A SUSPENSION SPECTRAL SEQUENCE FOR v_n-PERIODIC HOMOTOPY GROUPS

DAVID BLANC AND ROBERT D. THOMPSON

Abstract. We describe a category of towers of spaces in which the v-periodic homotopy groups of a space $\boldsymbol{X}$ with respect to any map $v : \Sigma^d \boldsymbol{M} \to \boldsymbol{M}$ are representable. Using this description we construct a spectral sequence converging to the v_n-periodic homotopy groups of the suspension of a space $\boldsymbol{X}$ of type n, with E^2-term depending only on $v_n^{-1}\pi_\star(\boldsymbol{X};\boldsymbol{M})$.

1. Introduction

Given a self map $v : \Sigma^d \boldsymbol{M} \to \boldsymbol{M}$ of some space $\boldsymbol{M}$, the t-th *v-periodic homotopy group* of a space $\boldsymbol{X}$, denoted $v^{-1}\pi_t(\boldsymbol{X};\boldsymbol{M})$, is defined to be the direct limit of

$$[\Sigma^t \boldsymbol{M}, \boldsymbol{X}] \xrightarrow{(\Sigma^t v)^\#} \dots [\Sigma^{t+rd}\boldsymbol{M}, \boldsymbol{X}] \xrightarrow{(\Sigma^{t+rd}v)^\#} [\Sigma^{t+(r+1)d}\boldsymbol{M}, \boldsymbol{X}] \dots \qquad (\text{for } t \geq 0).$$

The main case of interest is when $\boldsymbol{M}$ is a p-torsion finite complex of type n for some prime p and v is a v_n-self map (see §7.1 below). In the past decade there has been much work done on such v_n-periodic homotopy groups, both stable (cf. [R1, DHS, HS, Ma2]) and unstable (cf. [Ma1, T2, MT]). Recently Bousfield and Dror-Farjoun have developed a general framework for studying (unstable) periodic phenomena by means of localizations (see [Bo3, DF1, DF2], and section 7 below).

These localizations – and thus in particular the v_n-periodic homotopy groups – behave quite well with respect to products, fibration sequences, loops, and other homotopy (inverse) limits – see [DS, Theorems B,C] and [DF2]. However, there is no evident relation between the (unstable) v_n-periodic homotopy groups and homotopy *colimits*, even in the simplest cases, such as the suspension. (Unlike ordinary homotopy groups, there is not even a stable range in which such a relation exists).

1991 *Mathematics Subject Classification.* Primary 55Q70, Secondary 55T99, 55P60.

Key words and phrases. v_n-periodicity, Π-algebra, tower, suspension, spectral sequence.

Second author supported by the NSF.

Our interest in this subject was first raised by the following seemingly innocuous question: given a v_n-self map $v = \Sigma^d \boldsymbol{M} \to \boldsymbol{M}$ and a map $f : \boldsymbol{X} \to \boldsymbol{Y}$ which induces an isomorphism in $v^{-1}\pi_*(-;\boldsymbol{M})$, does $\Sigma f : \Sigma\boldsymbol{X} \to \Sigma\boldsymbol{Y}$ induce such an isomorphism, too? This appears to be a hard question, even if we replace the v-periodic groups by "homotopy groups with coefficients in $\boldsymbol{M}$": that is, by the groups $\pi_t(-;\boldsymbol{M}) \stackrel{Def}{=} [\Sigma^t\boldsymbol{M}, -]$ for $0 \leq t < \infty$ (see [BT]).

The question for the v_n-periodic groups has been answered in the meantime by Bousfield, under somwhat limiting assumptions (see section 7 below). Here we present a first approach to a more delicate question: in what way do the (unstable) v_n-periodic homotopy groups of a space $\boldsymbol{X}$ determine those of its suspension $\Sigma\boldsymbol{X}$? Our main result in this direction is:

Theorem A: *Let $v = \Sigma^d\boldsymbol{M} \to \boldsymbol{M}$ be a v_n-self map and $\boldsymbol{X}$ a sufficiently connected space $\boldsymbol{X}$ with $v_m^{-1}\pi_*(\boldsymbol{X};\boldsymbol{V}_{m-1}) = 0$ for $0 \leq m < n$ (where $\boldsymbol{V}_{m-1}$ is a suitable complex of type m); then there is a first quadrant spectral sequence converging to $v^{-1}\pi_*(\Sigma\boldsymbol{X};\boldsymbol{M})$, with E^2-term isomorphic to the derived functors of a certain (algebraic) functor $\hat{\Sigma}$ applied to $v^{-1}\pi_*(\boldsymbol{X};\boldsymbol{M})$.*

(This is stated more precisely in Theorem 10.1 below.) Once more the answer we give requires somewhat restrictive assumptions on $\boldsymbol{X}$. As with Bousfield's result, it is still not clear to what extent these are inherent in the problem, and to what extent they are dues to technical difficulties in the approach we take.

This approach is based on Stover's construction of the simplicial resolution of a space which he used to attack the analogous question of determining the homotopy groups of a wedge $\pi_\star(\boldsymbol{X} \vee \boldsymbol{Y})$ from $\pi_\star\boldsymbol{X}$ and $\pi_\star\boldsymbol{Y}$ – cf. [St]; as in his case, our results generalize to other homotopy colimits.

1.1. **outline.** In order to use Stover's approach, we need a version of periodic homotopy in which the periodic homotopy groups are *representable* in some (homotopy) category. The naive approach would be to use towers of spaces, as described in section 2 (see (4.3)). However, this does not quite suit our purpose, mainly because infinite wedges of towers do not constitute a categorical coproduct. So we are forced to extend our original category to one of "virtual towers" (in section 3): these should be thought of as a "cocompletion" of the category of (ordinary) towers with respect to homotopy colimits, an idea which may be of use in other contexts too. In this category the construction of [St] can be made to work: we construct a simplicial v-periodic resolution $\hat{\mathfrak{J}}_\bullet$ of an arbitrary tower $\mathfrak{X}$ in section 5.

Next, we need a Quillen spectral sequence relating a simplicial tower $\hat{\mathfrak{X}}_\bullet$ to its realization $\|\hat{\mathfrak{X}}_\bullet\|$ (which for $\hat{\mathfrak{X}}_\bullet = \hat{\mathfrak{J}}_\bullet$ should be closely related to the original $\mathfrak{X}$). This is done by means of mapping spaces for towers, which translate questions about a simplicial tower and its realization back into questions about simplicial spaces and their realizations. These are considered in section 6.

In section 7 we summarize some results of Bousfield on localization with respect to v_n-self maps; the most important is Corollary 7.9, which tells us in particular that if $f : \boldsymbol{X} \to \boldsymbol{Y}$ induces an isomorphism in $v_m^{-1}\pi_*(-; \boldsymbol{V}_{m-1})$ for $0 \leq m \leq n$ (and if $\boldsymbol{X}$, $\boldsymbol{Y}$ are sufficiently connected) then Σf does, too.

Section 8 then allows us to apply Bousfield's results to towers. It is the technical difficulties here which force us to restrict attention to spaces of type n; otherwise, it would appear that a full knowledge of $\{v_m^{-1}\pi_*(\boldsymbol{X}; \boldsymbol{V}_{m-1})\}_{m=1}^n$ should suffice to determine $v_n^{-1}\pi_*(\Sigma\boldsymbol{X}; \boldsymbol{V}_{n-1})\}$.

Finally, the appropriate analogue of the concept of a Π-algebra – which encodes the "algebraic" information regarding the v-periodic homotopy operations on $v^{-1}\pi_*(\boldsymbol{X}; \boldsymbol{M})$ needed to recover $v^{-1}\pi_*(\Sigma\boldsymbol{X}; \boldsymbol{M})$ – is defined in section 9. This is used in section 10 to describe the E^2-term of the suspension spectral sequence of Theorem A.

1.2. **conventions and notation.** Let $\mathcal{T}_\star$ denote the category of connected pointed CW-complexes, with base-point preserving maps. All spaces will be assumed to lie in $\mathcal{T}_\star$, unless otherwise stated.

$\boldsymbol{M}$ will denote a model space. For simplicity we shall assume it is a homotopy-commutative co-H space which is a finite-dimensional CW complex. We adopt the stable convention that $\boldsymbol{M}^r$ denotes the suspension of $\boldsymbol{M}$ with top cell(s) in dimension r (so that $\boldsymbol{M}^r$ does not necessarily exist for small r). The homotopy groups with coefficients in $\boldsymbol{M}$ of any space $\boldsymbol{X} \in \mathcal{T}_\star$ are $\pi_k(\boldsymbol{X}; \boldsymbol{M}) \stackrel{Def}{=} [\boldsymbol{M}^k, \boldsymbol{X}]$.

Let $v : \boldsymbol{M}^{d+r_0} \to \boldsymbol{M}^{r_0}$ be a fixed self-map (with $d > 0$ unless otherwise noted), which we shall assume to be a co-H map. We shall denote all its suspensions $\Sigma^{r-r_0}v : \boldsymbol{M}^{r+d} \to \boldsymbol{M}^r$ simply by $v : \boldsymbol{M}^d \to \boldsymbol{M}$, unless there is danger of confusion. We also assume that v is not nilpotent – i.e., that for all n, the composite $v \circ \Sigma^d v \circ \ldots \Sigma^{nd} v$ is not nullhomotopic.

With this notation, we define the t-th *v-periodic homotopy group* of $\boldsymbol{X}$ with coefficients in $\boldsymbol{M}$ to be

$$v^{-1}\pi_t(\boldsymbol{X};\boldsymbol{M}) \overset{Def}{=} \operatorname{colim}_n \{ [\boldsymbol{M}^t, \boldsymbol{X}] \xrightarrow{v^\#} \dots [\boldsymbol{M}^{t+nd}, \boldsymbol{X}] \xrightarrow{v^\#} [\boldsymbol{M}^{t+(n+1)d}, \boldsymbol{X}] \dots \}.$$

In any category, the direct limit, or *colimit*, over a diagram scheme I will be denoted by colim_I, while the (inverse) limit will be denoted by $\lim_I$. Similarly, a (pointed) homotopy colimit over I (in $\mathcal{T}_\star$) will be denoted by $\operatorname{hocolim}_I$.

Apology 1.4. We wish to apologize for the somewhat technical nature of what was originally intended to be a "conceptual" paper, and in particular for the large number of definitions. We have tried to indicate at each stage why these were forced upon us.

2. The category of towers

In order to represent v-periodic homotopy we first consider the naive choice – namely, the category of *towers* of spaces, which are clearly related to the idea of periodicity.

It should perhaps be remarked that Brayton Gray ([Gr2]) has considered an analogous concept, which he calls the category of *cospectra*. This terminology emphasizes a certain duality, which will be evident in this section, between unstable periodic homotopy, represented by towers, and ordinary stable homotopy, represented by spectra.

First, some definitions:

2.1. **towers of spaces.** The objects we shall be studying are *towers* in $\mathcal{T}_\star$ – i.e., sequences of spaces and maps:

$$\mathfrak{X} = \{ \dots \boldsymbol{X}[n+1] \xrightarrow{p_{n+1}} \boldsymbol{X}[n] \xrightarrow{p_n} \boldsymbol{X}[n-1] \xrightarrow{p_{n-1}} \dots \xrightarrow{p_1} \boldsymbol{X}[0] \},$$

where the space $\boldsymbol{X}[n]$ is called the n-th *level* of $\mathfrak{X}$ $(n \geq 0)$, and the map p_n is called the n-th *level map* of $\mathfrak{X}$. We use Gothic letters $(\mathfrak{X}, \mathfrak{Y}, \dots)$ to denote towers.

If $F : \mathcal{T}_\star \to \mathcal{T}_\star$ is any functor of spaces, we denote by $F\mathfrak{X}$ the result of applying F levelwise to the tower $\mathfrak{X}$. For example, $\Sigma^r\mathfrak{X}$ denotes $\{\dots \Sigma^r\boldsymbol{X}[n] \xrightarrow{\Sigma^r p_n} \Sigma^r\boldsymbol{X}[n-1] \to \dots \to \Sigma^r\boldsymbol{X}[0]\}$; and similarly for bifunctors such as $\mathfrak{X} \vee \mathfrak{Y}$.

In any case where we define a tower only from the n-th level and up, it may be extended to a full tower by considering the *"corrected" truncated tower*:

$$\sigma_n\mathfrak{X} \overset{Def}{=} \{\dots \boldsymbol{X}[n+1] \xrightarrow{p_{n+1}} \boldsymbol{X}[n] \xrightarrow{=} \boldsymbol{X}[n] \dots \xrightarrow{=} \boldsymbol{X}[n]\},$$

which may be obtained from any tower $\mathfrak{X}$ by replacing $\boldsymbol{X}[n-1] \to \ldots \to \boldsymbol{X}[0]$ at the bottom of the tower by n additional copies of $\boldsymbol{X}[n]$. Finally, for any space $\boldsymbol{X} \in \mathcal{T}_\star$ let $\mathfrak{C}(\boldsymbol{X})$ denote the constant tower: $\{\ldots \boldsymbol{X} \stackrel{=}{\to} \boldsymbol{X} \ldots\}$.

2.2. **maps between towers.** As in the case of spectra, the morphisms in our category are more complicated than the objects: let $\mathfrak{X} = \{\ldots \boldsymbol{X}[n] \xrightarrow{p_n} \boldsymbol{X}[n-1] \ldots\}$ and $\mathfrak{Y} = \{\ldots \boldsymbol{Y}[n] \xrightarrow{q_n} \boldsymbol{Y}[n-1] \to \ldots\}$ be two towers as above. Then:

a. A *strict map* $\mathfrak{f} : \mathfrak{X} \stackrel{st}{\to} \mathfrak{Y}$ between them is a sequence $\mathfrak{f} = \{f[k] : \boldsymbol{X}[k] \to \boldsymbol{Y}[k]\}_{k=0}^{\infty}$ of maps such that $q_k \circ f[k] = f[k-1] \circ p_k$ for $k > 0$. The set of all such strict maps between $\mathfrak{X}$ and $\mathfrak{Y}$ will be denoted $Hom^{st}_{\mathcal{T}ow}(\mathfrak{X}, \mathfrak{Y})$.

Note that such a sequence defined only for $k \geq n$ is equivalent to a strict map $\mathfrak{f} : \sigma_n\mathfrak{X} \to \mathfrak{Y}$.

b. The set of *(weak) maps* between $\mathfrak{X}$ and $\mathfrak{Y}$ is defined to be

$$Hom_{\mathcal{T}ow}(\mathfrak{X}, \mathfrak{Y}) \stackrel{Def}{=} \operatorname{colim}_n Hom^{st}_{\mathcal{T}ow}(\sigma_n\mathfrak{X}, \mathfrak{Y}). \tag{2.3}$$

Thus a (weak) map $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ is a "tail" of sequences $\{f[k] : \boldsymbol{X}[k] \to \boldsymbol{Y}[k]\}_{k=n}^{\infty}$ as above – i.e., the equivalence class of all such sequences which eventually agree. Each element in this equivalence class is called a *strict representative* of $\mathfrak{f}$. We denote by $n(\mathfrak{f})$ the least n for which a strict representative $\sigma_n\mathfrak{X} \to \mathfrak{Y}$ of $\mathfrak{f}$ exists.

The category of towers of (connected pointed) CW-complexes and (weak) maps between them will be denoted $\mathcal{T}ow$.

(c) In particular, a (weak) *homotopy* between two (weak) maps, $\mathfrak{F} : \mathfrak{f} \sim \mathfrak{g}$, is a (weak) map $\mathfrak{F} : \mathfrak{X} \times I \to \mathfrak{Y}$ such that $\mathfrak{F}|_{\mathfrak{X}\times\{0\}} = \mathfrak{f}$ and $\mathfrak{F}|_{\mathfrak{X}\times\{1\}} = \mathfrak{g}$, as usual. Note that equality here is that of weak maps – i.e., of equivalence classes – so a strict representative of $\mathfrak{F}$ need not be a strict homotopy between any two strict representatives $\mathfrak{f}$ and $\mathfrak{g}$, but merely between suitable tails thereof.

(d) As usual, the set of (weak) homotopy classes of maps between $\mathfrak{X}$ and $\mathfrak{Y}$ will be denoted by $[\mathfrak{X}, \mathfrak{Y}]_{\mathcal{T}ow}$, or simply $[\mathfrak{X}, \mathfrak{Y}]$.

We write $\pi_k(\mathfrak{Y}; \mathfrak{X})$ for $[\Sigma^k\mathfrak{X}, \mathfrak{Y}]$ $(k \geq 0)$, and call $\pi_*(\mathfrak{Y}; \mathfrak{X})$ the *homotopy groups of* $\mathfrak{Y}$ *with coefficients in* $\mathfrak{X}$. A tower map $\mathfrak{f} : \mathfrak{Y} \to \mathfrak{Z}$ will be called an $\mathfrak{X}$-*weak equivalence* if it induces an isomorphism in $\pi_*(-; \mathfrak{X})$.

(e) A diagram of towers $\mathfrak{F} : I \to \mathcal{T}ow$ will be called *strict* if there is an N such that for each morphism i of I, $n(\mathfrak{F}(i)) \leq N$. This means that the diagram $\sigma_N\mathfrak{F}$ can be written as a tower of diagrams of spaces (rather than merely a diagram of towers). Of course, every finite diagram of towers is strict.

Also, a tower map $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ will be called a *cofibration* if for some strict representative $\{f[k] : \boldsymbol{X}[k] \to \boldsymbol{Y}[k]\}_{k=n}^{\infty}$ each $f[k]$ is a cofibration.

$\mathcal{T}ow$ is in fact a *simplicial* category ([BK1, X, 3.1] – i.e., it has simplicial Hom-sets:

Definition 2.4. For any two towers $\mathfrak{X}$, $\mathfrak{Y}$ the *strict function complex* $\boldsymbol{map}^{st}_{\mathcal{T}ow}(\mathfrak{X},\mathfrak{Y})$ is defined (as for topological spaces or simplicial sets – see [M1, §6.4]) to be the simplicial set whose n-simplices are $\boldsymbol{map}^{st}_{\mathcal{T}ow}(\mathfrak{X},\mathfrak{Y})_n \stackrel{Def}{=} Hom^{st}_{\mathcal{T}ow}(\mathfrak{X}\times\boldsymbol{\Delta}[n],\mathfrak{Y})$. $\boldsymbol{\Delta}[n]$ denotes as usual the standard topological n-simplex.

In light of (2.3), if $\mathfrak{X}$ is a tower of epimorphisms (e.g., if each p_n is a fibration) we may define the *(weak) function complex*

$$\boldsymbol{map}_{\mathcal{T}ow}(\mathfrak{X},\mathfrak{Y}) \stackrel{Def}{=} \operatorname{colim}_n \boldsymbol{map}^{st}_{\mathcal{T}ow}(\sigma_n\mathfrak{X},\mathfrak{Y}), \tag{2.5}$$

(so that $\boldsymbol{map}_{\mathcal{T}ow}(\mathfrak{X},\mathfrak{Y})_n = Hom_{\mathcal{T}ow}(\mathfrak{X}\times\boldsymbol{\Delta}[n],\mathfrak{Y})$). Note that

$$\dots\boldsymbol{map}^{st}_{\mathcal{T}ow}(\sigma_{n-1}\mathfrak{X},\mathfrak{Y}) \hookrightarrow \boldsymbol{map}^{st}_{\mathcal{T}ow}(\sigma_n\mathfrak{X},\mathfrak{Y}) \hookrightarrow \dots$$

is a sequence of cofibrations, so the limit here is in fact a homotopy colimit. It is not hard to see that as usual:

$$[\Sigma^t\mathfrak{X},\mathfrak{Y}] \cong \pi_t(\boldsymbol{map}_*(\mathfrak{X},\mathfrak{Y})) \qquad \text{for all } t\geq 0. \tag{2.6}$$

Remark 2.7. Given towers of spaces $\mathfrak{X}$, $\mathfrak{Y}$ as above, the function complex $\boldsymbol{map}_{\mathcal{T}ow}(\mathfrak{X},\mathfrak{Y})$ may be described explicitly by means of limits if we assume $\mathfrak{Y}$ is a tower of fibrations, as follows:

Let $Hom^{[n]}_{\mathcal{T}ow}(\mathfrak{X},\mathfrak{Y})$ denote the set of strict maps between towers "through the n-th level" – that is, sequences $\mathfrak{f} = \{f[k] : \boldsymbol{X}[k] \to \boldsymbol{Y}[k]\}_{k=0}^{n}$ of maps such that $q_k \circ f[k] = f[k-1]\circ p_k$ for $0 < k \leq n$. Likewise we may define the "truncated function complexes" $\boldsymbol{map}^{[n]}_*(\mathfrak{X},\mathfrak{Y})$ by replacing $Hom^{st}_{\mathcal{T}ow}(-,-)$ by $Hom^{[n]}_{\mathcal{T}ow}(-,-)$ in definition 2.4 above. There is an obvious projection map $\pi : \boldsymbol{map}^{[n]}_*(\mathfrak{X},\mathfrak{Y}) \to \boldsymbol{map}_*(\boldsymbol{X}[n],\boldsymbol{Y}[n])$. Since $\mathfrak{Y}$ is a tower of fibrations, we have a pullback diagram
and $\boldsymbol{map}^{st}_{\mathcal{T}ow}(\mathfrak{X},\mathfrak{Y}) = \lim_n \boldsymbol{map}^{[n]}_*(\mathfrak{X},\mathfrak{Y})$.

$$\begin{array}{ccc}
\boldsymbol{map}_*^{[n]}(\mathfrak{X},\mathfrak{Y}) & \longrightarrow & \boldsymbol{map}_*(\boldsymbol{X}[n],\boldsymbol{Y}[n]) \\
\downarrow & & \downarrow (q_n)_\# \\
\boldsymbol{map}_*^{[n-1]}(\mathfrak{X},\mathfrak{Y}) & \xrightarrow{(p_n)^\# \circ \pi} & \boldsymbol{map}_*(\boldsymbol{X}[n],\boldsymbol{Y}[n-1
\end{array}$$

3. VIRTUAL TOWERS

As noted in §1.1, for the purposes of the next section we need to work in a cocomplete category – or at least, one which we can construct pushouts, infinite coproducts (wedges), and realizations of simplicial objects. The category $\mathcal{T}ow$ has finite colimits, but it is not hard to see that a wedge $\mathfrak{Y} = \bigvee_{i=1}^{\infty} \mathfrak{X}_i$ of infinitely many towers (as defined in §2.1) is *not* the categorical coproduct, in as much as the maps $\mathfrak{f} : \mathfrak{Y} \to \mathfrak{Z}$ are not in one-to-one correspondence with arbitrary collections of maps $\{\mathfrak{f}_i : \mathfrak{X} \to \mathfrak{Z}\}_{i=1}^{\infty}$. We now describe the "cocompletion of $\mathcal{T}ow$ under (homotopy) colimits", which we call the category of *virtual* towers; these are essentially filtered towers with a prescribed collection of filtrations, and the definition is motivated by our need to force $\bigvee_{i=1}^{\infty} \mathfrak{X}_i$ to become the categorical coproduct (see §3.6 below).

It should be pointed out that we are really interested in the homotopy theory of (virtual) towers, and thus would like a closed model category structure (in the sense of Quillen – cf. [Q1]) for the category of (virtual) towers (compare [EH, §3.3]). We hope to address this question in a future paper.

Definition 3.1. A *virtual tower* $\langle \hat{\mathfrak{X}}, F, \mathcal{F} \rangle$ is a sequence of spaces $\hat{\mathfrak{X}} = \{\boldsymbol{X}[n]\}_{n=0}^{\infty}$, together with a *filtration by towers* $F = \{F_k\hat{\mathfrak{X}}\}_{k=0}^{\infty}$ – that is, a sequence of tower maps $\mathfrak{i}_k : F_k\hat{\mathfrak{X}} \hookrightarrow F_{k+1}\hat{\mathfrak{X}}$ $(k \geq 0)$, each a cofibration, with $n(\mathfrak{i}_k) = k$, such that $\boldsymbol{X}[n] = F_n\boldsymbol{X}[n]$ for all n. (We allow the trivial tower $F_k\hat{\mathfrak{X}} = \mathfrak{C}(pt)$). Thus a virtual tower has *partial level maps* $F_n\boldsymbol{X}[n+1] \xrightarrow{F_n p_n} F_n\boldsymbol{X}[n] = \boldsymbol{X}[n]$.

In addition, we are given a set $\mathcal{F}$ of *allowable refinements* of the above maximal filtration F – i.e., filtrations by towers $F' = \{F'_0\hat{\mathfrak{X}} \overset{\mathfrak{i}'_0}{\hookrightarrow} F'_1\hat{\mathfrak{X}} \hookrightarrow \ldots\}$ together with cofibrations $\mathfrak{j}_k : F'_k\hat{\mathfrak{X}} \hookrightarrow F_k\hat{\mathfrak{X}}$ such that

$$\boldsymbol{X}[n] = \bigcup_{n(\mathfrak{j}_k) \leq n} F'_k\boldsymbol{X}[n] \qquad \text{for all } n.$$

This set $\mathcal{F}$ is assumed to be directed – i.e., any two filtrations $F', F'' \in \mathcal{F}$ have a common refinement $F''' \in \mathcal{F}$.

In most cases we shall allow any possible refinement of the given maximal F to belong to $\mathcal{F}$; and we shall often abbreviate $\langle\hat{\mathfrak{X}}, F, \mathcal{F}\rangle$ by $\hat{\mathfrak{X}}$, distinguishing $\mathcal{F}$ if necessary as $\mathcal{F}_{\hat{\mathfrak{X}}}$.

Alternatively, the virtual tower $\langle\hat{\mathfrak{X}}, F, \mathcal{F}\rangle$ may be thought of as the equivalence class $\mathcal{F}$ of sequences of filtrations by towers $F' = \{F'_0\hat{\mathfrak{X}} \hookrightarrow F'_1\hat{\mathfrak{X}} \hookrightarrow \ldots\}$, where $F' \sim F''$ if there is a third sequence F''' having both as refinements. We must place some restriction on the filtrations (e.g., to ensure that they form a set), and then the maximal filtration is $F_k\hat{\mathfrak{X}} = \bigcup_{F'\in\mathcal{F}} F'_k\hat{\mathfrak{X}}$.

Definition 3.2. A *virtual map* $\hat{\mathfrak{f}} : \langle\hat{\mathfrak{X}}, F, \mathcal{F}\rangle \to \langle\hat{\mathfrak{Y}}, G, \mathcal{G}\rangle$ between virtual towers is a sequence of (weak) tower maps $\mathfrak{f}_k : F'_k\hat{\mathfrak{X}} \to G'_k\hat{\mathfrak{Y}}$ for some $F' \in \mathcal{F}$ and $G' \in \mathcal{G}$, such that

$$\begin{array}{ccc} F'_k\hat{\mathfrak{X}} & \xrightarrow{\mathfrak{f}_k} & G'_k\hat{\mathfrak{Y}} \\ {\scriptstyle i'_k}\downarrow & & \downarrow{\scriptstyle i'_k} \\ F'_{k+1}\hat{\mathfrak{X}} & \xrightarrow{\mathfrak{f}_{k+1}} & G'_{k+1}\hat{\mathfrak{Y}} \end{array}$$

commutes. We say that $\hat{\mathfrak{f}}$ *is defined* with respect to the filtrations F' and G'. Of course, we can always take $G' = G$, the maximal filtration on $\hat{\mathfrak{Y}}$. Unless there is risk of confusion, we shall usually use F to denote the maximal filtration for all virtual towers at hand (and write $F_k\mathfrak{f}$ for $\mathfrak{f}_k$).

The category of virtual towers and maps will be denoted $v\mathcal{T}ow$. There is an embedding of categories $I : \mathcal{T}ow \hookrightarrow v\mathcal{T}ow$ with $I(\mathfrak{X}) = \langle\hat{\mathfrak{X}}, F, \mathcal{F}\rangle$, where $F_k\hat{\mathfrak{X}} = \mathfrak{X}$ for all k, and $\mathcal{F}$ consists of filtrations of the form $\mathfrak{C}(pt) = \ldots = \mathfrak{C}(pt) \subset \mathfrak{X} = \mathfrak{X}\ldots$; we shall often denote $I(\mathfrak{X})$ simply by $\mathfrak{X}$. For $\mathfrak{X} \in \mathcal{T}ow$ and $\hat{\mathfrak{Y}} \in v\mathcal{T}ow$ we then have

$$Hom_{v\mathcal{T}ow}(I(\mathfrak{X}), \hat{\mathfrak{Y}}) \cong \operatorname{colim}_k\, Hom_{\mathcal{T}ow}(\mathfrak{X}, F_k\hat{\mathfrak{Y}}). \tag{3.3}$$

Definition 3.4. Like $\mathcal{T}ow$ (§2.4), the category $v\mathcal{T}ow$ is also a simplicial category: for any two virtual towers $\hat{\mathfrak{X}}$, $\hat{\mathfrak{Y}}$, the *function complex* $\boldsymbol{map}_{v\mathcal{T}ow}(\hat{\mathfrak{X}}, \hat{\mathfrak{Y}})$, is again the simplicial set with

$$\boldsymbol{map}_{v\mathcal{T}ow}(\hat{\mathfrak{X}}, \hat{\mathfrak{Y}})_n \overset{Def}{=} Hom_{v\mathcal{T}ow}(\hat{\mathfrak{X}} \times \boldsymbol{\Delta}[n], \hat{\mathfrak{Y}}).$$

Again (3.3) implies that if $\mathfrak{X}$ is an ordinary tower then

$$\boldsymbol{map}_{v\mathcal{T}ow}(I(\mathfrak{X}), \hat{\mathfrak{Y}}) \;=\; \operatorname{colim}_k\, \boldsymbol{map}_{\mathcal{T}ow}(\mathfrak{X}, F_k\hat{\mathfrak{Y}}). \tag{3.5}$$

Example 3.6. If $\{\hat{\mathfrak{X}}_\alpha\}_{\alpha\in A}$ is some collection of virtual towers, define a virtual tower $\hat{\mathfrak{Y}} = \bigvee_{\alpha\in A}\hat{\mathfrak{X}}_\alpha$ as follows:

$\boldsymbol{Y}[n] = \bigvee_{\alpha\in A}\boldsymbol{X}_\alpha[n]$ and $F_k\hat{\mathfrak{Y}} = \bigvee_{\alpha\in A}(F_k)_\alpha\hat{\mathfrak{X}}_\alpha$, and the allowable refinements of F are those of the form $F'_k\hat{\mathfrak{Y}} = \bigvee_{\alpha\in B}(F'_k)_\alpha\hat{\mathfrak{X}}_\alpha$ where $B \subseteq A$ and $(F')_\alpha \in \mathcal{F}_{\hat{\mathfrak{X}}_\alpha}$. This is in fact the *coproduct* in the category of (pointed) virtual towers – i.e.,

$$Hom_{v\mathcal{T}ow}(\hat{\mathfrak{Y}}, \hat{\mathfrak{Z}}) \cong \prod_{\alpha\in A} Hom_{v\mathcal{T}ow}(\hat{\mathfrak{X}}_\alpha, \hat{\mathfrak{Z}}).$$

Example 3.7. Similarly, given two virtual maps $\hat{\mathfrak{f}} : \hat{\mathfrak{X}} \to \hat{\mathfrak{Y}}$ and $\hat{\mathfrak{g}} : \hat{\mathfrak{X}} \to \hat{\mathfrak{Z}}$, where $\hat{\mathfrak{g}}$ is a cofibration, say, we can define the *pushout* $\hat{\mathfrak{W}}$ as follows:

Choose a $F' \in \mathcal{F}_{\hat{\mathfrak{X}}}$ with respect to which $\hat{\mathfrak{f}}$ and $\hat{\mathfrak{g}}$ are both defined, so that

$$\begin{array}{ccc} F'_k\hat{\mathfrak{X}} & \xrightarrow{F_k\hat{\mathfrak{g}}} & F_k\mathfrak{Z} \\ {\scriptstyle F_k\hat{\mathfrak{f}}}\downarrow & & \\ F_k\hat{\mathfrak{Y}} & & \end{array} \tag{3.8}$$

is a compatible collection of weak tower maps (for all k).

For each k, let $n(k) = max\{n(F_k\hat{\mathfrak{f}}), n(F_k\hat{\mathfrak{g}})\}$, so (3.8) is a strict diagram in levels $\geq n(k)$. Define $F_k\hat{\mathfrak{W}}[n]$ to be the pushout of the n-th level of (3.8). This forms a tower $F_k\hat{\mathfrak{W}}$ (in levels $\geq n(k)$ – see §2.1), and there is a cofibration $\mathfrak{i}_k : F_k\hat{\mathfrak{W}} \hookrightarrow F_{k+1}\hat{\mathfrak{W}}$ for each k. Now set $\boldsymbol{W}[n] = F_n\hat{\mathfrak{W}}[n]$, and the allowable refinements of the given filtration $\{F_k\hat{\mathfrak{W}}\}$ are those obtained from compatible allowable refinements of $F_{\hat{\mathfrak{X}}}$, $F_{\hat{\mathfrak{Y}}}$, and $F_{\hat{\mathfrak{Z}}}$.

Definition 3.9. If $T : \mathcal{T}_\star \to \mathcal{T}_\star$ is a cofibration-preserving functor on spaces, applying it levelwise to each filtration of a virtual tower $\hat{\mathfrak{X}}$ yields a new virtual tower $T\hat{\mathfrak{X}}$ – for example, $\hat{\mathfrak{X}} \times I$ or $\Sigma\hat{\mathfrak{X}}$. Thus one has a concept of *virtual homotopies*, and as usual we denote by $[\hat{\mathfrak{X}}, \hat{\mathfrak{Y}}]$ the set of virtual homotopy classes of virtual maps between the virtual towers $\hat{\mathfrak{X}}$ and $\hat{\mathfrak{Y}}$ (or use the variant notation of §2.2(d) above). In light of (2.5) above, if $\mathfrak{X}$ is an ordinary tower we have

$$\pi_t(\hat{\mathfrak{Y}}; \mathfrak{X}) \cong \text{colim}_k\ \pi_t(F_k\hat{\mathfrak{Y}}; \mathfrak{X}) \tag{3.10}$$

Definition 3.11. A *simplicial tower* (resp. *virtual tower*) is a simplicial object in the category $\mathcal{T}ow$ (resp. $v\mathcal{T}ow$) – cf. [M1, §2.1]. In particular:

(a) A simplicial tower $\mathfrak{X}_\bullet$ is called *proper* (compare [M2, Definition 11.2]) if each degeneracy map $\mathfrak{s}_j[n] : \boldsymbol{X}_r[n] \to \boldsymbol{X}_{r+1}[n]$ is the inclusion of a sub-complex for each $n \geq n(\mathfrak{s}_j) = 0$.

(b) Given a proper simplicial tower $\mathfrak{X}_\bullet$, its *q-skeleton* $Sk_q\mathfrak{X}_\bullet$ is the simplicial tower defined:
$(Sk_q\mathfrak{X}_\bullet)_r = \hat{\mathfrak{X}}_r$ for $r \leq q$, and $(Sk_q\mathfrak{X}_\bullet)_r = \bigcup_{j=0}^r im(s_j|_{(Sk_q\mathfrak{X}_\bullet)_r}) \subseteq \mathfrak{X}_r[n]$ for $r \geq q$ and $n \geq N$, where N is the maximal $n(\lambda)$ for $\lambda : \mathfrak{X}_r \to \mathfrak{X}_{r\pm1}$ a face or degneracy map of towers in simplicial dimension $\leq n$. The simplicial identities guarantee that the restriction of the face and degeneracy maps of $\mathfrak{X}_\bullet$ to $Sk_q\mathfrak{X}_\bullet$ define a simplicial tower.

(c) A simplicial virtual tower $\langle\hat{\mathfrak{X}}_\bullet, F_\bullet, \mathcal{F}_\bullet\rangle$ is called *proper* if all face and degeneracy maps $\phi : \hat{\mathfrak{X}}_r \to \hat{\mathfrak{X}}_{r\pm1}$ are *defined* (§3.2) with respect to the given maximal filtrations F_r and $F_{r\pm1}$. This means that for all $k \geq 0$, $F_k\hat{\mathfrak{X}}_\bullet$ is an (ordinary) simplicial tower. We require also that it be proper in the sense of (a) above.

Definition 3.12. Let $\langle\hat{\mathfrak{X}}_\bullet, F_\bullet, \mathcal{F}_\bullet\rangle$ be a proper simplicial virtual tower; then its *realization* is the virtual tower $\|\hat{\mathfrak{X}}_\bullet\|$ defined as follows:

For each k, $F_k\hat{\mathfrak{X}}_\bullet$ is a proper simplicial tower, so its q-skeleton $Sk_qF_k\hat{\mathfrak{X}}_\bullet$ is in fact a strict simplicial tower (see §2.2(e)) – i.e., $Sk_qF_k\hat{\mathfrak{X}}_\bullet[n]$ is actually a simplicial space for $n \geq N(k,q)$, and these fit together into a tower of simplicial spaces. (Here $N(k,q)$ is the least level at which all face and degeneracy maps of $Sk_qF_k\hat{\mathfrak{X}}_\bullet$ have strict representatives).

Now recall (from [Se, §1] or [M2, §11.1]) that to a simplicial space $\boldsymbol{X}_\bullet$ we can associate a single space $\|\boldsymbol{X}_\bullet\|$, called its *realization*, or homotopy colimit. Thus realizing $Sk_qF_k\hat{\mathfrak{X}}_\bullet$ levelwise for each k, q yields together a bifiltered virtual tower, which we denote by $\|\hat{\mathfrak{X}}_\bullet\|$, with the *diagonal filtration* $F_k\|\hat{\mathfrak{X}}_\bullet\| = \|Sk_jF'_j\hat{\mathfrak{X}}_\bullet\|$, where j is maximal such that $N(j,j) \leq k$.

Definition 3.13. A diagram of virtual towers $\hat{\mathfrak{F}} : I \to v\mathcal{T}ow$ will be called *strict* if for each k, the diagram of ordinary towers $F_k\hat{\mathfrak{F}} : I \to \mathcal{T}ow$ is strict (§2.2(e)). Again, every finite diagram is strict.

In this case, we define the homotopy colimit of the diagram, written hocolim $\hat{\mathfrak{F}} \in v\mathcal{T}ow$, to be the virtual tower filtered by $F_k$hocolim $\hat{\mathfrak{F}} \stackrel{Def}{=}$ hocolim $F_k\hat{\mathfrak{F}} \in \mathcal{T}ow$.

One could in fact define homotopy colimits for arbitrary diagrams of virtual towers; we have done so only in the two cases we shall require, namely, the (infinte) coproduct (§3.6) and the realization of a simplicial virtual tower (§3.12).

4. v-periodicity

Now assume given a fixed model $\boldsymbol{M}$ with a self map $v : \boldsymbol{M}^{d+r} \to \boldsymbol{M}^{r}$, as in §1.2. We here make explicit the relation between towers and v-periodic homotopy groups, which motivated the previous two sections:

Definition 4.1. From v we can construct various towers of the form

$$\mathfrak{M} = \{\ldots \to \boldsymbol{M}^{k_n} \xrightarrow{v^{r_n}} \boldsymbol{M}^{k_{n-1}} \xrightarrow{v^{r_{n-1}}} \ldots \to \boldsymbol{M}^{k_o}\},$$

where $\{k_n\}_{n=0}^{\infty}$ is some increasing sequence of non-negative integers, and of course $r_n = (k_n - k_{n-1})/d$. Such a tower $\mathfrak{M}$ will be called a *v-model tower*. If $\mathfrak{M}'$ is another v-model tower, a map $\mathfrak{h} : \mathfrak{M}' \to \mathfrak{M}$ is called a *v-map* if at each level $\mathfrak{h}[n] = v^{e_n}$ for some $e_n \geq 0$. We denote by $\mathcal{M} = \mathcal{M}_v$ the set of all v-model towers; this is partially ordered by $\succeq$, where $\mathfrak{M}' \succeq \mathfrak{M}$ if there is a v-map $\mathfrak{h} : \mathfrak{M}' \to \mathfrak{M}$ (necessarily unique). In order to make use of the function complexes of (2.5), we can assume when necessary that all v-model towers are towers of fibrations (at the price of replacing $\boldsymbol{M}^{k_n} = \Sigma^{k_n - r_0}\boldsymbol{M}$ by a homotopy equivalent space, and v by a homotopic map; this may be done without affecting any of our arguments below).

Define the *v-periodic* homotopy groups of a virtual (or ordinary) tower $\hat{\mathfrak{X}}$ to be

$$v^{-1}\pi_t\hat{\mathfrak{X}} = \lim_{\langle \mathcal{M}, \succeq \rangle} \pi_t(\hat{\mathfrak{X}}; \mathfrak{M}),$$

where the limit is taken over all v-maps. Note that the graded group $v^{-1}\pi_\star\hat{\mathfrak{X}}$ is periodic in the sense that there is a natural isomorphism $v^{-1}\pi_t\hat{\mathfrak{X}} \cong v^{-1}\pi_{t+d}\hat{\mathfrak{X}}$ (induced by the obvious v-map $\mathfrak{v} : \Sigma^d\mathfrak{M} \to \mathfrak{M}$ for any v-model tower $\mathfrak{M}$). Thus in fact

$$v^{-1}\pi_t\hat{\mathfrak{X}} = \lim_{\langle \mathcal{M}, \succeq \rangle} \operatorname{colim}_k \pi_t(F_k\hat{\mathfrak{X}}; \mathfrak{M}). \tag{4.2}$$

Finally, for any space $\boldsymbol{X} \in \mathcal{T}_\star$ and any v-model tower $\mathfrak{M}$ there is an isomorphism:

$$\pi_t(\mathfrak{C}(\boldsymbol{X}); \mathfrak{M}) \;\cong\; v^{-1}\pi_t(\boldsymbol{X}; \boldsymbol{M}), \tag{4.3}$$

so that in this case $v^{-1}\pi_\star\mathfrak{C}(\boldsymbol{X}) \cong v^{-1}\pi_*(\boldsymbol{X}; \boldsymbol{M})$ are actually periodic in the above sense, though for an arbitrary (ordinary or virtual) tower $\mathfrak{X}$, the $\mathfrak{M}$-homotopy groups $\pi_\star(\mathfrak{X}; \mathfrak{M})$ need not be – i.e., in general there will be no $q > 0$ such that $v^{-1}\pi_{t+q}\mathfrak{X} \cong v^{-1}\pi_t\mathfrak{X}$ for all t.

Definition 4.4. An (ordinary) tower $\mathfrak{X} = \{\ldots \to \boldsymbol{X}[n] \xrightarrow{p_n} \boldsymbol{X}[n-1]\ldots\}$ will be called *v-regular* if each p_n induces an isomorphism in $v^{-1}\pi_*(-;\boldsymbol{M})$. A virtual tower $\langle \hat{\mathfrak{X}}, F, \mathcal{F}\rangle$ will be called *v-regular* if each $F_k\hat{\mathfrak{X}}$ is such.

For an ordinary tower $\mathfrak{X}$ let $Q_t(\mathfrak{X}) \stackrel{Def}{=} \lim_n v^{-1}\pi_t(\boldsymbol{X}[n];\boldsymbol{M})$, where we think of $v^{-1}\pi_*(-;\boldsymbol{M})$ as a $\mathbb{Z}/d$-graded abelian group. An element $\beta \in Q_t(\mathfrak{X})$ is represented by a collection of maps $f_n : \boldsymbol{M}^{t+j_n d} \to \boldsymbol{X}[n]$ $(n \geq 0)$ with $p_n \circ f_n = f_{n-1} \circ v^{j_n - j_{n-1}}$ for some increasing non-negative sequence $\{j_n\}_{n=0}^{\infty}$. For a virtual tower $\hat{\mathfrak{X}} = \langle \hat{\mathfrak{X}}, F, \mathcal{F}\rangle$ we define $Q_t(\hat{\mathfrak{X}}) \stackrel{Def}{=} \operatorname{colim}_k Q_t(F_k\hat{\mathfrak{X}})$.

Lemma 4.5. *For any self-map v and tower $\hat{\mathfrak{X}}$ there are natural homomorphisms $\Phi_t : v^{-1}\pi_t\hat{\mathfrak{X}} \to Q_t(\hat{\mathfrak{X}})$ (for each $t \geq 0$), such that Φ_t is an isomorphism if $\hat{\mathfrak{X}}$ is v-regular.*

Proof. Φ_t is induced by homomorphisms $\Phi_{(t,F_k\hat{\mathfrak{X}},\mathfrak{M})} : \pi_t(F_k\hat{\mathfrak{X}};\mathfrak{M}) \to Q_t(F_k\hat{\mathfrak{X}})$ for any v-model tower $\mathfrak{M} = \{\ldots \to \boldsymbol{M}^{k_n d} \xrightarrow{v^{r_n}} \boldsymbol{M}^{k_{n-1} d}\ldots\}$, natural in $\mathfrak{M}$ and $F_k\hat{\mathfrak{X}}$. To define these, note that any $\alpha \in \pi_t(F_k\hat{\mathfrak{X}};\mathfrak{M})$ is represented by a sequence of maps $f_n : \boldsymbol{M}^{t+k_n} \to F_k\boldsymbol{X}[n]$, with $p_n \circ f_n = f_{n-1} \circ v^{r_n}$ – i.e., $(F_k p_n)_\#[f_n] \equiv [f_{n-1}]$ in $v^{-1}\pi_t(F_k\boldsymbol{X}[n-1];\boldsymbol{M})$, so $\{[f_n]\}_{n=0}^{\infty}\}$ is a well-defined element $\Phi_{(t,F_k\hat{\mathfrak{X}},\mathfrak{M})}(\alpha)$ in $Q_t(F_k\hat{\mathfrak{X}})$. Note that in $Q_t(F_k\hat{\mathfrak{X}})$ the degree t is only considered modulo d.

Now let $\hat{\mathfrak{X}}$ be v-regular:

First, assume $\Phi_t([\gamma]) = 0$ for some $\gamma \in v^{-1}\pi_t\hat{\mathfrak{X}}$. Then $\Phi_{(t,F_k\hat{\mathfrak{X}},\mathfrak{M})}([\mathfrak{g}]) = 0$ for some v-model tower $\mathfrak{M}$ and $\mathfrak{g} : \mathfrak{M} \to F_k\hat{\mathfrak{X}}$ representing γ. This means that for each n there is an e_n such that $(v^{e_n})^\# g_n \simeq 0$. Define another v-model tower $\mathfrak{M}'$ by $\mathfrak{M}'[n] = \Sigma^{e_n d}\boldsymbol{M}[n]$, with the obvious v-map $\mathfrak{h} : \mathfrak{M}' \to \mathfrak{M}$; then $\mathfrak{h}^\#[\mathfrak{g}] = 0$, so $\gamma = 0 \in v^{-1}\pi_t F_k\hat{\mathfrak{X}}$ and thus Φ_t is a monomorphism in the limit.

Next, given an element $\beta \in Q_t(\hat{\mathfrak{X}})$, represent it by a sequence of maps $f_n : \boldsymbol{M}^{t+j_n d} \to F_k\boldsymbol{X}[n]$ with $p_n \circ f_n = f_{n-1} \circ v^{j_n - j_{n-1}}$. Defining a tower $\mathfrak{M}$ by $\boldsymbol{M}[n] = \boldsymbol{M}^{j_n d}$ we have $\mathfrak{f} : \Sigma^k\mathfrak{M}' \to \hat{\mathfrak{X}}$ with $\Phi_{(t,F_k\hat{\mathfrak{X}},\mathfrak{M})}([\mathfrak{f}]) = \beta$, so again Φ_t is an epimorphism in the limit. □

Remark 4.6. Of course, if an ordinary tower $\mathfrak{X}$ is v-regular, then

$$v^{-1}\pi_t\mathfrak{X} \cong Q_t(\mathfrak{X}) \cong v^{-1}\pi_t(\boldsymbol{X}[n];\boldsymbol{M}) \qquad \text{for each } n \geq 0;$$

similarly, for v-regular virtual towers $v^{-1}\pi_t\hat{\mathfrak{X}} \cong \operatorname{colim}_k v^{-1}\pi_t(F_k\boldsymbol{X}[n];\boldsymbol{M})$.

5. The v-periodic resolution

Now that we have the proper set-up, given a self map v as above, we can use the Stover construction to define, for any virtual tower of spaces $\hat{\mathfrak{X}}$, a simplical tower $\hat{\mathfrak{J}}_\bullet$ which serves as the "v-periodic resolution" of $\hat{\mathfrak{X}}$ (in a sense to be made precise below). We shall actually only need the case where $\hat{\mathfrak{X}} = \mathfrak{C}(\boldsymbol{X})$ is the constant tower of some space $\boldsymbol{X}$, though the construction works in general.

5.1. **the mapping cotriple.** As in [St, §2], we define a functor $\mathcal{J}_v : v\mathcal{T}ow \to v\mathcal{T}ow$ by setting $\mathcal{J}_v(\hat{\mathfrak{X}}) = \bigvee_{\mathfrak{M}\in\mathcal{M}} \hat{\mathfrak{Z}}_{\mathfrak{M}}$, where $\hat{\mathfrak{Z}}_{\mathfrak{M}}$ is defined to be the functorial pushout (§3.7) of the diagram

$$
\begin{array}{ccc}
\bigvee_{\mathfrak{F}\in Hom_{v\mathcal{T}ow}(C\mathfrak{M},\hat{\mathfrak{X}})} \mathfrak{M}_{\mathfrak{F}} & \xrightarrow{\bigvee_{\mathfrak{F}} \mathfrak{i}_{\mathfrak{F}}} & \bigvee_{\mathfrak{F}\in Hom_{v\mathcal{T}ow}(C\mathfrak{M},\hat{\mathfrak{X}})} C\mathfrak{M}_{\mathfrak{F}} \\
\Big\downarrow{\scriptstyle \varphi^*} & & \vdots \\
\bigvee_{\mathfrak{f}\in Hom_{v\mathcal{T}ow}(\mathfrak{M},\hat{\mathfrak{X}})} \mathfrak{M}_{\mathfrak{f}} & \cdots\cdots\to & \hat{\mathfrak{Z}}_{\mathfrak{M}}
\end{array}
\tag{5.2}
$$

Here $\mathfrak{i} : \mathfrak{M} \hookrightarrow C\mathfrak{M}$ is the natural inclusion (at each level) of the space $\boldsymbol{M}[n]$ into its cone $C\boldsymbol{M}[n]$, and φ^* takes the copy of $\mathfrak{M}$ indexed by a map $\mathfrak{F} : C\mathfrak{M} \to \hat{\mathfrak{X}}$ in the upper right-hand coproduct isomorphically to the copy indexed by $\mathfrak{i}^{\#}\mathfrak{F}$ in the lower right-hand coproduct.

$\mathcal{J}_v$ is clearly a cotriple on the category of virtual towers, with the obvious counit $\varepsilon : \mathcal{J}_v(\hat{\mathfrak{X}}) \to \hat{\mathfrak{X}}$ – namely, "evaluation", with $\varepsilon|_{\mathfrak{M}_{\mathfrak{f}}} = \mathfrak{f}$ and $\varepsilon|_{C\mathfrak{M}_{\mathfrak{F}}} = \mathfrak{F}$ – and comultiplication $\mu : \mathcal{J}_v(\hat{\mathfrak{X}}) \to \mathcal{J}_v(\mathcal{J}_v(\hat{\mathfrak{X}}))$ – where $\mu|_{\mathfrak{M}_{\mathfrak{f}}}$ is an isomorphism onto the copy of $\mathfrak{M}$ in $\mathcal{J}_v(\mathcal{J}_v(\hat{\mathfrak{X}}))$ indexed by the inclusion $\mathfrak{M}_{\mathfrak{f}} \hookrightarrow \mathcal{J}_v(\hat{\mathfrak{X}})$, for any $\mathfrak{f} : \mathfrak{M} \to \hat{\mathfrak{X}}$; and similarly for $C\mathfrak{M}_{\mathfrak{F}}$.

The filtration on $\mathcal{J}_v(\hat{\mathfrak{X}})$ is by "level of origin of indices" – i.e., $\mathfrak{M}_{\hat{\mathfrak{f}}} \subseteq F_k\mathcal{J}_v(\hat{\mathfrak{X}}) \Leftrightarrow \hat{\mathfrak{f}} : \mathfrak{M} \to \hat{\mathfrak{X}}$ has a strict representative $\mathfrak{f} : \sigma_k\mathfrak{M} \to F_k\hat{\mathfrak{X}}$, and similarly for $C\mathfrak{M}_{\hat{\mathfrak{F}}}$. This clearly implies that $\varepsilon : \mathcal{J}_v(\hat{\mathfrak{X}}) \to \hat{\mathfrak{X}}$ is *defined* (§3.2) with respect to $F_{\mathcal{J}_v(\hat{\mathfrak{X}})}$, $F_{\hat{\mathfrak{X}}}$.

Now given $\hat{\mathfrak{X}} \in v\mathcal{T}ow$, one may define a functorial simplicial virtual tower $\hat{\mathfrak{J}}_\bullet$ by setting $\hat{\mathfrak{J}}_n = \mathcal{J}_v^{n+1}\hat{\mathfrak{X}}$, with face and degeneracy maps induced by the counit and comultiplication respectively (cf. [Go, Appendix, §3]). The counit also induces an augmentation $\varepsilon : \hat{\mathfrak{J}}_\bullet \to \hat{\mathfrak{X}}$.

Fact 5.3. The virtual simplicial tower $\hat{\mathfrak{J}}_\bullet$ defined above is proper (see definition 3.11).

Proof. Note that any degeneracy map $\mathfrak{s}_j : \hat{\mathfrak{J}}_r \to \hat{\mathfrak{J}}_{r+1}$, and all face maps $\mathfrak{d}_i : \hat{\mathfrak{J}}_r \to \hat{\mathfrak{J}}_{r-1}$ except for $\mathfrak{d}_0$, are isomorphisms on the coproduct summands in the description of $\hat{\mathfrak{J}}_r = \mathcal{J}_v(\hat{\mathfrak{J}}_{r-1})$ above, and thus are defined with respect to the filtration on the $\hat{\mathfrak{J}}_r$'s (cf. §3.6 & 3.7 above). Since $\mathfrak{d}_0 : \hat{\mathfrak{J}}_r \to \hat{\mathfrak{J}}_{r+1}$ is just $\varepsilon : \mathcal{J}_v(\hat{\mathfrak{J}}_{r-1}) \to \hat{\mathfrak{J}}_{r-1}$, it too is defined with respect to given filtration on $\mathcal{J}_v(\hat{\mathfrak{J}}_{r-1})$. □

Remark 5.4. For any fixed $c \geq 0$, we may assume if necessary that all the level spaces $F_k\hat{\mathfrak{J}}_r[n]$ are c-connected, by replacing $\mathfrak{M}$ by $\sigma_n\mathfrak{M}$ throughout Definition 5.1. This is because the connectivities of the spaces $\boldsymbol{M}[n]$ are increasing, since we assume $d > 0$ (§1.2).

Lemma 5.5. *For any $t \geq 0$ and v-tower $\mathfrak{M}$, the augmented simplicial group*

$$\pi_t(\hat{\mathfrak{J}}_\bullet; \mathfrak{M}) \xrightarrow{\varepsilon_\#} \pi_t(\mathfrak{X}; \mathfrak{M})$$

is acyclic – that is, $\pi_s(\pi_t(\hat{\mathfrak{J}}_\bullet; \mathfrak{M})) = 0$ for $s \geq 1$, and $\pi_0(\pi_t(\hat{\mathfrak{J}}_\bullet; \mathfrak{M})) \cong \pi_t(\mathfrak{X}; \mathfrak{M})$.

Proof (cf. [St, Proposition 2.6]): write $P_\bullet$ for the simplicial group $\pi_t(\hat{\mathfrak{J}}_\bullet; \mathfrak{M})$; then using normalized chains $(N_*P_\bullet, \partial)$ (cf. [M1, §17]), one may represent any $\gamma \in N_kP_\bullet$ by a (weak) map $\mathfrak{g} : \Sigma^t\mathfrak{M} \to \hat{\mathfrak{J}}_k$, with $d_j[\mathfrak{g}] \simeq 0$ for $0 \leq j \leq k$.

But any map $\mathfrak{f} : \Sigma^t\mathfrak{M} \to \hat{\mathfrak{Y}}$ has a corresponding wedge summand $\Sigma^t\mathfrak{M}_\mathfrak{f} \overset{i_\mathfrak{f}}{\hookrightarrow} \mathcal{J}_v(\hat{\mathfrak{Y}})$, with $\varepsilon \circ i_\mathfrak{f} \simeq \mathfrak{f}$. (This is how the comultiplication $\mu : \mathcal{J}_v \to \mathcal{J}_v \circ \mathcal{J}_v$ was defined). Applying this to $\hat{\mathfrak{Y}} = \hat{\mathfrak{J}}_k$, we obtain an element $[i_\mathfrak{g}] \in P_{k+1}$ represented by $i_\mathfrak{g} : \Sigma^t\mathfrak{M}_\mathfrak{g} \to \mathcal{J}_v(\hat{\mathfrak{J}}_k)$, with $d_0[i_\mathfrak{g}] = \varepsilon_\#[i_\mathfrak{g}] = [\mathfrak{g}]$ and $d_j[i_\mathfrak{g}] = [i_{d_{j-1}[\mathfrak{g}]}]$ for $1 \leq j \leq k+1$.

Since $d_{j-1}[\mathfrak{g}] \simeq 0$ for $1 \leq j \leq k+1$, by construction each wedge summand $\Sigma^t\mathfrak{M}_{d_{j-1}\circ\mathfrak{g}} \hookrightarrow \hat{\mathfrak{J}}_k$ extends to a cone $C\Sigma^t\mathfrak{M}_{\mathfrak{F}_j} \hookrightarrow \hat{\mathfrak{J}}_k$ (presumably in more than one way), so that $i_{d_{j-1}\circ\mathfrak{g}} : \Sigma^t\mathfrak{M}_{d_{j-1}\circ\mathfrak{g}} \to \hat{\mathfrak{J}}_k$ is nullhomotopic.

We have thus found $[i_\mathfrak{g}] \in N_{k+1}P_\bullet$ with $\partial[i_\mathfrak{g}] = [d_0 \circ i_\mathfrak{g}] = \gamma$ – i.e., $\pi_kP_\bullet = 0$ for $k \geq 1$.

For $k = 0$ we have shown that if $\gamma \in P_0 = \pi_t(\mathcal{J}_v\mathfrak{X}; \mathfrak{M})$ has $\varepsilon^\#\gamma = 0$, then $\gamma \in Im\{\partial : P_1 \to P_0\}$. Thus $\pi_0P_\bullet = P_0/Ker(\varepsilon^\#) \cong Im(\varepsilon^\#) = \pi_t(\mathfrak{X}; \mathfrak{M})$, since $\varepsilon^\#$ is clearly an epimorphism. □

Corollary 5.6. *For any $t \geq 0$ the augmented simplicial group $v^{-1}\pi_t\hat{\mathfrak{J}}_\bullet \xrightarrow{\varepsilon_\star} v^{-1}\pi_t\mathfrak{X}$ is acyclic, too.*

Proof. It is a colimit of acyclic simplicial abelian groups. □

6. REALIZATIONS AND MAPPING SPACES

The category of virtual towers was needed to define the v-periodic simplicial resolution, since the construction requires that the v-periodic homotopy groups be representable (as homotopy classes of morphisms). However, since it is still more convenient to work with topological spaces, we need a mechanism for passing back from (simplicial) towers to spaces, while still preserving information about morphisms. This is provided by the mapping space functor of §2.4 & 3.4.

In order to relate this to the realization of simplicial objects, recall from [BF, Theorem B.5], [Q2] that for each simplicial space (or bisimplicial set) $\boldsymbol{X}_\bullet$ there is a spectral sequence converging to the homotopy groups of the realization (resp., diagonal), with

$$E^2_{s,t} = \pi_s(\pi_t \boldsymbol{X}_\bullet) \Rightarrow \pi_{s+t}\|\boldsymbol{X}_\bullet\|.$$

(As above, applying π_t dimensionwise to the $\boldsymbol{X}_n$'s yields a simplicial group $P[t]_\bullet = \pi_t \boldsymbol{X}_\bullet$, and $E^2_{s,t} = \pi_s(P[t]_\bullet)$.)

Lemma 6.1. *If $\boldsymbol{X}_\bullet$ is a proper simplicial space with each $\boldsymbol{X}_n$ $r-1$-connected, then for any r-dimensional CW-complex $\boldsymbol{M}$ the natural map $\gamma : \|\boldsymbol{map}_*(\boldsymbol{M}, \boldsymbol{X}_\bullet)\| \to \boldsymbol{map}_*(\boldsymbol{M}, \|\boldsymbol{X}_\bullet\|)$ is a (weak) homotopy equivalence.*

Proof. First, for $\boldsymbol{M} = \boldsymbol{S}^k$ $(k \leq r)$, we know by [M2, Theorem 12.3] that $\|\Omega \boldsymbol{X}_\bullet\| \to \Omega\|\boldsymbol{X}_\bullet\|$ is a weak equivalence; so $\|\Omega^k \boldsymbol{X}_\bullet\| \to \Omega^k\|\boldsymbol{X}_\bullet\|$ is, too. Now consider the cofibration sequence $\boldsymbol{S}^k \to \boldsymbol{K} \to \boldsymbol{M}$ (where by induction on the CW filtration of $\boldsymbol{M}$ we may assume the Lemma holds for $\boldsymbol{K}$, and of course $\boldsymbol{S}^k$). This induces a fibration sequence $\boldsymbol{map}_*(\boldsymbol{M}, \boldsymbol{X}_n) \longrightarrow \boldsymbol{map}_*(\boldsymbol{K}, \boldsymbol{X}_n) \longrightarrow \Omega^k \boldsymbol{X}_n$ for each $n \geq 0$; by [A], we obtain a fibration sequence of the realizations

$$\|\boldsymbol{map}_*(\boldsymbol{M}, \boldsymbol{X}_\bullet)\| \longrightarrow \|\boldsymbol{map}_*(\boldsymbol{K}, \boldsymbol{X}_\bullet)\| \longrightarrow \|\Omega^k \boldsymbol{X}_\bullet\|,$$

which maps to the fibration sequence

$$\boldsymbol{map}_*(\boldsymbol{M}, \|\boldsymbol{X}_\bullet\|) \longrightarrow \boldsymbol{map}_*(\boldsymbol{K}, \|\boldsymbol{X}_\bullet\|) \longrightarrow \Omega^k\|\boldsymbol{X}_\bullet\|.$$

By induction and the Five Lemma we conclude that the Lemma holds for $\boldsymbol{M}$, too. □

As a consequence we note the following:

Corollary 6.2. *For any r-dimensional CW-complex $\boldsymbol{M}$ and $(r-1)$-connected simplicial space $\boldsymbol{X}_\bullet$ as above, there is a first quadrant spectral sequence*

$$E^2_{s,t} = \pi_s(\pi_t(\boldsymbol{X}_\bullet; \boldsymbol{M})) \Rightarrow \pi_{s+t}(\|\boldsymbol{X}_\bullet\|; \boldsymbol{M}) \ .$$

Proof. Apply the spectral sequence of [BF] to the simplicial space $\boldsymbol{map}_*(\boldsymbol{M}, \boldsymbol{X}_\bullet)$. □

Definition 6.3. Given a model space $\boldsymbol{M}$, define an $\boldsymbol{M}$-*CW complex* to be a conic space obtained by a process of "attaching $\boldsymbol{M}$-cells", in a manner precisely analogous to the usual definition (cf. [W, II, §1]), with spheres replaced by suspensions of $\boldsymbol{M}$. The theory of CW-complexes carries over essentially without change, as long as we use $\pi_\star(-;\boldsymbol{M})$ to replace $\pi_\star(-)$ throughout, in particular in the definition of weak equivalences in [W, IV, (7.12)].

Thus, for example, if $f : \boldsymbol{X} \to \boldsymbol{Y}$ is a map between $\boldsymbol{M}$-CW-complexes which is an $\boldsymbol{M}$-*weak equivalence* (i.e., induces an isomorphism in $\pi_\star(-;\boldsymbol{M})$, then f is a homotopy equivalence. (This is the analogue of the Whitehead Theorem – cf. [W, V, Theorem 3.8]).

Fact 6.4. For every model $\boldsymbol{M}$ there is a functorial $\boldsymbol{M}$-CW-approximation (or $\boldsymbol{M}$-*colocalization* – see [Bo1, 7.5]) functor $CW_M : \mathcal{T}_\star \to \mathcal{T}_\star$ such that $CW_M\boldsymbol{X}$ is an $\boldsymbol{M}$-CW-complex, together with a natural transformation $\theta : CW_M \to Id$ such that $(\theta_{\boldsymbol{X}})_\# : \pi_\star(CW_M\boldsymbol{X};\boldsymbol{M}) \cong \pi_\star(\boldsymbol{X};\boldsymbol{M})$.

Compare [Bo1, CPP, DF2].

There is also a v-periodic version of the Quillen spectral sequence, for a sufficiently connected simplicial space:

Proposition 6.5. *For a self map* $v : \boldsymbol{M}^{d+r} \to \boldsymbol{M}^d$ *as in §1.2, and an* $(r+d-1)$-*connected simplicial space* $\boldsymbol{X}_\bullet$, *there is a first and fourth quadrant periodic spectral sequence*

$$E^2_{s,t} = \pi_s(v^{-1}\pi_t(\boldsymbol{X}_\bullet;\boldsymbol{M})) \Rightarrow v^{-1}\pi_{s+t}(\|\boldsymbol{X}_\bullet\|;\boldsymbol{M}) \ .$$

Proof. Set $\boldsymbol{Z}_\bullet \stackrel{Def}{=} \boldsymbol{map}_*(\boldsymbol{M}, \boldsymbol{X}_\bullet)$, so that $\|\boldsymbol{Z}_\bullet\| \simeq \boldsymbol{map}_*(\boldsymbol{M}, \|\boldsymbol{X}_\bullet\|)$ (Lemma 6.1), and in fact $\|\Omega^d\boldsymbol{Z}_\bullet\| \simeq \|\boldsymbol{map}_*(\boldsymbol{M}^{d+r}, \boldsymbol{X}_\bullet)\| \simeq \boldsymbol{map}_*(\boldsymbol{M}^{d+r}, \|\boldsymbol{X}_\bullet)\|$, too, though in general we only know $\Omega^{k-d-r}\|\boldsymbol{Z}_\bullet\| \simeq \boldsymbol{map}_*(\boldsymbol{M}^k, \|\boldsymbol{X}_\bullet)\|$ for $k > d+r$. Now consider the Quillen specral sequence for the simplicial space (or bisimplicial group) $\boldsymbol{Z}_\bullet$, with

$$E^1_{s,t} = \pi t\boldsymbol{Z}_s = \pi_{t+r}(\boldsymbol{X}_s;\boldsymbol{M}) \Rightarrow \pi s + t\|\boldsymbol{Z}_\bullet\| = \pi_{s+t+r}(\|\boldsymbol{X}_\bullet\|;\boldsymbol{M}).$$

If we reindex so that

$$\hat{E}^1_{s,t} = \pi_{t+k}(\boldsymbol{X}_s;\boldsymbol{M}) \Rightarrow \pi_{s+t}\Omega^{k-r}\|\boldsymbol{Z}_\bullet\| \qquad (t \geq r-k),$$

we can think of this as a (first and fourth quadrant) spectral sequence for $\pi_\star\Omega^{k-r}\|\boldsymbol{Z}_\bullet\| = \pi_\star\boldsymbol{map}_*(\boldsymbol{M}^k, \|\boldsymbol{X}_\bullet)\|$. We need to either disregard those terms in $\hat{E}^\infty_{s,t}$ with $s+t < k-r$, or set $\hat{E}^\ell_{s,t} = 0$ whenever

$s+t<k-r$ and "fringe" the remainder (cf. [BK2, §4.2]) so that those $\hat{E}^{\ell}_{s,t}$'s which supported differentials into the missing terms are suitably reduced.

By naturality, the map of mapping spaces induced by v also commutes with realization: $\|v^{\#}\| : \|\boldsymbol{Z}_{\bullet}\| \to \|\Omega^d \boldsymbol{Z}_{\bullet}\|$ is homotopic to $v^{\#} : \boldsymbol{map}_*(\boldsymbol{M}^r, \|\boldsymbol{X}_{\bullet}\| \to \boldsymbol{map}_*(\boldsymbol{M}^{r+d}, \|\boldsymbol{X}_{\bullet}\|)$, so more generally

$$\Omega^{k-r}\|v^{\#}\| : \Omega^{k-r}\|\boldsymbol{Z}_{\bullet}\| \to \Omega^{k-r+d}\|\boldsymbol{Z}_{\bullet}\|$$

is

$$v^{\#} : \boldsymbol{map}_*(\boldsymbol{M}^k, \|\boldsymbol{X}_{\bullet})\| \to \boldsymbol{map}_*(\boldsymbol{M}^{k+d}, \|\boldsymbol{X}_{\bullet})\|.$$

Thus we get a map of simplicial spaces $v^{\#} : \boldsymbol{Z}_{\bullet} \to \Omega^d \boldsymbol{Z}_{\bullet}$ inducing $v^{\#} : \pi_{t+r}(\boldsymbol{X}_s; \boldsymbol{M}) \to \pi_{t+r+d}(\boldsymbol{X}_s; \boldsymbol{M})$ on $E^1_{s,t}$, and also $v^{\#} : \pi_{t+s+r}(\|\boldsymbol{X}_{\bullet}\|; \boldsymbol{M}) \to \pi_{t+s+r+d}(\|\boldsymbol{X}_{\bullet}\|; \boldsymbol{M})$ on E^{∞}.

Since homology commutes with sequential direct limits of chain complexes (cf. [Mi]), the spectral sequence for a simplicial space also commutes with sequential homotopy direct limits.

Thus if we define the *telescope* $Tel(\boldsymbol{Y})$ of a space $\boldsymbol{Y}$ (with respect to a self map $v : \boldsymbol{M}^{d+r} \to \boldsymbol{M}$ as above) to be the homotopy colimit of

$$\boldsymbol{map}_*(\boldsymbol{M}^r, \boldsymbol{Y}) \xrightarrow{v^{\#}} \boldsymbol{map}_*(\boldsymbol{M}^{r+d}, \boldsymbol{Y}) \xrightarrow{v^{\#}} \dots \boldsymbol{map}_*(\boldsymbol{M}^{r+kd}, \boldsymbol{Y}) \to \dots,$$

then $\pi_t Tel(Y) \cong v^{-1}\pi_t(\boldsymbol{Y}; \boldsymbol{M})$ by [Gr1, Proposition 15.9], and we get a spectral sequence with $E^1_{s,t} =$

$$\lim_k \{\pi_{t+r}(\boldsymbol{X}_s; \boldsymbol{M}) \xrightarrow{v^{\#}} \pi_{t+r+d}(\boldsymbol{X}_s; \boldsymbol{M}) \xrightarrow{v^{\#}} \pi_{t+r+2d}(\boldsymbol{X}_s; \boldsymbol{M}) \xrightarrow{v^{\#}} \dots\} = v^{-1}\pi_t(\boldsymbol{X}_s; \boldsymbol{M}),$$

and thus the E^2-term is as stated, converging to $v^{-1}\pi_{t+s}(\|\boldsymbol{X}_{\bullet}\|; \boldsymbol{M})$. □

7. LOCALIZATIONS

For the rest of the discussion we shall need to make use of some technical results of Bousfield. This is the first place where our approach will no longer work for general towers, or even for those constructed from an arbitrary self map $v : \boldsymbol{M}^{d+r} \to \boldsymbol{M}$ as above; we are now forced to restrict attention to v_n-periodic self maps (to be defined immediately). However, these appear to be the only examples of real interest (and in fact these are known to be the only examples stably – cf. [DHS]). First, we shall need some

7.1. **notation and terminology.** Fix a prime p, and for each $n \geq 0$ choose a finite r_n-dimensional CW complex $\boldsymbol{V}_{n-1}$ with a self-map $v_n : \Sigma^{d_n}\boldsymbol{V}_{n-1} \to \boldsymbol{V}_{n-1}$. For simplicity we assume that v_n (and so in particular $\boldsymbol{V}_{n-1}$) is a suspension. We require that $\boldsymbol{V}_{n-1}$ be of type n – that is, the m-th Morava K-theory $K(m)_*\boldsymbol{V}_n = 0$ for $m < n$ and $K(n)_*\boldsymbol{V}_n \neq 0$ – and v_n is a v_n-self map – that is, v_n induces an isomorphism in $K(n)_*$, and 0 in $K(m)_*$ for $m \neq n$. (See [R2, §1.5]).

In particular, we shall assume that $\boldsymbol{V}_{-1} = \boldsymbol{S}^2$, with $v_0 : \boldsymbol{V}_{-1} \to \boldsymbol{V}_{-1}$ the degree p map, for some prime p. Then $\boldsymbol{V}_0 = \boldsymbol{S}^3 \cup_p \boldsymbol{e}^4$ is the 4-dimensional mod p Moore space, and we take $v_1 : \Sigma^{2(p-1)+1}\boldsymbol{V}_0 \to \Sigma\boldsymbol{V}_0$ to be the Adams map for $p > 2$ (or its 4-fold iterate for $p = 2$ – see [CN]). In general we let $\boldsymbol{V}_n$ be (a suitable suspension of) the cofiber of the map $v_n : \Sigma^d\boldsymbol{V}_{n-1} \to \boldsymbol{V}_{n-1}$ (cf. [HS, Theorem 5.12] or [D]).

We are interested in the v_n-periodic homotopy groups $v_n^{-1}\pi_*(\boldsymbol{X}; \boldsymbol{V}_{n-1})$, defined as in §1.2; note that $v_0^{-1}\pi_*(-; \boldsymbol{V}_{-1})$ is just $\pi_\star(-) \otimes \mathbb{Z}[1/p]$.

Definition 7.2. A map $f : \boldsymbol{A} \to \boldsymbol{B}$ is called a v_n-*periodic weak equivalence*, or $v_n^{-1}\pi_\star$-*w.e.*, if it induces an isomorphism $f_* : v_n^{-1}\pi_*(\boldsymbol{A}; \boldsymbol{V}_{n-1}) \cong v_n^{-1}\pi_*(\boldsymbol{B}; \boldsymbol{V}_{n-1})$.

One of the basic tools for dealing with v_n-periodic phenomena is the concept of *localization* with respect to a pointed space $\boldsymbol{W}$ (or more generally, a map $f : \boldsymbol{A} \to \boldsymbol{B}$), first considered by Dror-Farjoun ([DF1]) and Bousfield ([Bo1]):

Definition 7.3. Recall from [Bo3, p. 3] that, given a fixed space $\boldsymbol{W}$, a space $\boldsymbol{X}$ is called $\boldsymbol{W}$-*local* (or $\boldsymbol{W}$-*periodic*) if $\boldsymbol{W} \to \star$ induces a homotopy equivalence $\boldsymbol{X} \xrightarrow{\sim} \boldsymbol{map}(\boldsymbol{W}, \boldsymbol{X})$. A map $f : \boldsymbol{A} \to \boldsymbol{B}$ is called a $\boldsymbol{W}$-*weak equivalence*, or $\boldsymbol{W}$-*w.e.*, if $\boldsymbol{map}(\boldsymbol{B}, \boldsymbol{X}) \xrightarrow{f^\#} \boldsymbol{map}(\boldsymbol{A}, \boldsymbol{X})$ is a homotopy equivalence for every $\boldsymbol{W}$-local space $\boldsymbol{X}$. Finally, a map $\varphi : \boldsymbol{X} \to \hat{\boldsymbol{X}}$ is a $\boldsymbol{W}$-*localization* (or $\boldsymbol{W}$-*periodization*) if $\hat{\boldsymbol{X}}$ is $\boldsymbol{W}$-local and φ is a $\boldsymbol{W}$-weak equivalence.

Such localizations exist for any $\boldsymbol{W}$ (and in fact the definition generalizes, replacing $\boldsymbol{W} \to \star$ by an arbitrary map $f : \boldsymbol{A} \to \boldsymbol{B}$). A functorial version of $\boldsymbol{W}$-localization is denoted by $P_W\boldsymbol{X}$ (the notation $L_W(\boldsymbol{X})$ is also used). See [DF1] and [Bo3, §2].

Remark 7.4. We are interested in the case $\boldsymbol{W} = \boldsymbol{V}_n$ as in §7.1. In particular, a $\boldsymbol{V}_n$-weak equivalence will be called a P_{v_n}-*equivalence*. It turns out (cf. [Bo3, Theorem 9.15]) that this concept does not depend on the precise choices of the spaces $\boldsymbol{V}_m$ (or the v_m-self maps), but only on the connectivity of $\boldsymbol{V}_n$.

In [Bo3, §10.1] Bousfield defines (non-constructively) an increasing sequence of integers $c(n) \geq n+2$ (with equality conjectured), such that each $c(n) \leq 1+$ the connectivity of $\boldsymbol{V}_n$. (Thus $c(0)=2$ and $c(1)=3$ for odd p). Note that of course

$$r_n - 1 \geq c(n) \qquad \text{for } n \geq 1. \tag{7.5}$$

Bousfield then proves the following generalization of [T1, Theorem 1.2]:

Theorem 7.6. [Bo3, Theorem 11.14]*: For each* $n \geq 0$ *the maps* $v_n : \Sigma^{d_n}\boldsymbol{V}_{n-1} \to \boldsymbol{V}_{n-1}$ *are* $v_n^{-1}\pi_\star$*-w.e.'s, after at most* 2 *suspensions.*

as well as:

Theorem 7.7. [Bo3, Theorem 13.3]*: If* $\boldsymbol{X}, \boldsymbol{Y}$ *are* $c(n)$*-connected spaces, then a map* $f : \boldsymbol{X} \to \boldsymbol{Y}$ *is a* P_{v_n}*-equivalence if and only if it is a* $v_k^{-1}\pi_\star$*-w.e.'s for each* $0 \leq k \leq n$*.*

Corollary 7.8. *The maps* $\Sigma v_n : \Sigma^{d_n+1}\boldsymbol{V}_{n-1} \to \Sigma\boldsymbol{V}_{n-1}$ *are in fact* P_{v_n}*-equivalences.*

Proof. Any space $\boldsymbol{X}$ of type n – so in particular $\boldsymbol{V}_{n-1}$ – has $v_m^{-1}\pi_*(\boldsymbol{X};\boldsymbol{V}_{m-1}) = 0$ for $0 \leq m < n$. □

Corollary 7.9. *Let* $\{\boldsymbol{X}_\alpha\}_{\alpha\in A}$ *and* $\{\boldsymbol{Y}_\alpha\}_{\alpha\in A}$ *be two diagrams of* $c(n)$*-connected spaces, and* $\{f_\alpha : \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha\}_{\alpha\in A}$ *a map of diagrams with each* f_α *a* $v_m^{-1}\pi_\star$*-w.e. for* $0 \leq m \leq n$*. Then* $hocolim_\alpha\ (f_\alpha) : hocolim_\alpha\ (\boldsymbol{X}_\alpha) \to hocolim_\alpha\ (\boldsymbol{Y}_\alpha)$ *is a* $v_m^{-1}\pi_\star$*-w.e. for* $0 \leq m \leq n$*, too.*

Proof. It is evident from the definition (see [Bo3, §2.5]) that (pointed) homotopy colimits preserve $\boldsymbol{W}$-w.e.'s for any $\boldsymbol{W}$; we may use Theorem 7.7 to restate this in terms of $v_m^{-1}\pi_\star$-w.e.'s. □

Finally we record the following two facts:

Lemma 7.10. [Bo3, Theorem 11.5]*): For any pointed connected space* $\boldsymbol{X}$*, the localization map* $\varphi_{\boldsymbol{X}} : \boldsymbol{X} \to P_{V_n}\boldsymbol{X}$ *is a* $v_m^{-1}\pi_\star$*-w.e. for all* $1 \leq m \leq n$*.* □

This is in fact true with $P_{V_n}\boldsymbol{X}$ replaced by $P_{\Sigma^s V_n}\boldsymbol{X}$ for any $s \geq 0$.

By considering the long exact sequence for the cofibration sequence

$$\Sigma^{d_n}\boldsymbol{V}_{n-1} \xrightarrow{v_n} \boldsymbol{V}_{n-1} \to \boldsymbol{V}_n$$

we immediately deduce from the the above and the definition of the localization $P_{V_n}\boldsymbol{X}$ the isomorphisms

$$v_n^{-1}\pi_t(\boldsymbol{X};\boldsymbol{V}_{n-1}) \cong v_n^{-1}\pi_t(P_{V_{\boldsymbol{X}}};\boldsymbol{V}_{n-1}) \cong \pi_t(P_{V_n}\boldsymbol{X};\boldsymbol{V}_{n-1}) \tag{7.11}$$

for all $\boldsymbol{X}$ and $t \geq 1$.

8. P_{v_n}-EQUIVALENCES AND TOWERS

In this and the following sections we shall assume we are given a v_n-self map $v : \Sigma^{d_n+r_n}\boldsymbol{V}_{n-1} \to \boldsymbol{V}_{n-1}$ of the form described in §7.1, which we shall denote simply by $v : \Sigma^{d+r}\boldsymbol{M} \to \boldsymbol{M}$, with $\mathfrak{M}$ a corresponding v-model tower.

As noted in the introduction, it is the proof of Proposition 8.5 which actually forces us to restrict attention to spaces of type n – i.e., those for which $v_m^{-1}\pi_*(\boldsymbol{X};\boldsymbol{V}_{m-1}) = 0$ for $0 \leq m < n$. (It is not clear whether the Proposition is in fact false for diagrams of arbitrary P_{v_n}-regular towers, though the proof provided evidently will not carry through in greater generality.)

Definition 8.1. A map of virtual towers $\hat{\mathfrak{f}} : \hat{\mathfrak{X}} \to \hat{\mathfrak{Y}}$ will be called a P_{v_n}-*equivalence* if it induces an isomorphism $\hat{\mathfrak{f}}_\# : v_m^{-1}\pi_t\hat{\mathfrak{X}} \xrightarrow{\cong} v_m^{-1}\pi_t\hat{\mathfrak{Y}}$ for each $0 \leq m \leq n$.

(Compare Theorem 7.7).

Definition 8.2. An (ordinary) tower $\mathfrak{X} = \{\ldots \to \boldsymbol{X}[n] \xrightarrow{p_n} \boldsymbol{X}[n-1]\ldots\}$ will be called P_{v_n}-*regular* if each level space $\boldsymbol{X}[N]$ is $c(n)$-connected (§7.4) and each level map p_n is a P_{v_n}-equivalence. A virtual tower $\langle\hat{\mathfrak{X}}, F, \mathcal{F}\rangle$ will be called P_{v_n}-*regular* if each $F_k\hat{\mathfrak{X}}$ is such (compare Definition 4.4).

Fact 8.3. As in §4.6, if an ordinary tower $\mathfrak{X}$ is P_{v_n}-regular, then

$$v^{-1}\pi_t\hat{\mathfrak{X}} \cong Q_t(\mathfrak{X}) \cong v_m^{-1}\pi_t(\boldsymbol{X}[N];\boldsymbol{V}_{m-1}),$$

where $Q_t(-)$ and $v^{-1}\pi_t(-)$ are taken with respect to any of the $v_m : \boldsymbol{V}_{m-1}^{d_n} \to \boldsymbol{V}_{m-1}$ of §7.1 for $0 \leq m \leq n$ and $N \geq 0$; and similarly for P_{v_n}-regular virtual towers $\hat{\mathfrak{X}}$ we have $v^{-1}\pi_t\hat{\mathfrak{X}} \cong \operatorname{colim}_k v_m^{-1}\pi_t(F_k\boldsymbol{X}[N];\boldsymbol{V}_{m-1})$ for each $0 \leq m \leq n$ and $N, k \geq 0$.

Proposition 8.4. *Let v be a v_n-self map as in §7.1, and $\hat{\mathfrak{X}}$ a virtual tower. Let $\hat{\mathfrak{J}}_\bullet = \hat{\mathfrak{J}}^v_\bullet \to \hat{\mathfrak{X}}$ be the v-periodic resolution of §5.1; then $\|\hat{\mathfrak{J}}_\bullet\|$ is $v^{-1}\pi_\star$-w.e. to $\hat{\mathfrak{X}}$.*

Proof. By §5.4 we may assume all level spaces $F_k\hat{\mathfrak{J}}_r[n]$ of $\hat{\mathfrak{J}}_\bullet$ are (r_n+d_n-1)-connected. By construction, each v-tower $\mathfrak{M}$ is P_{v_n}-regular (by Corollary 7.8), so each $F_k\hat{\mathfrak{J}}_r$ is, too, by Corollary 7.9. If we set $\mathfrak{Z}^k_\bullet = \sigma_{N(k,k)}\mathit{Sk}_k F_k\hat{\mathfrak{X}}_\bullet$, then each tower $\mathfrak{Z}^k_r$ is P_{v_n}-regular, so by Corollary 7.9 again $\|\mathfrak{Z}^k_\bullet\| = F_k\|\hat{\mathfrak{J}}_\bullet\|$ is also P_{v_n}-regular, which by definition means $\|\hat{\mathfrak{J}}_\bullet\|$ is, too.

Therefore, if we let $\boldsymbol{Z}^k_\bullet = \mathfrak{Z}^k_\bullet[N]$, for any k and $N \geq N(k,k)$, by Fact 8.3 above

$$v^{-1}\pi_t\hat{\mathfrak{J}}_r \cong \operatorname{colim}_k\, v_n^{-1}\pi_t(\boldsymbol{Z}^k_r;\boldsymbol{V}_{n-1})$$

and similarly $v^{-1}\pi_t\|\hat{\mathfrak{J}}_\bullet\| \cong \operatorname{colim}_k\, v_n^{-1}\pi_t(\|\boldsymbol{Z}^k_\bullet\|;\boldsymbol{V}_{n-1})$. Applying the v-periodic Qullen spectral sequence of Proposition 6.5 to each simplicial space $\boldsymbol{Z}^k_\bullet$, we get

$$E^2_{s,t} = \pi_s(v^{-1}\pi_t(\boldsymbol{Z}^k_\bullet;\boldsymbol{M})) \cong \pi_s(v^{-1}\pi_t(F_k\hat{\mathfrak{J}}_\bullet;\boldsymbol{M}))$$

converging to $v^{-1}\pi_{s+t}(\|\boldsymbol{Z}^k_\bullet\|;\boldsymbol{M}) \cong v^{-1}\pi_{s+t}(\|F_k\hat{\mathfrak{J}}_\bullet\|;\boldsymbol{M})$. Since homology, and thus spectral sequences, commute with sequential direct limits (cf. [Mi]), taking the direct limit as $k\to\infty$ yields a spectral sequence with

$$\hat{E}^2_{s,t} \cong \pi_s(v^{-1}\pi_t\hat{\mathfrak{J}}_\bullet) = \begin{cases} v^{-1}\pi_t\hat{\mathfrak{X}} & \text{if } s=0 \\ 0 & \text{if } s>0 \end{cases}$$

by Corollary 5.6, converging to $\operatorname{colim}_k v_n^{-1}\pi_{s+t}(\|\boldsymbol{Z}^k_\bullet\|;\boldsymbol{V}_{n-1}) \cong v^{-1}\pi_{s+t}(\|\hat{\mathfrak{J}}_\bullet\|;\boldsymbol{M})$, and the proposition follows. □

Proposition 8.5. *Let* $\{\hat{\mathfrak{f}} : \hat{\mathfrak{X}}_\alpha \to \hat{\mathfrak{C}}(\boldsymbol{Y}_\alpha)\}_{\alpha\in A}$ *be a map of strict diagrams of virtual towers (§3.13) with each* f_α *a* P_{v_n}*-equivalence, and assume each virtual tower* $\hat{\mathfrak{X}}_\alpha$ *is* P_{v_n}*-regular with* $v_m^{-1}\pi_*(F_k\boldsymbol{X}_\alpha[N];\boldsymbol{V}_{m-1}) = 0$ *for each* $0\leq m<n$ *and all* N, k, *and that each tower* $\hat{\mathfrak{C}}(\boldsymbol{Y}_\alpha)$ *is constant with* $\boldsymbol{Y}_\alpha$ $c(n)$*-connected. Then*

$$\mathit{hocolim}_\alpha\, \hat{\mathfrak{f}}_\alpha : \mathit{hocolim}_\alpha\, \hat{\mathfrak{X}}_\alpha \to \mathit{hocolim}_\alpha\, \hat{\mathfrak{C}}(\boldsymbol{Y}_\alpha)$$

is a P_{v_n}*-equivalence, too.*

Proof. For every index α there is a filtration F' on $\hat{\mathfrak{X}}_\alpha$ with respect to which $\hat{\mathfrak{f}}_\alpha$ is defined, and

(8.6)

$$v^{-1}\pi_t\hat{\mathfrak{X}}_\alpha \cong \operatorname{colim}_k \lim_n v^{-1}\pi_*(F'_k\boldsymbol{X}_\alpha[n];\boldsymbol{M}) \xrightarrow{(\hat{\mathfrak{f}}_\alpha)_\#} v^{-1}\pi_t\hat{\mathfrak{C}}(\boldsymbol{Y})_\alpha \cong v^{-1}\pi_*(\boldsymbol{Y}_\alpha;\boldsymbol{M})$$

is an isomorphism. Without loss of generality assume $n(F'_k\hat{\mathfrak{f}}_\alpha) = n(F'_k\mathfrak{i}_k) = k$ (for $F'_k\hat{\mathfrak{X}}_\alpha \xrightarrow{F'_k\mathfrak{i}_k} F'_{k+1}\hat{\mathfrak{X}}_\alpha \xrightarrow{F'_{k+1}\hat{\mathfrak{f}}_\alpha} \boldsymbol{Y}_\alpha$), so we have a diagram as in Figure 1 to describe $\mathfrak{f}$:

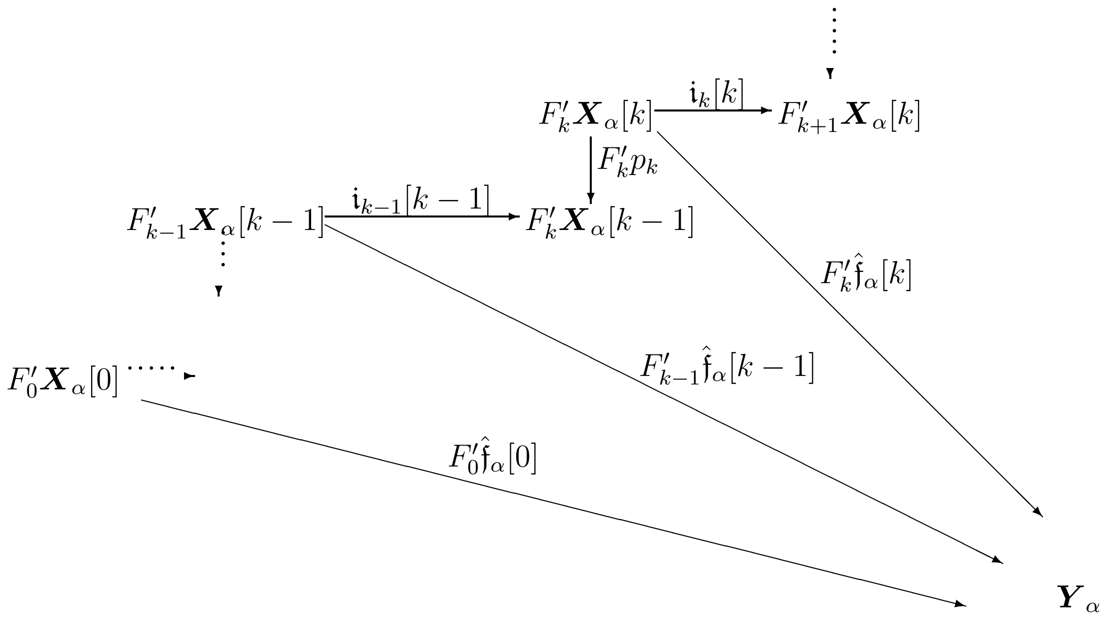

FIGURE 1. explicit description of $\hat{\mathfrak{f}}_\alpha$

Note that each $F'_k p_k$ is a P_{v_n}-equivalence by assumption, so after applying P_V this is still true by Lemma 7.10. Thus $P_V F'_k p_k$ is a $\Sigma\boldsymbol{M}$-weak equivalence by (7.11). (Recall that $\boldsymbol{M} = \boldsymbol{V}_{m-1}$ and $\boldsymbol{V} = \boldsymbol{V}_m$.) Now apply the functor $CW_{\Sigma M}$ of §6.4 to Figure 1: then $CW_{\Sigma M}(P_V F'_k p_k)$ is a homotopy equivalence (§6.3), so it has a homotopy inverse $r_k : CW_{\Sigma M}(P_V F'_k\boldsymbol{X}_\alpha[k-1]) \to CW_{\Sigma M}(P_V F'_k\boldsymbol{X}_\alpha[k])$. We thus get a diagram

$$\begin{array}{ccccc}
\ldots \to CW_{\Sigma M}P_V F'_k\boldsymbol{X}_\alpha[k] & & \xrightarrow{r_{k+1}\circ\mathfrak{i}_k[k]} & & CW_{\Sigma M}P_V F'_{k+1}\boldsymbol{X}_\alpha[k+1] \to \ldots \\
& {\scriptstyle F'_k\hat{\mathfrak{f}}_\alpha[k]\circ\theta} \searrow & & \swarrow {\scriptstyle F'_{k+1}\hat{\mathfrak{f}}_\alpha[k+1]\circ\theta} & \\
& & \boldsymbol{Y}_\alpha & &
\end{array} \tag{8.7}$$

Changing the horizontal maps into cofibrations we get

$$\begin{array}{ccccccc} \dots \longrightarrow & \boldsymbol{Z}^k_\alpha & & \xrightarrow{\;j^k_\alpha\;} & & \boldsymbol{Z}^{k+1}_\alpha & \longrightarrow \dots \\ & & {\scriptstyle f^k_\alpha} \searrow & & \swarrow {\scriptstyle f^{k+1}_\alpha} & & \\ & & & \boldsymbol{Y}_\alpha & & & \end{array} \tag{8.8}$$

with each $\boldsymbol{Z}^k_\alpha \simeq CW_{\Sigma M} P_V F'_k \boldsymbol{X}_\alpha[k]$ a $\boldsymbol{V}$-local ΣM-CW-complex which is ΣM-weak equivalent to $P_V F'_k \boldsymbol{X}_\alpha[k]$, and thus $v^{-1}\pi_\star$-w.e. to $F'_k \boldsymbol{X}_\alpha[k]$ by Lemma 7.10. If $\boldsymbol{Z}_\alpha = \mathrm{colim}_k\ \boldsymbol{Z}^k_\alpha$ is the (homotopy) colimit of the horizontal maps in (8.8), then $\pi_t(\boldsymbol{Z}_\alpha; \boldsymbol{M}) \cong \mathrm{colim}_k\ \pi_t(\boldsymbol{Z}^k_\alpha; \boldsymbol{M})$ (cf. [Gr1, Proposition 15.9]), and $f_\alpha : \boldsymbol{Z}_\alpha \to \boldsymbol{Y}_\alpha$ induces an isomorphism in $\pi_t(-; \boldsymbol{M})$ for $t \geq 1$ (and so in $v^{-1}\pi_*(-; \boldsymbol{M})$) by (7.11). Therefore, if we set

$$\boldsymbol{Z} \overset{Def}{=} \mathrm{hocolim}_\alpha\ \boldsymbol{Z}_\alpha, \quad \boldsymbol{Y} \overset{Def}{=} \mathrm{hocolim}_\alpha\ \boldsymbol{Y}_\alpha, \quad \text{and} \quad f \overset{Def}{=} \mathrm{hocolim}_\alpha\ f_\alpha,$$

by Corollary 7.9 we see $f : \boldsymbol{Z} \to \boldsymbol{Y}$ is a $v^{-1}\pi_\star$-w.e..

Now since the given diagram $(\hat{\mathfrak{X}}_\alpha)_{\alpha \in A}$ is strict (§2.2(e)), $\mathrm{hocolim}_\alpha\ \hat{\mathfrak{X}}_\alpha$ is defined and filtered by $\mathrm{hocolim}_\alpha\ F'_k \hat{\mathfrak{X}}_\alpha$, where $\mathrm{hocolim}_\alpha\ F'_k \hat{\mathfrak{X}}_\alpha)[n] = \mathrm{hocolim}_\alpha\ F'_k \hat{\mathfrak{X}}_\alpha[n]$ for $n \geq N$ (see §3.13). But

$$\mathrm{hocolim}_\alpha\ \boldsymbol{Z}_\alpha = \mathrm{hocolim}_\alpha\ \mathrm{colim}_k\ \boldsymbol{Z}^k_\alpha \simeq \mathrm{colim}_k\ \mathrm{hocolim}_\alpha\ \boldsymbol{Z}^k_\alpha$$

since homotopy colimits (of spaces) commute with each other (cf. [Vo, Theorem 2.4]), and since $\boldsymbol{Z}^k_\alpha$ is $v^{-1}\pi_\star$-w.e. to $F'_k \boldsymbol{X}_\alpha[k]$ for all α, k and $\mathrm{hocolim}_\alpha\ F'_k \hat{\mathfrak{X}}_\alpha$ is still P_{v_n}-regular we have

$$v^{-1}\pi_t \mathrm{hocolim}_\alpha\ \hat{\mathfrak{X}}_\alpha = \mathrm{colim}_k\ v^{-1}\pi_t F'_k \mathrm{hocolim}_\alpha\ \hat{\mathfrak{X}}_\alpha =$$

$$\mathrm{colim}_k\ v^{-1}\pi_t \mathrm{hocolim}_\alpha\ F'_k \hat{\mathfrak{X}}_\alpha \cong \mathrm{colim}_k\ v^{-1}\pi_t(\mathrm{hocolim}_\alpha\ \boldsymbol{Z}^k_\alpha; \boldsymbol{M}) \cong$$

$$v^{-1}\pi_t(\mathrm{colim}_k\ \mathrm{hocolim}_\alpha\ \boldsymbol{Z}^k_\alpha; \boldsymbol{M}) = v^{-1}\pi_t(\mathrm{hocolim}_\alpha\ \mathrm{colim}_k\ \boldsymbol{Z}^k_\alpha; \boldsymbol{M}) =$$

$$v^{-1}\pi_t(\mathrm{hocolim}_\alpha\ \boldsymbol{Z}_\alpha; \boldsymbol{M}) \cong v^{-1}\pi_t(\boldsymbol{Z}; \boldsymbol{M}) \cong$$

$$v^{-1}\pi_t(\boldsymbol{Y}; \boldsymbol{M}) \cong v^{-1}\pi_t(\mathrm{hocolim}_\alpha\ \boldsymbol{Y}_\alpha; \boldsymbol{M}) =$$

$$v^{-1}\pi_t \hat{\mathfrak{C}}(\mathrm{hocolim}_\alpha\ \boldsymbol{Y}_\alpha) \overset{Def}{=} v^{-1}\pi_t \mathrm{hocolim}_\alpha\ \hat{\mathfrak{C}}(\boldsymbol{Y}_\alpha)$$

which completes the proof of the Proposition since all spaces (and towers) in question have $v_m^{-1}\pi_*(-; \boldsymbol{V}_{m-1}) = 0$ for $0 \leq m < n$. □

9. Π-algebras

In this section we provide the "algebraic" underpinning needed to describe the E^2-term of the spectral sequence which we shall set up in the next section.

Recall (e.g., from [Bl, §3.1] or [St, §4]) that a Π-*algebra* is an algebraic object modeled on the homotopy groups of a space, together with the action of the primary homotopy operations ([W, XI, §1]) on them. We have analogous concepts for other representable functors:

Definition 9.1. First, one may replace the spheres representing ordinary homotopy groups by some other model space $\boldsymbol{M}$, to get $\boldsymbol{M}$-*homotopy operations* corresponding to each homotopy class $\alpha \in \pi_r(\boldsymbol{M}^{n_1} \vee \ldots \vee \boldsymbol{M}^{n_k}; \boldsymbol{M})$ (subject to the universal relations among such operations, corresponding to compositions of maps among wedges of $\boldsymbol{M}^{n}$'s).

We then define an $\boldsymbol{M}$-Π-*algebra* to be a graded set $\{X_i\}_{i=0}^{\infty}$, together with an action of the $\boldsymbol{M}$-homotopy operations on them. As usual, the *free* $\boldsymbol{M}$-Π-algebras are those isomorphic to $\pi_\star(\bigvee_{\alpha\in A} \boldsymbol{M}^{r_\alpha}; \boldsymbol{M})$ for some (possibly infinite) wedge of model spaces (cf. [Bl, §3.1.2]).

Now if $v : \boldsymbol{M}^d \to \boldsymbol{M}$ is a self map of our model space, the situation for $v^{-1}\pi_*(-; \boldsymbol{M})$-homotopy operations is somewhat different:

Definition 9.2. A primary *v-periodic homotopy operation of type* $(n_1, \ldots, n_k; r)$ is defined as usual to be a natural transformation

$$\vartheta : v^{-1}\pi_{n_1}(-; \boldsymbol{M}) \times \ldots \times v^{-1}\pi_{n_k}(-; \boldsymbol{M}) \to v^{-1}\pi_r(-; \boldsymbol{M}).$$

It is natural to extend this definition to *v-tower homotopy operations*, where we consider $v^{-1}\pi_{n_i}(-) : v\mathcal{T}ow \to Abgp$ rather than $v^{-1}\pi_{n_i}(-; \boldsymbol{M}) : \mathcal{T}_\star \to Abgp$. In light of (4.3), any v-tower homotopy operation is in particular a v-periodic one (though not necessarily conversely!).

We then have the following analogue of [W, XI, Theorem 1.3]:

Lemma 9.3. *The v-tower homotopy operations of type* $(n_1, \ldots, n_k; r)$ *are in one-to-one correspondence with elements of*

$$\mathcal{P}_{(n_1,\ldots,n_k;r)} = \lim_{\overleftarrow{\mathfrak{M}\in\mathcal{M}}} v^{-1}\pi_r(\Sigma^{n_1}\mathfrak{M} \vee \ldots \vee \Sigma^{n_k}\mathfrak{M}).$$

Proof. Given a v-tower homotopy operation

$$\vartheta : v^{-1}\pi_{n_1}(-) \times \ldots \times v^{-1}\pi_{n_k}(-) \to v^{-1}\pi_r((-); \boldsymbol{M}),$$

for any $\mathfrak{M} \in \mathcal{M}$ let $\mathfrak{W} = \Sigma^{n_1}\mathfrak{M} \vee \ldots \vee \Sigma^{n_k}\mathfrak{M}$, with inclusions $\mathfrak{j}_i : \Sigma^{n_i}\mathfrak{M} \hookrightarrow \mathfrak{W}$. Then $\vartheta([\mathfrak{j}_1], \ldots, [\mathfrak{j}_k]) \in v^{-1}\pi_r\mathfrak{W}$ may be represented by a tower map $\mathfrak{f}_\vartheta : \mathfrak{M}' \to \mathfrak{W}$ for some $\mathfrak{M}' \in \mathcal{M}$, with the obvious compatibility conditions with respect to v-tower maps $\mathfrak{M}' \to \mathfrak{M}''$, and so on.

Now any $(\gamma_1, \dots, \gamma_k) \in v^{-1}\pi_{n_1}\hat{\mathfrak{X}} \times \dots \times v^{-1}\pi_{n_k}\hat{\mathfrak{X}}$ may be represented by $\mathfrak{g}_i : \Sigma^{n_i}\mathfrak{M} \to \hat{\mathfrak{X}}$ $(i = 1, \dots, k)$, and so by $\mathfrak{g} : \mathfrak{W} \to \hat{\mathfrak{X}}$, with $\mathfrak{g}_i = \mathfrak{f} \circ \mathfrak{j}_i$. Then $\vartheta(\gamma_1, \dots, \gamma_k) = \vartheta([\mathfrak{g} \circ \mathfrak{j}_1], \dots, [\mathfrak{g} \circ \mathfrak{j}_k]) = \mathfrak{g}_\#\vartheta([\mathfrak{j}_1], \dots, [\mathfrak{j}_k]) = \mathfrak{g}_\#[\mathfrak{f}_\vartheta] = \mathfrak{f}_\vartheta^\#(\gamma_1, \dots, \gamma_k)$. The converse direction is obvious. □

Corollary 9.4. *If* $v : \boldsymbol{M}^d \to \boldsymbol{M}$ *is as in §7.1, then the v-tower homotopy operations of type* $(n_1, \dots, n_k; r)$ *are in one-to-one correspondence with elements of*

$$v_n^{-1}\pi_r(\boldsymbol{M}^{n_1} \vee \dots \vee \boldsymbol{M}^{n_k}; \boldsymbol{M})$$

Proof. In this case each v-model tower is P_{v_n}-regular by Theorem 7.6, as is each finite coproduct $\mathfrak{W} = \Sigma^{n_1}\mathfrak{M} \vee \dots \vee \Sigma^{n_k}\mathfrak{M}$ (Corollary 7.9), and thus

$$v^{-1}\pi_r\mathfrak{W} \cong Q_r(\mathfrak{W}) \overset{Def}{=} \lim_{\overleftarrow{n}} v^{-1}\pi_t(\boldsymbol{W}[n]; \boldsymbol{M}) \cong v^{-1}\pi_r(\boldsymbol{M}^{n_1} \vee \dots \vee \boldsymbol{M}^{n_k}; \boldsymbol{M})$$

by Lemma 4.5. Also, each v-tower map $\mathfrak{h} : \mathfrak{M}' \to \mathfrak{M}$ then induces an isomorphism between $Q_r(\mathfrak{M}')$ and $Q_r(\mathfrak{M})$, so the inverse system in the definition of $\mathcal{P}_{(n_1,\dots,n_k;r)}$ is constant. □

Thus in this case the v-periodic homotopy operations are the same as the v-tower ones.

Definition 9.5. As in §9.1, we define a v-Π-*algebra* to be a $\mathbb{Z}/d$-graded set $\{X_i\}_{i=0}^{d-1}$, together with an action of the v-tower homotopy operations on them (subject to the universal relations holding among the $\mathcal{P}_{(n_1,\dots,n_k;r)}$'s under composition). This is a category of ($\mathbb{Z}/d$-graded) *universal algebras* (cf. [BS, §2]), or a *variety of algebras* in the terminology of [Mc, V, §6].

Examples of v-Π-algebras include $v^{-1}\pi_*(\boldsymbol{X}; \boldsymbol{M})$, for any space $\boldsymbol{X}$, or more generally $v^{-1}\pi_\star\mathfrak{X}$ for any tower $\mathfrak{X}$.

Remark 9.6. If $n = 1$ and p is odd, with $v = v_1$ the Adams map between mod p Moore spaces, then by Corollary 9.4 the *free* v-Π-*algebras* (cf. [Bl, §3.1.2]) are those isomorphic to $v^{-1}\pi_*(\boldsymbol{W}; \boldsymbol{M})$ for any (possibly infinite) wedge $\boldsymbol{W}$ of mod p Moore spaces.

$v^{-1}\pi_*(\boldsymbol{M}^k; \boldsymbol{M})$ has been fully calculated by the second author in [T1, Theorem 1.1], so the Hilton-Milnor theorem (cf. [W, XI, Theorem 6.7]), together with the fact that $\boldsymbol{M}^j \wedge \boldsymbol{M}^k \simeq \boldsymbol{M}^{j+k-1} \vee \boldsymbol{M}^{j+k}$ (cf. [N, Corollary 6.6]), give an explicit description of the free v-Π-algebras in this case.

Remark 9.7. A stricter analogy with §9.1 would require a choice of a *specific* v-model tower $\mathfrak{M}$; we then consider $\mathfrak{M}$-*homotopy operations*

$$\vartheta : \pi_{n_1}(-;\mathfrak{M}) \times \ldots \times \pi_{n_k}(-;\mathfrak{M}) \to \pi_r(-;\mathfrak{M})$$

for towers, which are obviously in one-to-one correspondence with elements of

$$\pi_r(\Sigma^{n_1}\mathfrak{M} \vee \ldots \vee \Sigma^{n_k}\mathfrak{M};\mathfrak{M}).$$

One then has a concept of $\mathfrak{M}$-Π-algebras, as before, modeled on $\pi_\star(\hat{\mathfrak{X}};\mathfrak{M})$ for any virtual tower $\hat{\mathfrak{X}}$. Since $\pi_\star(\hat{\mathfrak{C}}(\boldsymbol{X});\mathfrak{M}) \cong v^{-1}\pi_*(\boldsymbol{X};\boldsymbol{M})$ for any space $\boldsymbol{X}$, the $\mathfrak{M}$-homotopy operations are in particular v-periodic ones, so we can think of a $\mathfrak{M}$-Π-algebra as a "simplified" v-Π-algebra.

9.8. **derived functors.** Note that the simplicial virtual tower $\hat{\mathfrak{J}}_\bullet$ of §5.1 has each $F_k\hat{\mathfrak{J}}_r$ homotopy equivalent to $\mathfrak{W}_k$, where $\mathfrak{W}_k$ is an (infinite) coproduct of $\Sigma^t\mathfrak{M}$'s. Moreover, the tower map $\mathfrak{i}_k : F_k\hat{\mathfrak{J}}_r \hookrightarrow F_{k+1}\hat{\mathfrak{J}}_r$ is homotopy equivalent to a projection on a sub-coproduct, followed by the inclusion of a summand, so by (2.3) and Lemma 4.5 $v^{-1}\pi_\star\hat{\mathfrak{J}}_n = \mathrm{colim}_k\ v^{-1}\pi_*(gW_k;\boldsymbol{M}) \cong \mathrm{colim}_k\ v^{-1}\pi_*(\mathfrak{W}_k;\boldsymbol{M})$ is a free v-Π-algebra. In fact, by Corollary 5.6 the augmented simplicial v-Π-algebra $v^{-1}\pi_{\hat{\mathfrak{J}}_\bullet} \to v^{-1}\pi_\star\hat{\mathfrak{X}}$ is a free simplicial resolution of $v^{-1}\pi_\star\hat{\mathfrak{X}}$ (which is just $v^{-1}\pi_*(\boldsymbol{X};\boldsymbol{M})$ if $\hat{\mathfrak{X}} = \hat{\mathfrak{C}}(\boldsymbol{X})$).

Now given any (not necessarily additive!) functor $T : v\text{-}\Pi\text{-}Alg \to \mathcal{C}$ into a suitable ([BS, §2.1]) category $\mathcal{C}$, the n-th derived functor (in the sense of Quillen) of T, applied to $v^{-1}\pi_*(\boldsymbol{X};\boldsymbol{M}) \in v\text{-}\Pi\text{-}Alg$, is isomorphic to the n-th homotopy group of the simplicial $\mathcal{C}$-object $T(v^{-1}\pi_\star\hat{\mathfrak{J}}_\bullet)$. It is denoted $L_nT(v^{-1}\pi_*(\boldsymbol{X};\boldsymbol{M}))$ (see, e.g., [BS, §2.2]).

In particular, let $F : \mathcal{T}_\star \to \mathcal{T}_\star$ be any functor which preserves $v^{-1}\pi_\star$-w.e.'s. Then we can define a functor $\bar{F} : v\text{-}\Pi\text{-}Alg \to grAbgp$ as follows: if Θ is a free v-Π-algebra, then $\Theta \cong v^{-1}\pi_*(\mathfrak{X};\boldsymbol{M})$ for some $\boldsymbol{X} \overset{w.e.}{\simeq} \bigvee_i \Sigma^{r_i}\boldsymbol{M}$, and we may let $\bar{F}(\Theta) = v^{-1}\pi_*(F(\boldsymbol{X});\boldsymbol{M})$. Note that if also $\Theta \cong v^{-1}\pi_*(\boldsymbol{X}';\boldsymbol{M})$, then $\boldsymbol{X}$ and $\boldsymbol{X}'$ are actually weakly equivalent, so $\boldsymbol{X}$ and $\boldsymbol{X}'$ are certainly $v^{-1}\pi_\star$-w.e., and thus $\bar{F}$ is well defined on free $\mathfrak{M}$-Π-algebras (compare [Bl, §7.1.2]). Now we *extend* $\bar{F}$ to an arbitrary $\mathfrak{M}$-Π-algebra Ψ by setting $\bar{F}\Psi = (L_0\bar{F})\Psi$ (cf. [BS, §2.2.4]).

Example 9.9. Consider the functor $v^{-1}\pi_*(\Sigma(-);\boldsymbol{M})$ from spaces to graded abelian groups; this takes $v^{-1}\pi_\star$-w.e.'s to isomorphisms by Corollary 7.9, so we have the induced functor

$$\hat{\Sigma} : v\text{-}\Pi\text{-}Alg \to grAbgp.$$

10. The suspension spectral sequence

We are now in a position to construct the v_n-periodic spectral sequence for the suspension of a space $\boldsymbol{X} \in \mathcal{T}_\star$:

Theorem 10.1. *Let* $v = v_n : \Sigma^d \boldsymbol{V}_{n-1} \to \boldsymbol{V}_{n-1}$ *be a self map as in* §*7.1* $(n \geq 1)$*; then for any* $c(n)$*-connected space* $\boldsymbol{X}$ *with* $v_m^{-1}\pi_*(\boldsymbol{X}; \boldsymbol{V}_{m-1}) = 0$ *for all* $0 \leq m < n$ *there is a first quadrant spectral sequence with*

$$E^2_{s,t} \;=\; (L_s\hat{\Sigma})(v^{-1}\pi_*(\boldsymbol{X};\boldsymbol{M}))_t \Rightarrow v^{-1}\pi_{s+t}(\Sigma\boldsymbol{X};\boldsymbol{M}),$$

where $\hat{\Sigma} : v\text{-}\Pi\text{-}Alg \to grAbgp$ *is the functor of* §*9.9 above.*

Proof. Let $\hat{\mathfrak{J}}_\bullet$ be the v-periodic simplicial resolution of $\hat{\mathfrak{C}}(\boldsymbol{X})$ of §5.1, where by §5.4 we may assume all level spaces are $c(n)$-connected, too. By Proposition 8.4 we know $\|\hat{\mathfrak{J}}_\bullet\|$ is $v^{-1}\pi_\star$-w.e. to $\hat{\mathfrak{C}}(\boldsymbol{X})$, so $\Sigma\|\hat{\mathfrak{J}}_\bullet\|$ is P_{v_n}-equivalent, and thus in particular $v^{-1}\pi_\star$-w.e. to $\Sigma\hat{\mathfrak{C}}(\boldsymbol{X}) = \hat{\mathfrak{C}}(\Sigma\boldsymbol{X})$ by Theorem 7.7 and Proposition 8.5 since we assumed $v_m^{-1}\pi_*(\boldsymbol{X}; \boldsymbol{V}_{m-1}) = 0$ for all $0 \leq m < n$.

But as for any proper simplicial virtual tower (or space), $\|\Sigma\hat{\mathfrak{J}}_\bullet\| \cong \Sigma\|\hat{\mathfrak{J}}_\bullet\|$, and by Proposition 8.5 $\Sigma\|\hat{\mathfrak{J}}_\bullet\|$ is $v^{-1}\pi_\star$-w.e. to $\Sigma\hat{\mathfrak{C}}(\boldsymbol{X}) = \hat{\mathfrak{C}}(\Sigma\boldsymbol{X})$. Thus the v-periodic Quillen spectral sequence for $\Sigma\hat{\mathfrak{J}}_\bullet$ (Proposition 6.5) converges to

$$v^{-1}\pi_\star\|\Sigma\hat{\mathfrak{J}}_\bullet\| \cong v^{-1}\pi_\star\Sigma\|\hat{\mathfrak{J}}_\bullet\| \cong v^{-1}\pi_\star\hat{\mathfrak{C}}(\Sigma\boldsymbol{X}) \cong v^{-1}\pi_*(\Sigma\boldsymbol{X};\boldsymbol{M}).$$

To identify the E^2-term, note that $v^{-1}\pi_\star\hat{\mathfrak{J}}_\bullet = v^{-1}\pi_\star\hat{\mathfrak{J}}_\bullet$ is a free v-Π-algebra resolution of $v^{-1}\pi_\star\hat{\mathfrak{C}}(\boldsymbol{X}) = v^{-1}\pi_*(\boldsymbol{X};\boldsymbol{M})$, and thus

$$E^2_{s,\star} = \pi_s(v^{-1}\pi_\star\Sigma\hat{\mathfrak{J}}_\bullet) = \pi_s(v^{-1}\pi_\star\Sigma\hat{\mathfrak{J}}_\bullet) = (L_s\hat{\Sigma})(v^{-1}\pi_*(\boldsymbol{X};\boldsymbol{M})),$$

as defined in §9.9 above. □

References

[A] D.W. Anderson, "Fibrations and geometric realizations", *Bull. AMS* **84** (1978) No. 5, pp. 765-788.

[Bl] D. Blanc, "A Hurewicz spectral sequence for homology", *Trans. AMS* **318** (1990) No. 1, pp. 335-354.

[BS] D. Blanc & C.R. Stover, "A generalized Grothendieck spectral sequence", in *Adams Memorial Symp. on Alg. Top., I*, (ed. N. Ray & G. Walker), LMS Lec. Notes Ser. **175**, Camb. U. 1992, pp. 145-161.

[BT] D. Blanc & R.D. Thompson, "$\boldsymbol{M}$-equivalences and homotopy colimits", *preprint* 1993.

[Bo1] A.K. Bousfield, "Construction of factorization systems in categories", *J. Pure Appl. Alg.*, **9** (1977) pp. 207-220.

[Bo2] A.K. Bousfield, "The Boolean algebra of spectra", *Comm. Math. Helv.* **54** (1979) pp. 368-377.

[Bo3] A.K. Bousfield, "Localization and periodicity in unstable homotopy theory", *J. AMS* **7** (1994) No. 4, pp. 831-873.

[BF] A.K. Bousfield & E.M. Friedlander, "Homotopy theory of Γ-spaces, spectra, and bisimplicial sets", in *Geometric Applications of Homotopy Theory, II*, ed. M.G. Barratt & M.E. Mahowald, Springer-Verlag *Lec. Notes Math.* **658**, Berlin-New York 1978, pp. 80-130.

[BK1] A.K. Bousfield & D.M. Kan, *Homotopy limits, Completions, and Localizations*, Springer-Verlag *Lec. Notes Math.* 304, Berlin-New York 1972.

[BK2] A.K. Bousfield & D.M. Kan, "The homotopy spectral sequence of a space with coefficients in a ring", *Topology* **11** (1972) No. 1, pp. 79-106.

[CPP] C. Casacuberta, G. Peschke & M. Pfenniger, "On orthogonal pairs in categories and localization", *Adams Memorial Symp. on Alg. Top., I*, (ed. N. Ray & G. Walker), LMS Lec. Notes Ser. **175**, Camb. U. 1992, pp. 211-223.

[CN] F. Cohen & J. Neisendorfer, "A note on desuspending the Adams map", *Math. Proc. Camb. Phil. Soc.* **99** (1986), pp. 59-64.

[D] E.S. Devinatz, "Small ring spectra", *J. Pure Appl. Alg.*, **81** (1992) No. 1, pp. 11-16.

[DHS] E.S. Devinatz, M.J. Hopkins & J.H. Smith, "Nilpotence and Stable Homotopy theory I", *Ann. Math.* **128** (1988) pp. 207-241.

[DF1] E. Dror Farjoun, "Homotopy Localization and v_1-Periodic Spaces", in *Algebraic Topology - Homotopy and Group Cohomology*, ed. J. Aguadé, M. Castellet & F.R. Cohen, Springer-Verlag *Lec. Notes Math.* 304, Berlin-New York 1992, pp. 104-114.

[DF2] E. Dror-Farjoun, "Localizations, fibrations, and conic structures", *preprint* 1991.

[DS] E. Dror-Farjoun & J.H. Smith, "Homotopy localization nearly preserves fibrations", *preprint* 1992.

[EH] D.A. Edwards & H.M. Hastings, *Čech and Steenrod Homotopy Theories*, Springer-Verlag *Lec. Notes Math.* **542**, Berlin-New York 1976.

[Go] R. Godement, *Topologie algébrique et théorie des faisceaux*, Act. Sci. & Ind. No. 1252, Publ. Inst. Math. Univ. Strasbourg **XIII**, Hermann, Paris 1964.

[Gr1] B. Gray, *Homotopy Theory: An Introduction to Algebraic Topology*, Academic Press, New York, 1975.

[Gr2] B. Gray, *in preparation.*

[HS] M.J. Hopkins & J.H. Smith, "Nilpotence and Stable Homotopy theory II", *preprint* 1992.

[Mc] S. Mac Lane, *Categories for the working mathematician*, GTM 5, Springer-Verlag, Berlin-New York 1971.

[Ma1] M.E. Mahowald, "The Image of J in the EHP sequence", *Ann. Math.*, **116** (1982) No. 1, pp. 65-112.

[Ma2] M.E. Mahowald, "*bo* resolutions", *Pacific J. Math.*, **92** (1981) pp. 365-383.

[MT] M.E. Mahowald & R.D. Thompson, "The K-theory localization of an unstable sphere", *Topology* **31** (1992) No. 1, pp. 133-141.

[M1] J.P. May, *Simplicial Objects in Algebraic Topology*, U. of Chicago Press, Chicago-London, 1967.

[M2] J.P. May, *The geometry of iterated loop spaces*, Springer-Verlag *Lec. Notes Math.* 271, Berlin- New York 1972.

[Mi] J.W. Milnor, "On axiomatic homology theory", *Pac. J. Math.* **12** (1962), pp. 337-341.

[N] J.A. Neisendorfer, *Primary homotopy theory*, Mem. AMS **25** (No. 232), AMS, Providence, RI, 1980.

[Q1] D.G. Quillen, *Homotopical Algebra*, Springer-Verlag *Lec. Notes Math.* 20, Berlin-New York 1963.

[Q2] D.G. Quillen, "Spectral sequences of a double semi-simplicial group", *Topology* **5** (1966), pp. 155-156.

[R1] D.C. Ravenel, "Localization with respect to certain periodic homology theories", *Am. J. Math.* **106** (1984) No. 2, pp. 351-414.

[R2] D.C. Ravenel, *Nilpotence and Periodicity in Stable Homotopy Theory*, Ann. Math. Studies **128**, Princeton U. Press, Princeton, NJ, 1992.

[Se] G. Segal, "Classifying spaces and spectral sequences", *Inst. Haut. Et. Sci.* Publ. Math. N^o. **34** (1968), pp. 105-112.

[St] C.R. Stover, "A Van Kampen spectral sequence for higher homotopy groups", *Topology*, **29** (1990) No. 1, pp. 9-26.

[T1] R.D. Thompson, "Unstable v_1-periodic homotopy groups of a Moore space", *Proc. AMS* **107** (1989) No. 3, pp. 833-845.

[T2] R.D. Thompson, "The v_1-periodic homotopy groups of an unstable sphere at odd primes", *Trans. AMS* **319** (1990) No. 2, pp. 535-559.

[Vo] R.M. Vogt, "Commuting homotopy limits", *Math. Z.*, **153** (1977), pp. 59-82.

[W] G.W. Whitehead, *Elements of homotopy theory*, GTM **61**, Springer-Verlag, Berlin-New York 1971.

The Hebrew University of Jerusalem

Northwestern University

Current address: University of Haifa, 31905 Haifa, Israel

Current address: Hunter College & CUNY Graduate Center, New York, NY 10021, USA

E-mail address: `blanc@mathcs2.haifa.ac.il`

E-mail address: `thompson@math.hunter.cuny.edu`