O}} \to Z^{\mathcal{O}}$, so that $f = k^{\mathcal{O}} f'$.

That means that the functor $(-)^{\mathcal{O}}$ has a left adjoint. We will call this left adjoint *realization* and denote it by $\underset{\sim}{Z} \mapsto |\underset{\sim}{Z}|$.

More explicitly, the left adjoint to the functor $X \mapsto X^{\mathcal{O}}$ is the functor which assigns to $\underset{\sim}{Y} \in \mathcal{S}^{\mathcal{D}}$ the coend $\mathrm{Inc} \times_{\mathcal{D}} \underset{\sim}{Y}$, where $\mathrm{Inc} : \mathcal{D}^{\mathrm{op}} = \mathcal{S} = \mathcal{O} \hookrightarrow \mathcal{M}ap(\mathcal{S})$ is the inclusion of the full subcategory of orbits (for notational reasons, let $\mathcal{D} = \mathcal{S}^{\mathrm{op}}$). Of course on the face of it this is a large coend, but it actually gives a functor for the following reason. Since $\underset{\sim}{Y}$ is a small diagram, there is a small full subcategory $i : \mathcal{I} \subset \mathcal{D}$ such that $\underset{\sim}{Y}$ is the left Kan extension of $i^*(\underset{\sim}{Y})$ over i. It then follows from associativity properties of coends that $\mathrm{Inc} \times_{\mathcal{D}} \underset{\sim}{Y}$ is isomorphic to the small coend $\mathrm{Inc} \times_{\mathcal{I}} i^*(\underset{\sim}{Y})$.

Remark 4.3. A similar realizarion functor was constructed in [6, 3.11]. The main difference of our construction is that the domain category of our functor is the category of *small* functors, whether in [6] the author talks about the category of *all* contravariant functors from spaces to spaces.

Proposition 4.4. *The functors $X \mapsto X^{\mathcal{O}}$ and $\underset{\sim}{Y} \mapsto |\underset{\sim}{Y}|$ form a Quillen pair, which give a Quillen equivalence between $\mathcal{S}^{\mathcal{S}^{\mathrm{op}}}$ and $\mathcal{M}ap(\mathcal{S})_{eq}$.*

Proof. To produce the Quillen pair, it will suffice by [14, 8.5.3] to show that any (acyclic) fibration in $\mathcal{M}ap(\mathcal{S})_{\mathrm{eq}}$ is preserved by the functor $X \mapsto X^{\mathcal{O}}$; this, however, follows immediately from definition of the model category structures on $\mathcal{M}ap(\mathcal{S})_{\mathrm{eq}}$ (4.1) and on $\mathcal{S}^{\mathcal{S}^{\mathrm{op}}}$ (3.1).

In order to show that the pair of functors is a Quillen equivalence we have to prove that for any cofibrant diagram $\underset{\sim}{Y} \in \mathcal{S}^{\mathcal{S}^{\mathrm{op}}}$ and for any fibrant $X \in \mathcal{M}ap(\mathcal{S})$, a map $f : \underset{\sim}{Y} \to X^{\mathcal{O}}$ is a weak equivalence if and only if the adjoint map $f^\sharp : |\underset{\sim}{Y}| \to X$ is a weak equivalence. But, by the definition of weak equivalences, $f^\sharp$ is a weak equivalence if and only if $(f^\sharp)^{\mathcal{O}} : |\underset{\sim}{Y}|^{\mathcal{O}} \to X^{\mathcal{O}}$ is a weak equivalence, so it will suffice, by the 2-out-of-3 property, to show that the unit of the adjunction induces a weak equivalence $g : \underset{\sim}{Y} \to |\underset{\sim}{Y}|^{\mathcal{O}}$ for every cofibrant diagram $\underset{\sim}{Y}$. This we now do.

From the small object argument (§3) we know that $\underset{\sim}{Y}$ is a retract of $\Gamma_I^\infty(\emptyset \to \underset{\sim}{Y})$, where $\emptyset$ is the empty diagram. Since retracts are preserved by all functors, and retracts of weak equivalences are weak equivalences, we can assume that $\underset{\sim}{Y} =$

$\Gamma_I^\infty(\emptyset \to \underset{\sim}{Y})$. Let $\underset{\sim}{Y}_n = \Gamma_I^n(\emptyset \to \underset{\sim}{Y})$; then

$$\underset{\sim}{Y} = \operatorname{colim}(\underset{\sim}{Y}_1 \to \underset{\sim}{Y}_2 \to \cdots \to \underset{\sim}{Y}_n \to \cdots),$$

where $\underset{\sim}{Y}_n \to \underset{\sim}{Y}_{n+1}$ is obtained by a pushout

$$\begin{tikzcd}
\coprod_\alpha R^{A_\alpha} \otimes \partial\Delta^n \arrow[r] \arrow[d] & \underset{\sim}{Y}_n \arrow[d] \\
\coprod_\alpha R^{A_\alpha} \otimes \Delta^n \arrow[r] & \underset{\sim}{Y}_{n+1} .
\end{tikzcd}$$

Since left adjoints commute with colimits we obtain:

$$|\underset{\sim}{Y}| = \operatorname{colim}(|\underset{\sim}{Y}_1| \to |\underset{\sim}{Y}_2| \to \cdots \to |\underset{\sim}{Y}_n| \to \cdots),$$

where $|\underset{\sim}{Y}_n| \to |\underset{\sim}{Y}_{n+1}|$ is obtained by a pushout

$$\begin{tikzcd}
\coprod_\alpha |R^{A_\alpha}| \otimes \partial\Delta^n \arrow[r] \arrow[d] & |\underset{\sim}{Y}_n| \arrow[d] \\
\coprod |R^{A_\alpha}| \otimes \Delta^n \arrow[r] & |\underset{\sim}{Y}_{n+1}|.
\end{tikzcd}$$

But $|R^A| \cong O_A = (A \to *)$, as can be verified by using above coend description of the realization functor, or by a simple adjointness verification. Hence, the pushout diagram above becomes

$$\begin{tikzcd}
\coprod_\alpha O_{A_\alpha} \otimes \partial\Delta^n \arrow[r] \arrow[d] & |\underset{\sim}{Y}_n| \arrow[d] \\
\coprod_\alpha O_{A_\alpha} \otimes \Delta^n \arrow[r] & |\underset{\sim}{Y}_{n+1}|.
\end{tikzcd}$$

But it was shown in [7] and [4] that the functor $(-)^{\mathcal{O}}$ commutes up to a weak equivalence with all colimits of the above form. This immediately leads the conclusion that the natural map $\underset{\sim}{Y} \to |\underset{\sim}{Y}|^{\mathcal{O}}$ is a weak equivalence in the $\mathcal{S}^{\mathcal{S}^{\mathrm{op}}}$. □

5. Application: homotopy localization of spaces

In this section we take $\mathcal{D} = \mathcal{S}$ and we present an application of the projective model structure on $\mathcal{S}^{\mathcal{S}}$ to homotopy localization in the category of spaces.

We first recall some basic notions. Suppose that $f\colon A \to B$ is a cofibration of spaces. A space Z is said to be *f-local* if Z is fibrant and $f^* : \hom(B, Z) \to \hom(A, Z)$ is a weak equivalence in $\mathcal{S}$; a map $X \to Y$ is an *f-equivalence* if $\hom(Y, Z) \to \hom(X, Z)$ is a weak equivalence in $\mathcal{S}$ for every f-local Z. Finally, an f-equivalence $X \to X'$ is an *f-localization map* if X' is f-local.

It is well known (see [2] and [8]) that there exists a homotopy idempotent, coaugmented, simplicial homotopy functor $L_f\colon \mathcal{S} \to \mathcal{S}$ which has the following two properties:

(1) for any $X \in \mathcal{S}$ the coaugmentation $\eta_X\colon X \to L_f(X)$ is an f-localization map, and

(2) for every map $g\colon X \to Z$, where $Z \in \mathcal{S}$ is f-local, there exists a factorization of g

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "\eta_X"'] & & Z \\ & L_f X, \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

and in this factorization the map h is unique up to simplicial homotopy.

We produce a localization functor which is weakly equivalent to the one above, but which has a stronger universal property. Assume as usual that $f\colon A \to B$ is a cofibration of spaces.

Theorem 5.1. *There exists a homotopy idempotent, coaugmented, small, simplicial, homotopy functor $L_f : \mathcal{S} \to \mathcal{S}$ with the following two properties:*

(1) *for any $X \in \mathcal{S}$ the coaugmentation $\eta_X\colon X \to L_f(X)$ is an f-localization map, and*

(2) *for every coaugmented functor $\underset{\sim}{L}\colon \mathcal{S} \to \mathcal{S}$ taking f-local values, there exists a factorization*

$$\begin{tikzcd} \mathrm{Id} \arrow[rr, "\varepsilon"] \arrow[dr, "\eta"'] & & \underset{\sim}{L}. \\ & L_f \arrow[ur, dashed, "\zeta"'] & \end{tikzcd}$$

In this factorization the natural transformation ζ is unique up to a simplicial homotopy (of natural transformation).

Proof. We sketch the proof, with references. Given a cofibration $f\colon A \hookrightarrow B$, consider the class N of maps in $\mathcal{S}^{\mathcal{S}}$ given by $N = \{R^C \times f \mid C \in \mathcal{S}\}$. Then N is locally small in $\mathcal{M}ap(\mathcal{S}^{\mathcal{S}})$ (3.7), and just as in the fixed-point-wise situation of [4], the class

$$\mathrm{Hor}(N) = \left\{ (R^C \times f) \,\square \begin{pmatrix} \partial\Delta^n \\ \downarrow \\ \Delta^n \end{pmatrix} \,\middle|\, C \in \mathcal{S}, n \geq 0 \right\}$$

$$= \left\{ R^C \times A \times \Delta^n \coprod_{R^C \times A \times \partial\Delta^n} R^C \times B \times \partial\Delta^n \to R^C \times B \times \Delta^n \,\middle|\, C \in \mathcal{S}, n \geq 0 \right\}.$$

permits the generalized small object argument [5]. (Note that although the functors $\hom(A,–)$ and $\hom(B,–)$ do not necessarily commute with sequential colimits in $\mathcal{S}$, they do commute with well–ordered colimits of sufficiently high cofinality.) Observe now that the identity functor Id and the constant functor $*$ on $\mathcal{S}$ are small and in fact representable; one is R^C for $C = *$ and the other for C being the empty diagram. Therefore, taking $\mathcal{L} = \mathrm{Hor}(N) \cup J$ and applying the generalized small object technique, we can factor the map $\mathrm{Id} \to *$ into a composite $\mathrm{Id} \to \underset{\sim}{K} \to *$ in which, by construction, $\underset{\sim}{K}$ is a small simplicial functor.

There are three properties of this factorization to notice. First, by the very nature of the generalized small object technique, the map $\underset{\sim}{K} \to *$ has the right lifting property with respect to the maps in $\mathcal{L}$. This amounts to the assertion that for each $C \in \mathcal{S}$ the space $\underset{\sim}{K}(C)$ is fibrant and has the right lifting property with respect to the maps in $\mathrm{Hor}(N)$, i.e., to the assertion that $\underset{\sim}{K}(C)$ is f-local. Secondly, if $\underset{\sim}{L}\colon \mathcal{S} \to \mathcal{S}$ is a functor which takes on f-local values, then the induced map $\hom(\underset{\sim}{K}, \underset{\sim}{L}) \to \hom(\mathrm{Id}, \underset{\sim}{L})$ is an acyclic fibration in $\mathcal{S}$. This follows from the way in which $\underset{\sim}{K}$ is constructed from iterated pushouts of the maps in $\mathcal{L}$, and the fact that for any $g\colon U \to V$ in $\mathrm{Hor}(N)$ and any f-local space Z, the restriction map $\hom(V, Z) \to \hom(U, Z)$ is an acyclic fibration. By picking $C \in \mathcal{S}$ and applying this observation to the coextended diagram $\underset{\sim}{L}$ given by $\underset{\sim}{L}(X) = \hom(\hom(X, C), Z)$, one sees that for any f-local space Z, $\hom(K(C), Z) \to \hom(C, Z)$ is an acyclic fibration. In particular $C \to K(C)$ is an f-equivalence. Finally, $\underset{\sim}{K}$ is a homotopy functor; in fact the above considerations show that $\underset{\sim}{K}$ is an f-localization functor, and it follows formally from the definition that such functors take weak equivalences between spaces into simplicial homotopy equivalences between fibrant spaces. To finish the proof, it is enough to take $L_f = \underset{\sim}{K}$. We leave it to the reader to deduce from 5.1(2) that L_f is homotopy idempotent. □

6. Example: A functorial A-Postnikov tower

It is well known that there exists a construction of the classical Postnikov tower, which is functorial 'as a tower'. However, this construction, due to Moore, is *ad hoc* and does not allow for a natural generalization. Our new construction of localizations provides a general method of obtaining functorial towers.

Example 6.1. E. Farjoun discussed a Postnikov tower with respect to a space A [8]. This is a construction that associates to every space X a tower of spaces $\cdots \to P_{\Sigma^n A}X \to P_{\Sigma^{n-1}A}X \to \cdots \to P_{\Sigma A}X \to P_A X$, where $P_B = L_{B\to *}$ is the nullification functor. The classical construction of localizations ensured that each level in this tower is a functor in X, but not the whole tower. We take an advantage of localizations with the functorial universal property in order to obtain an equivalent tower functorial in X.

Let f_n be the map $\Sigma^n A \to *$ for all $n \geq 0$. From now on denote by $P_{\Sigma^n A} = L_{f_n}$ the localization functor with functorial universal property constructed in the previous section. A fibrant simplicial set X is f_n-local iff $* \simeq \hom(\Sigma^n A, X) = \hom(\Sigma^{n-1}A, \Omega X) = \Omega \hom(\Sigma^{n-1}A, X)$. Therefore if a fibrant X is f_{n-1}-local, then X is f_n-local. By Theorem 5.1 for each $n > 0$ there exists a natural transformation $\zeta_n\colon L_{f_n} \to L_{f_{n-1}}$. Combining ζ_n for all $n > 0$ we obtain the required tower of functors $\cdots \to P_{\Sigma^n A} \xrightarrow{\zeta_n} P_{\Sigma^{n-1}A} \xrightarrow{\zeta_{n-1}} \cdots \to P_{\Sigma A} \xrightarrow{\zeta_1} P_A$. If $A = S^0$, then we recover a new construction of the classical Postnikov tower.

Remark 6.2. The construction of the localization functor with a stronger functorial property can be carried out also in the stable model cateogry of spectra. As an application we can obtain the functorial decomposition of spectra into a tower of chromatic layers.

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D-MATH, ETH Zentrum, 8059 Zurich, Switzerland

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA
E-mail address: chorny@math.ethz.ch
E-mail address: dwyer.1@nd.edu