REALIZING COALGEBRAS OVER THE STEENROD ALGEBRA

DAVID BLANC

Abstract. We describe algebraic obstruction theories for realizing an abstract (co)algebra K_* over the mod p Steenrod algebra as the (co)homology of a topological space, and for distinguishing between the p-homotopy types of different realizations. The theories are expressed in terms of the Quillen cohomology of K_*.

1. Introduction

The question of which graded R-algebras can occur as the cohomology ring of a space $\mathbf{X}$ with coefficients in R was first raised explicitly by Steenrod in [St2], but it goes back to Hopf, for $R = \mathbb{Q}$ – see [Ho]. When $R = \mathbb{F}_p$, the cohomology $H^*(\mathbf{X}; \mathbb{F}_p)$ also has a compatible action of the Steenrod algebra, so it is natural to ask:

Which algebras over the Steenrod algebra can be realized as the cohomology of a space, and in how many different ways?

This question has been addressed repeatedly in the past – see, for example, [A, AW, Ad, ABN, CE, DMW, DW, Sm, SS] and the survey in [Ag]. Two related algebraic questions have also often been considered: which $\mathbb{F}_p$-algebras can be provided with a compatible action of the Steenrod algebra (see, e.g., [DKW, ST, Th]), and conversely, which unstable modules over the Steenrod algebra can be provided with a compatible algebra structure, or directly: which unstable modules are realizable – see, e.g., [CS, Ku, Sc2]. The analogous *stable* question of whether a given module over the Steentrod algebra can be realized by a spectrum, which has also been extensively studied (e.g., [BM, BG, BP]), can be answered in terms of suitable Ext groups (see [Ma, Ch. 16, 3]), and an unstable version of this for the Massey-Peterson case was developed by Harper (see [Ha]).

However, we shall not be concerned with these variants here: our goal is to describe a general obstruction theory for the original realization problem, which can be stated purely algebraically, in terms of the Quillen cohomology of the given algebra – analogous to the stable theory. This answers a question of Lannes, Miller and others in the 1980's, asking for an unstable analogue of the stable obstruction theory, which was also (independently) one of the motivations for the project begun by Dwyer, Kan and Stover in [DKS1, DKS2] (see also [**?**]).

It turns out to be more natural to consider of the dual question, that of realizing a *coalgebra* over the Steenrod algebra as the *homology* of a space. This is because the cohomology of a space in general has the structure of a *profinite* unstable algebra;

Date: Revised: March 1, 1999.

1991 *Mathematics Subject Classification.* Primary 55S10; Secondary 18G55.

Key words and phrases. Steenrod algebra, realization, unstable coalgebra, (co)homology operations, cohomology ring, cosimplicial resolution, Quillen cohomology.

it is only when the space is of finite type that it is an unstable algebra, and the realizability of an unstable algebra of finite type is of course strictly equivalent to that of the corresponding coalgebra (its vector space dual).

Part the theory we describe here actually works over a more general ground ring R, but to obtain its full force we restrict attention to the case when R is a field. Thus, if we define an *unstable R-coalgebra* to be a graded coalgebra over R equipped with a compatible action of the unstable R-homology operations, we have:

Theorem A. *For $R = \mathbb{F}_p$ or $\mathbb{Q}$, let K_* be a connected unstable R-coalgebra, such that either $K_1 = 0$, or K_* has finite projective dimension. Then there is a sequence of cohomology classes $\chi_n \in H^{n+2}(K_*; \Sigma^n K_*)$ such that χ_n is defined whenever $\chi_1 = \ldots = \chi_{n-1} = 0$, and all the classes vanish if and only if $K_* \cong H_*(\mathbf{X}; R)$ for some space $\mathbf{X}$.*

(See Theorems 5.10 and 6.3 below). The proof involves showing that, for any space $\mathbf{X}$, any (algebraic) cosimplicial resolution of the unstable coalgebra $H_*(\mathbf{X}; R)$ can be realized as a cosimplicial space, and conversely. As a side benefit, when $R = \mathbb{F}_p$, this provides a way of constructing minimal "unstable Adams resolutions" (see Remark 6.6 below).

There is a similar theory for distinguishing between different realizations:

Theorem B. *For any two simply-connected spaces $\mathbf{X}$ and $\mathbf{Y}$ of finite type such that $H_*(\mathbf{X}; \mathbb{F}_p) \cong H_*(\mathbf{Y}; \mathbb{F}_p) \cong K_*$ as unstable coalgebras over the Steenrod algebra, there is a sequence of cohomology classes $\delta_n \in H^{n+1}(K_*; \Sigma^n K_*)$ whose vanishing implies that $\mathbf{X}$ and $\mathbf{Y}$ are p-equivalent.*

(See Theorems 7.3 and 7.6 below). This result is less satisfactory, in that $\mathbf{X}$ and $\mathbf{Y}$ may be p-equivalent even if the cohomology classes do not all vanish (which simply expresses the fact that the homotopy theory of cosimplicial spaces is richer than that of topological spaces). Note that in this the classes δ_n resemble the usual k-invariants, and indeed Theorem B can be thought of as providing a system of algebraic "invariants" for the p-type of a space, dual to those of [Bl5] or [**?**] (see §7.8).

Theorem B also holds for $R = \mathbb{Q}$; in this case we simply recover the homology version of the obstruction theory of [HS] and [F].

1.1. **Notation and conventions.** $\mathcal{T}$ will denote the category of topological spaces, and $\mathcal{T}_*$ that of pointed connected topological spaces with base-point preserving maps. We denote objects in $\mathcal{T}$ or $\mathcal{T}_*$ by boldface letters: $\mathbf{X}, \mathbf{Y}$, and so on, to help distinguish them from the various algebraic objects we consider.

$\mathbb{Q}$ denotes the rationals, and for p prime, $\mathbb{F}_p$ denotes the field with p elements. For a ring R (always assumed to be commutative with unit), $R\text{-}Mod$ denotes the category of R-modules, and $R\langle X_*\rangle$ the free R-module on a (possibly graded) set of generators X_*. Tensor products of R-modules will always be over the ground ring R, unless otherwise stated, and the *dual* module of $A \in R\text{-}Mod$ is denoted by $A^\star := \mathrm{Hom}_{R\text{-}Mod}(A, R)$.

$H_*(\mathbf{X}; R) \in R\text{-}Mod$ is the homology of a topological space (or simplicial set) $\mathbf{X}$ with coefficients in R. We write $f_\# : H_*(\mathbf{X}; R) \to H_*(\mathbf{Y}; R)$ for the graded homomorphism induced by $f : \mathbf{X} \to \mathbf{Y}$. $R_\infty\mathbf{X}$ is the Bousfield-Kan R-completion of $\mathbf{X}$ (cf. [BK1, I, 4.2]).

For an abelian category $\mathcal{M}$, we let $c_*\mathcal{M}$ denote the category of chain complexes over $\mathcal{M}$ (in non-negative degrees); similarly, $c^*\mathcal{M}$ denotes the category of cochain complexes.

For any category $\mathcal{C}$, we denote by $\operatorname{gr}\mathcal{C}$ the category of *non-negatively graded objects* over $\mathcal{C}$, with $|x| = n \Leftrightarrow x \in X_n$ for $X_* \in \operatorname{gr}\mathcal{C}$. Given a (fixed) object $B \in \mathcal{C}$, we denote by $\mathcal{C}\backslash B$ the category of objects under B (cf. [M2, II, §6]).

1.2. **Definition.** A *cosimplicial object* $X^\bullet$ over any category $\mathcal{C}$ is a sequence of objects $X^0, X^1, \dots, X^n, \dots$ in $\mathcal{C}$ equipped with *coface* and *codegeneracy* maps $d^i : X^n \to X^{n+1}$, $s^j : X^{n+1} \to X^n$ $(0 \leq i, j \leq n)$ satisfying the cosimplicial identities

$$d^j d^i = d^{i+1} d^j \qquad \text{if } i \geq j$$

$$s^j d^i = \begin{cases} d^i s^{j-1} & \text{if } i < j \\ \mathrm{Id} & \text{if } i = j, j+1 \\ d^{i-1} s^j & \text{if } i \geq j+2 \end{cases} \tag{1.3}$$

$$s^j s^i = s^i s^{j+1} \qquad \text{if } i \leq j$$

(cf. [BK1, X, §2.1]).

We denote by $c\mathcal{C}$ the category of cosimplicial objects over $\mathcal{C}$. If we restrict attention to $X^0, X^1, \dots, X^n$, with their coface and codegeneracy maps, we have an *n-cosimplicial* object; the category of such will be denoted by $c^{\langle n\rangle}\mathcal{C}$.

Dually, we denote by $s\mathcal{C}$ the category of *simplicial* objects over $\mathcal{C}$ (cf. [M, §2]). The category of simplicial sets, however, will be denoted simply by $\mathcal{S}$ (rather than $s\mathcal{S}et$), and that of *pointed* simplicial sets by $\mathcal{S}_*$. Objects in these two categories will again be denoted by boldface letters. The standard n simplex in $\mathcal{S}$ is denoted by $\Delta[n]$, generated by $\sigma_n \in \Delta[n]_n$, and $\Lambda^k_n \in \mathcal{S}$ is the sub-simplicial set of $\Delta[n]$ generated by $d_i\sigma_n$ for $i \neq k$.

Since we shall be dealing for the most part with simplicial sets as our model for the homotopy category of topological spaces, we shall call cosimplicial pointed simplicial sets – i.e., objects in $c\mathcal{S}_*$ – simply *cosimplicial spaces.*

1.4. **Example.** The cosimplicial space $\mathbf{\Delta}^\bullet \in c\mathcal{S}$ has the standard simplicial n-simplex $\Delta[n]$ in cosimplicial dimension n, with coface and codegeneracy maps being the standard inclusions and projections (cf. [BK1, I, 3.2]).

1.5. **Definition.** If $\mathcal{C}$ has enough limits, the obvious truncation functor $\operatorname{tr}_n : c\mathcal{C} \to c^{\langle n\rangle}\mathcal{C}$ has a right adjoint ρ_n, and the composite $\operatorname{cosk}^n := \rho_n \circ \operatorname{tr}_n : c\mathcal{C} \to c\mathcal{C}$ is called the *n-coskeleton* functor. (This is dual to *n-skeleton* functor $\operatorname{sk}_n : s\mathcal{C} \to s\mathcal{C}$.)

1.6. **Organization.** In section 2 we recall some basic facts about coalgebras over the Steenrod algebra, and in section 3 we show how certain convenient CW resolutions for such coalgebras may be constructed. Section 4 deals with the coaction of the fundamental group of a cosimplicial coalgebra, and the Quillen cohomology of unstable coalgebras. In section 5 we describe the cohomology classes which determine whether a given algebraic resolution may be realized topologically (Theorem 5.10), and in section 6 we apply this to the original question of realizing an abstract coalgebra (Theorem 6.3). Finally, in section 7 a similar theory is developed for distinguishing between different possible realizations of a given algebraic resolution (Theorem 7.3), and thus for determining the p-type of a space (Theorem 7.6).

1.7. **Acknowledgements.** I would like to thank Pete Bousfield, Bill Dwyer, and Brooke Shipley for several useful conversations, and Dan Kan for suggesting that I consider the problem of realizing algebras over the Steenrod algebra, which was one of the motivations for the project begun in [DKS1] and [DKS2]. I would also like to thank the referee for many helpful comments and suggestions.

2. Unstable coalgebras

We first recall some basic facts about the category of the coalgebras over the Steenrod algebra:

2.1. **Definition.** For a field R, let $coAlg_R$ denote the category of graded coalgebras over R: an object in $coAlg_R$, which we shall call simply an *R-coalgebra*, is thus a (non-negatively) graded R-module $V_* \in \operatorname{gr} R\text{-}Mod$, equipped with a coassociative *diagonal* (or *comultiplication*) map $\Delta : V_* \to V_* \otimes V_*$, with $(\mathrm{Id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{Id}) \circ \Delta$, and an *augmentation* (or *counit*) map $\varepsilon : V_* \to R$, with $(\mathrm{Id} \otimes \varepsilon) \circ \Delta$ and $(\varepsilon \otimes \mathrm{Id}) \circ \Delta$ equal respectively to the natural isomorphisms $V_* \to V_* \otimes R$ and $V_* \to R \otimes V_*$. We require the comultiplication to be *cocommutative*, in the graded sense – i.e., $\Delta \circ \tau = \Delta$, where $\tau(a \otimes b) := -1^{|a||b|} b \otimes a$ is the graded switch map. See [Sw, §1.0] and [MM, §2.1].

We assume all our graded coalgebras $C_* \in coAlg_R$ are *connected* – that is, $C_0 \cong R$; $C_* \in coAlg_R$ is called *simply-connected* if in addition $C_1 = 0$. The coalgebra C_* is of *finite type* if C_k is finite dimensional vector space over R for each $k \geq 0$. We can pass from $coAlg_R$ to the category $\mathcal{A}lg_R$ of connected, unital, graded-commutative *algebras* over R by taking the vector-space dual: $(C_*)^\star \in \mathcal{A}lg_R$ (and if $C^* \in \mathcal{A}lg_R$ is of finite type, we can of course pass back to $coAlg_R$ in the same way).

2.2. **Unstable coalgebras.** As usual, the homology $H_*(\mathbf{X}; R)$ of a space $\mathbf{X}$ with coefficients in field R is an R-coalgebra, with Δ induced by the diagonal $\mathbf{X} \to \mathbf{X} \times \mathbf{X}$ (composed with inverse of the Künneth isomorphism $H_*(\mathbf{X}; R) \otimes H_*(\mathbf{X}; R) \cong H_*(\mathbf{X} \times \mathbf{X}; R)$). However, $H_*(\mathbf{X}; R)$ also comes equipped with an action of the primary R-homology operations: these are natural transformations $H_i(-; R) \to H_k(-; R)$, dual to the corresponding cohomology operations, and they vanish if $k > i$ (see [St1, §9]).

For any field R, an *unstable coalgebra* (over R) is a non-negatively graded R-module equipped with an action of primary R-homology operations (which include the coalgebra structure), satisfying the universal identities for these operations. We denote the category of such unstable R-coalgebras by $\mathcal{CA}_R$.

The simplest case is when $R = \mathbb{Q}$: $\mathcal{CA}_\mathbb{Q} \approx coAlg_\mathbb{Q}$, since there are no non-trivial primary $\mathbb{Q}$-homology operations besides the coproduct (see [Q2, I, §1]). The next simplest is $R = \mathbb{F}_p$:

2.3. **Definition.** For any prime p, an *unstable module over the mod p Steenrod algebra*, $\mathcal{A}_p$, is a non-negatively graded $\mathbb{F}_p$-vector space K_*, equipped with a right action of $\mathcal{A}_p$ – i.e., a graded homomorphism

$$\lambda : K_* \otimes \mathcal{A}_p \to K_* \tag{2.4}$$

where $|x\,\mathrm{Sq}^i| = |x| - i$ if $p = 2$, and $|x\mathcal{P}^i| = |x| - 2(p-1)i$, $|x\beta| = |x| - 1$ if $p > 2$ (where we write $x\,\mathrm{Sq}^i$ for $\lambda(x \otimes \mathrm{Sq}^i)$, etc.). The action is required to be *unstable* in the sense that $x\,\mathrm{Sq}^i = 0$ if $2i > |x|$ (for $p = 2$) and $x\mathcal{P}^i = 0$ if $2pi > |x|$ (for $p > 2$). The category of such unstable modules will be denoted by $\mathcal{M}od\text{-}\mathcal{A}_p$.

The category $\mathcal{M}od\text{-}\mathcal{A}_p$ is dual to the more familiar category $\mathcal{U}$ of unstable "cohomology-like" modules over the Steenrod algebra (see, e.g., [Sc1, §1.3]).

2.5. **Definition.** For any prime p, the category $\mathcal{CA}_{\mathbb{F}_p}$ of graded coalgebras over the mod p Steenrod algebra $\mathcal{A}_p$, has as objects non-negatively graded coalgebras C_* over $\mathbb{F}_p$, which are at the same time unstable $\mathcal{A}_p$-modules. The two structures are related by the Cartan formula, which says that the λ of (2.4) is a homomorphism of coalgebras (see [Mn, §4]) – dual to: $\mathrm{Sq}^n(a \cdot b) = \sum_k \mathrm{Sq}^k\, a \cdot \mathrm{Sq}^{n-k}\, b$ for $p = 2$, and $\mathcal{P}^n(a \cdot b) = \sum_k \mathcal{P}^k a \cdot \mathcal{P}^{n-k} b$, $\beta(a \cdot b) = (\beta a) \cdot b + (-1)^{|a|} a \cdot \beta b$ for $p > 2$. There is also a Verschiebung formula, dual to fact that that the top Steenrod operation equals the Frobenius – i.e., for $|a| = n$, we have $\mathrm{Sq}^n\, a = a^2$ if $p = 2$, and $\mathcal{P}^{n/2} a = a^p$ if $p > 2$ and n is even. See [BC, §5].

In particular, we can think of $H_*(\mathbf{X}; \mathbb{F}_p)$ as an object in either $\mathcal{M}od\text{-}\mathcal{A}_p$ or $\mathcal{CA}_{\mathbb{F}_p}$ for any space $\mathbf{X}$. Again, $\mathcal{CA}_{\mathbb{F}_p}$ is dual to the more familiar category $\mathcal{K} = \mathcal{K}_{\mathbb{F}_p}$ of unstable algebras over the Steenrod algebra (cf. [Sc1, §1.4]). However, taking vector space duals yields a strict equivalence of categories only when dealing with (co)algebras of *finite type*, and our approach is more naturally presented in terms of coalgebras, as noted above.

2.6. **(Co)abelian (co)algebras.** Recall that any *abelian* R-algebra (i.e., abelian group object in $\mathcal{A}lg_R$) must have a trivial multiplication (for any ring R). When $R = \mathbb{Q}$, the subcategory $(\mathcal{K}_{\mathbb{Q}})_{ab}$ of abelian objects in $\mathcal{K}_{\mathbb{Q}}$ is actually equivalent to the category of graded vector spaces over $\mathbb{Q}$. $(\mathcal{K}_{\mathbb{F}_p})_{ab}$ is equivalent to a subcategory of $\mathcal{M}od\text{-}\mathcal{A}_p$ (viewed as algebras with a trivial product): for $p = 2$, $(\mathcal{K}_{\mathbb{F}_2})_{ab} = \Sigma\mathcal{U}$ is the category of $\mathcal{A}_2$-modules with $Sq^i x = 0$ for $|x| \leq i$; for $p > 2$, $(\mathcal{K}_{\mathbb{F}_p})_{ab} = \mathcal{V}$ is the category of $\mathcal{A}_p$-modules with $\mathcal{P}^i x = 0$ for $|x| \leq 2i$ (cf. [Ml, §1]). In all these cases the abelianization functor $()_{ab} : \mathcal{K}_R \to (\mathcal{K}_R)_{ab}$ assigns to any algebra $A^* \in \mathcal{K}_R$ its "module of indecomposables", $Q(A^*)$.

Dualizing, we see that the *coabelian* objects (i.e., abelian cogroup objects) in $\mathcal{CA}_R$ must have trivial comultiplication, so $(\mathcal{CA}_R)_{co\text{-}ab}$ is equivalent to the appropriate subcategory of $\mathcal{M}od\text{-}\mathcal{A}_p$; the coabelianization functor is just the R-module of primitives $\mathcal{P}(A_*) \hookrightarrow A_*$, for any $A_* \in \mathcal{CA}_R$ (see [Bo2, §8.6], and compare [L, §2]).

2.7. **Functors and limits of coalgebras.** The underlying-set functor $coAlg_R \to \mathrm{gr}\,\mathcal{S}et$ factors through $\hat{U} : coAlg_R \to \mathrm{gr}\,R\text{-}Mod$, with right adjoint $\hat{G} : grR\text{-}Mod \to coAlg_R$, where $\hat{G}(V_*)$ is the (cocommutative) *cofree coalgebra* on V_* (cf. [Sw, §6.4]). Moreover, the functor $\hat{G}$ creates all colimits in $coAlg_R$, in the sense of [M2, V, §1], and the pair $(\hat{U}, \hat{G})$ *produces* all limits in $coAlg_R$ in the sense of [Bl2, §3.3]. The same is true for the right adjoint $G : grR\text{-}Mod \to \mathcal{CA}_R$ of the "underlying graded R-module" functor $U : \mathcal{CA}_R \to \mathrm{gr}\,R\text{-}Mod$. See [Bl2, Prop. 7.5] and [Bo2, §8.2].

Note also that any R-coalgebra, as well as any unstable R-coalgebra, is isomorphic to the colimit of its finite sub-coalgebras (see [G, 1.1]), and this allows one to describe the product of an arbitrary collection of coalgebras $(C_*^{(i)})_{i\in I}$ in terms of the partially ordered collection $\mathcal{F}$ of finite subsets $J \subseteq I$ as

$$\prod_{i\in I} C_*^{(i)} = \operatorname{colim}_{J\in\mathcal{F}} \operatorname{colim}_{\alpha\in A_J} (\otimes_{i\in J} C_*^{(i)})_\alpha, \tag{2.8}$$

where $(\otimes_{i\in J} C_*^{(i)})_\alpha$ $(\alpha \in A_J)$ runs over all *finite* sub-coalgebras of the finite tensor product $\otimes_{i\in J} C_*^{(i)}$ (see [G, 1.2]).

2.9. *Remark.* In fact, any *cofree unstable R-coalgebra* $G_* = G(V_*) \in \mathcal{CA}_R$ is of the form $G_* \cong H_*(\mathbf{K}(R\langle X_*\rangle); R)$, where $X_* \in \operatorname{gr} \mathcal{S}et$ is a graded set and $\mathbf{K}(R\langle X_*\rangle)$ is the GEM (generalized Eilenberg-Mac Lane object) $\prod_{n=1}^\infty \prod_{x\in X_n} \mathbf{K}(R, n)$. By (2.8) we have $G_* \cong \prod_{n=1} \prod_{x\in X_n} H_*(\mathbf{K}(R, n); R)$.

Note that each $H_*(\mathbf{K}(R, n); R)$, and thus G_*, is an abelian Hopf algebra (see [Bo1, §4.4]), and for any map of graded sets $f : X_* \to Y_*$, the map $G(f) : G(X_*) \to G(Y_*)$ is a morphism of Hopf algebras (in particular, an algebra homomorphism).

3. Resolutions of coalgebras

We now prove some basic facts about cofree resolutions for coalgebras:

3.1. **Definition.** For any concrete cocomplete category $\mathcal{C}$, the *co-matching object* functor $M : \mathcal{S}^{op} \times c\mathcal{C} \to \mathcal{C}$, written $M^K X^\bullet$ for a finite simplicial set $K \in \mathcal{S}$ and any $X^\bullet \in c\mathcal{C}$, is defined by requiring that $M^{\Delta[n]} X^\bullet := X^n$, and if $K = \operatorname{colim}_i K_i$ then $M^K X^\bullet = \operatorname{colim}_i M^{K_i} X^\bullet$ (cf. [Bo1, §6] and [DKS2, §2.1]).

In particular, write $M^n X^\bullet$ for $M^{\operatorname{sk}_{n-1}\Delta[n]} X^\bullet$. Note that each coface map $d^k : X^{n-1} \to X^n$ factors through the map $\xi^n : M^n X^\bullet \to X^n$ induced by the inclusion $\operatorname{sk}_{n-1}\Delta[n] \hookrightarrow \Delta[n]$. A cosimplicial space $\mathbf{X}^\bullet \in c\mathcal{S}_*$ will be called *cofibrant* if each of these maps ξ^n is a cofibration. This concept refers to the resolution model category structure on $c\mathcal{S}_*$ (see §3.13 below), rather than the Reedy model category structure of [BK1, X,§4].

3.2. **Definition.** Given a cosimplicial object $X^\bullet$ over a concrete complete category $\mathcal{C}$, the analogous construction for the codegeneracies yields $L^n X^\bullet$ is defined (in the cases of interest to us) by

$$L^n X^\bullet := \{(x_0, \ldots, x_{n-1}) \in (X^{n-1})^{\times n} \mid s^i x_j = s^{j-1} x_i \text{ for all } 0 \le i < j \le n\}.$$

Again, each codegeneracy map $s^i : X^n \to X^{n-1}$ equals the natural map $\varsigma^n : X^n \to L^n X^\bullet$, composed with the projection onto the i-th factor.

$L^n X^\bullet$ has been called the n-th "co-latching object" for $X^\bullet$ – cf. [DKS1, §2.3]. In [BK1, X, §4.5] it is denoted by $M^n \mathbf{X}^\bullet$; the notation we have here was chosen to be consistent with that of [DKS1, DKS2] and [Bl5].

3.3. **Definition.** If $\mathbf{X}^\bullet \in c\mathcal{S}_*$ is cofibrant, its *n-cochains* object, written $C^n \mathbf{X}^\bullet$, is defined to be the cofiber of $\xi_0^n : M^{\Lambda_n^0} \mathbf{X}^\bullet \to \mathbf{X}^n$ (§1.2), so $C^n \mathbf{X}^\bullet = \mathbf{X}^n / (\cup_{i=1}^n d^i \mathbf{X}^{n-1})$. Similarly, the object $B^n \mathbf{X}^\bullet$ is defined to be the cofiber of $\xi^n : M^n \mathbf{X}^\bullet \to \mathbf{X}^n$, so that $B^n \mathbf{X}^\bullet = \mathbf{X}^n / (\cup_{i=0}^n d^i \mathbf{X}^{n-1})$.

These all fit into the commutative diagram of Figure 1:

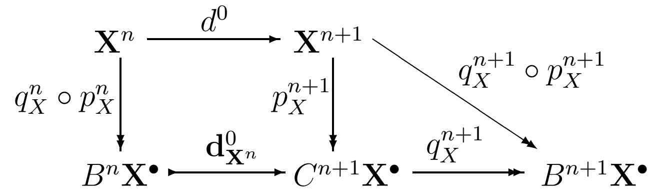

FIGURE 1

in which $p_X^n : \mathbf{X}^n \twoheadrightarrow C^n\mathbf{X}^\bullet$ and $q_X^n : C^n\mathbf{X}^\bullet \twoheadrightarrow B^n\mathbf{X}^\bullet$ are quotient maps, and the bottom row is a cofibration sequence (with the cofibration $\mathbf{d}^0 = \mathbf{d}^0_{X^n} : B^n\mathbf{X}^\bullet \to C^{n+1}\mathbf{X}^\bullet$ induced from $\xi^n : M^n\mathbf{X}^\bullet \to X^n$ by cobase change). We will call $\mathbf{d}^0$ the n-th *principal face map* for $\mathbf{X}^\bullet$.

3.4. **Definition.** The same definitions may be applied to a cosimplicial object $A^\bullet \in c\mathcal{C}$ over any suitable category $\mathcal{C}$ (e.g., if $\mathcal{C}$ has a faithful "underlying object" functor into an abelian category). In this case the kernel of $\mathbf{d}^0_A = B^nA^\bullet \to C^{n+1}A^\bullet$ is defined as usual to be the *n-th cohomotopy object* of $A^\bullet$, and denoted by $\pi^nA^\bullet$ (see [BK1, X, §7.1]). When $\mathcal{C}$ is an abelian category, $A^\bullet \in c\mathcal{C}$ is equivalent under the Dold-Kan equivalence (cf. [Do, Thm 1.9]) to a cochain complex A^*, and $\pi^nA^\bullet \cong H^nA^*$. However, in most cases $\pi^nA^\bullet$ will have additional structure – e.g., it will be a (coabelian) object in $\mathcal{C}$ (see [BS, §5]).

3.5. *Remark.* Let $\Sigma^n\mathbf{K}(R,k) \in c\mathcal{S}_*$ denote the cosimplicial space consisting of the usual simplicial Eilenberg-Mac Lane space $\mathbf{K}(R,k)$ in cosimplicial dimension n, a single point in lower dimensions, and $L^p\Sigma^n\mathbf{K}(R,k)$ (defined inductively) in dimension $p > n$; similarly $C\Sigma^n\mathbf{K}(R,k)$ is obtained from $\Sigma^n\mathbf{K}(R,k)$ by attaching a single copy of $\mathbf{K}(R,k)$ in cosimplicial dimension $n+1$ (see §3.9 and compare [DKS2, §3.6]). Write $\mathcal{E}M^n(R,k)_{\mathcal{CA}}$ for the cosimplicial unstable coalgebra $G(R\langle\Sigma^n\mathbf{K}(R,k)\rangle)$ (and similarly $C\mathcal{E}M^n(R,k)_{\mathcal{CA}} := G(R\langle C\Sigma^n\mathbf{K}(R,k)\rangle))$. Then for any $A^\bullet_* \in c\mathcal{CA}_R$ we have $\mathrm{Hom}_{c\mathcal{CA}_R}(A^\bullet_*, \mathcal{E}M^n(R,k)_{\mathcal{CA}}) \cong B^nA^\bullet_k$ and $\mathrm{Hom}_{c\mathcal{CA}_R}(A^\bullet_*, C\mathcal{E}M^n(R,k)_{\mathcal{CA}}) \cong C^{n+1}A^\bullet_k$, so it makes sense to denote $\pi^nA^\bullet_k$ by $[A^\bullet_*, \mathcal{E}M^n(R,k)_{\mathcal{CA}}]$ – and these are in fact the homotopy classes of maps into $\mathcal{E}M^n(R,k)_{\mathcal{CA}}$ in the model category structure on $c\mathcal{CA}_R$ described in [Bl2, §7].

The following statement is dual to [Bl5, Lemma 2.29]:

3.6. **Proposition.** *For any field R and cofibrant $\mathbf{X}^\bullet \in c\mathcal{S}_*$, the inclusion $\iota : C^n\mathbf{X}^\bullet \hookrightarrow \mathbf{X}^n$ induces an isomorphism $\iota_\star : H_*(C^n\mathbf{X}^\bullet; R) \cong C^n(H_*(\mathbf{X}^\bullet; R))$ for each $n \geq 0$.*

Proof. The free R-module functor $F : \mathcal{S}_* \to sR\text{-}Mod$ is a left adjoint, so preserves all colimits, and thus $C_n(F\mathbf{X}^\bullet) \cong F(C^n\mathbf{X}^\bullet)$ (we need $\mathbf{X}^\bullet$ to be cofibrant in order for $C^n\mathbf{X}^\bullet$ to be meaningful). If $K : sR\text{-}Mod \xrightarrow{\cong} c_*R\text{-}Mod$ is the Dold-Kan equivalence functor, then for any $\mathbf{X} \in \mathcal{S}_*$, $H_*(\mathbf{X}; R)$ is the homology of the chain complex $KF\mathbf{X}$, so it suffices to show that for any cosimplicial chain complex $A^\bullet_*(= KF\mathbf{X}^\bullet)$ the map $\iota_\star : H_k(C^nA^\bullet_*) \cong C^n(H_kA^\bullet_*)$ is an isomorphism for all k, n.

Now for a cosimplicial object $A^\bullet$ over any abelian category, we can define the *Moore cochain complex* by $N^nA^\bullet := \cap_{j=0}^{n-1}\mathrm{Ker}(s^j) \subseteq A^n$ (with differential $\delta := \sum_{i=0}^n(-1)^id^i$). We claim that the composite $N^nA^\bullet \hookrightarrow A^n \twoheadrightarrow C^nA^\bullet$ is an isomorphism $\Phi : N^nA^\bullet \xrightarrow{\cong} C^nA^\bullet$.

First note that $N^nA^\bullet \cap \bigcup_{i=1}^n \mathrm{Im}(d^i) = 0$, since if $\alpha = \sum_{i=\ell}^n d^i(x_i)$ (which we may assume by induction on $0 \leq \ell \leq n$) and $s^j\alpha = 0$ for $0 \leq j \leq n$, then $0 = s^{\ell-1}\alpha = x_\ell + \sum_{i=\ell+1}^n d^{i-1}s^{\ell-1}(x_i)$, so $d^\ell x_\ell = -\sum_{i=\ell+1}^n d^i d^\ell s^{\ell-1}(x_i)$. Thus α is in fact of the form $\sum_{i=\ell+1}^n d^i(x_i')$ – which implies that $\alpha = 0$ (for $\ell = n$). This shows that Φ is one-to-one.

Next, given $\alpha \in A^n$ with $s^j(\alpha) = 0$ for $0 \leq j < \ell$ (which we again assume by induction on $0 \leq \ell < n$), then there is an $\alpha' \in A^n$ with $[\alpha] = [\alpha'] \in C^nA^\bullet$ such that $s^j(\alpha') = 0$ for $0 \leq j \leq \ell$ – namely, $\alpha' := \alpha - d^{\ell+1}s^\ell\alpha$. This shows that Φ is onto.

Thus the Proposition will follow if we show $\iota_\star : H_k(N^nA^\bullet_*) \cong N^n(H_kA^\bullet_*)$ is an isomorphism: given $\langle\alpha\rangle \in N^n(H_kA^\bullet_*)$ represented by $\alpha \in A^n_k$ with $\partial_k(\alpha) = 0$, (where ∂ is the boundary map of the chain complex A^n_*), we assume that $s^j(\alpha) = 0$ for $0 \leq j < \ell$, and that there are $b_\ell, \dots, b_{n-1} \in A^{n-1}_{k+1}$ such that $s^j(\alpha) = \partial_{k+1}(b_j)$ for $\ell \leq j < n$. Replacing α by $\alpha' := \alpha - \partial_{k+1}(d^{\ell+1}(b_\ell))$, we see by induction on $0 \leq \ell \leq n-1$ that we may choose a representative for $\langle\alpha\rangle$ with $s^j(\alpha) = 0$ for all j. Thus $\iota_\star$ is surjective. Finally, if $\langle\alpha\rangle = 0 \in N^n(H_kA^\bullet_*)$ for $\alpha \in N^nA^\bullet_k$, there is a $\beta \in A^n_{k+1}$ such that $\partial_{k+1}(\beta) = \alpha$, with $s^j(\beta) = 0$ for $0 \leq j \leq \ell$; setting $\beta' := \beta - d^{\ell+1}s^{\ell+1}(\beta)$, we see that we can assume $\beta \in N^nB_{k+1}$, so $\iota_\star$ is one-to-one. □

3.7. **Definition.** A cosimplicial coalgebra $A^\bullet_*$ is called *cofree* if for each $n \geq 0$ there is a graded set T^n_* of elements in A^n_* such that $A^n_* = G(T^n_*)$ (cf. §2.7), and

(3.8) each codegeneracy map $s^j : A^n_* \to A^{n-1}_*$ takes T^n_* to T^{n-1}_*.

3.9. **Definition.** A *CW complex* over a pointed category $\mathcal{C}$ is a cosimplicial object $C^\bullet \in s\mathcal{C}$, together with a sequence of objects $\bar{C}^n \in \mathcal{C}$ $(n = 0, 1, \dots)$ – called a *CW basis* for $C^\bullet$ – such that $C^n = \bar{C}^n \times L^nC^\bullet$ for each $n \geq 0$, with projection $\mathrm{proj}_{\bar{C}^n} : C^n \to \bar{C}^n$, such that $\mathrm{proj}_{\bar{C}^n} \circ d^i = 0$ for $1 \leq i \leq n$ (compare [Ka, §3] and [Bl1, §5]).

The coface map $\bar{d}^0_n := \mathrm{proj}_{\bar{C}^n} \circ d^0 : C^{n-1} \to \bar{C}^n$ is called the *attaching map* for $\bar{C}^n$, and it is readily verified that the attaching maps $\bar{d}^0_n$ $(n = 0, 1, \dots)$, together with the cosimplicial identities (1.3), determine all the face and degeneracy maps of $C^\bullet$. Note that $\bar{d}^0_n$ factors through a map $B^{n-1}C^\bullet \to \bar{C}^n$.

In particular, we require that a CW basis for a free cosimplicial algebra $A^\bullet_* \in \mathcal{C}A_R$ be a sequence of *cofree* colagebras $(\bar{A}^n_*)_{n=0}^\infty$.

On the other hand, for a cosimplicial object over $\mathcal{S}_*$, it will be convenient to require only that $C^n \simeq \bar{C}^n \times L^nC^\bullet$ for all n.

For any field R, the category $c\mathcal{C}A_R$ of cosimplicial unstable R-coalgebras has a model category structure in the sense of Quillen (cf. [Q1, I, §1]), induced from that of cR-$Mod \approx c^*R$-Mod by the obvious pair of adjoint functors (see [Bl2, §7]). All we shall need from this are the following:

3.10. **Definition.** A *cofree cosimplicial resolution* of an unstable R-coalgebra K_* is defined to be a cofree cosimplicial coalgebra $A^\bullet_*$, equipped with a coaugementation $K_* \to A^0_*$, such that in each degree $k \geq 1$, the cohomotopy groups of the cosimplicial R-module $(A^\bullet)_k$ (i.e., the cohomology groups of the corresponding cochain complex over R-$\mathcal{M}od$) vanish in dimensions $n \geq 1$, and the coaugmentation induces an isomorphism $K_* \cong \pi^0 A^\bullet_*$.

3.11. **Constructing CW resolutions.** As usual, such a resolution is simply a fibrant (and cofibrant) object in $c\mathcal{CA}_R$ which is weakly equivalent to the constant cosimplicial object $c(K_*)^\bullet$, in the model category structure mentioned above. In particular, such resolutions always exist, for any $K_* \in \mathcal{CA}_R$; there are a number of ways to construct them, including the (very large) canonical monad resolution described in [BK2, §11.4] when R is a field (see [Bl2, §7.8]).

We shall be interested in a particular type, namely, those equipped with a CW basis, which will be called *CW resolutions*, since these can be chosen to be small (e.g., minimal). Their construction is straightforward: starting with $A^{-1}_* := K_*$ and $B^{-1}A^\bullet_* := A^{-1}_*$, we assume that we have constructed $A^\bullet_*$ through cosimplicial dimension $n-1$; then we simply choose some cofree unstable coalgebra $\bar{A}^n_* \in \mathcal{CA}_R$ with a one-to-one map $\bar{d}^0_n : B^{n-1}A^\bullet_* \hookrightarrow \bar{A}^n_*$. These always exist, by the universal property of cofree coalgebras – e.g., one could take $\bar{A}^n_* := GU(B^{n-1}A^\bullet_*)$. Setting $A^n_* := \bar{A}^n_* \times L^n A^\bullet_*$ completes the inductive stage.

The dual construction, for simplicial groups, algebras, and so on, is classical: see [Ka], [Ta] and [And, I, §6]. Note, however, the following analogue of [Bl4, Prop. 3.18]:

3.12. **Proposition.** *Any cofree cosimplicial unstable coalgebra $A^\bullet_* \in c\mathcal{CA}_R$ has a CW basis $(\bar{A}^n_*)^\infty_{n=0}$.*

Proof. Start with $\bar{A}^0_* := A^0_*$, and note that (3.8) of definition 3.7 implies (by induction on n) that $L^n A^\bullet_* \cong G(R\langle Y_*\rangle)$ for some $Y_* \in \operatorname{gr} \mathcal{S}et$.

Now because $\mathcal{CA}_R$ has the "underlying structure" of an abelian category, we may define a homomorphism of the underlying abelian groups $\psi^n : A^n_* \to A^n_*$ by

$$\psi^n(\alpha) := \sum_{k=1}^{[(n+1)/2]} \sum_{(I,J)\in\mathcal{L}^n_k} (-1)^{|I|+|J|+1} d^I s^J \alpha,$$

where $\mathcal{L}^n_k = \{(I,J) \in \mathbb{N}^k \times \mathbb{N}^k \mid j_{k-t} > i_t > j_{k+1-t} \text{ for all } 1 \leq t \leq k\}$ (and we let $j_0 := n$).

It then follows from the cosimplicial identities (1.3) that $s^j \psi^n(\alpha) = s^j \alpha$ for $0 \leq j \leq n-1$. Since the definition of ψ^n depends only on $(s^0\alpha, \ldots, s^{n-1}\alpha) \in L^n A^\bullet_*$, we see that $\varsigma^n : A^n_* \to L^n A^\bullet_*$ – and in fact even $\bar{\varsigma}^n := \varsigma^n|_{\bigcup_{i=1}^n \operatorname{Im}(d^i)}$ – are epimorphisms.

Moreover, given $\alpha = \sum_{i=1}^n d^k a_k \in \bigcup_{i=1}^n \operatorname{Im}(d^i)$ such that $s^j\alpha = 0$ for $0 \leq j \leq n-1$, the identities (1.3) imply that

$$\alpha = \sum_{p=1}^{n} \sum_{q=0}^{p-1} d^p d^q \Big(\sum_{i=q}^{p-2} (-1)^{p-i-1} s^i a_q + \sum_{j=0}^{q-1} (-1)^{p-j} s^j (a_{q+1} + a_p)\Big) \in \bigcup_{i=1}^{n-1} \operatorname{Im}(d^i),$$

so by induction on n we see $\alpha = 0$ – and thus $\bar{\varsigma}^n$ is one-to-one, so in fact it is an isomorphism of unstable coalgebras. We thus have $A^\bullet_* = G(R\langle X_* \amalg Y_*\rangle)$ for some $X_* \in \operatorname{gr} \mathcal{S}et$, where $\amalg$ denotes the disjoint union, and we may assume $\bigcup_{i=1}^n \operatorname{Im}(d^i) = G(R\langle Y_*\rangle)$.

Finally, for each $x \in X_*$, set $x_0 := x$, and define x_k inductively by $x_{k+1} := x_k - d^{k+1}s^k x_k$ $(0 \leq k < n)$. We see that $s^j x_k = 0$ for $0 \leq j < k$, so $\hat{x} := x_n$

has $s^j\hat{x}=0$ for all j, i.e., $\varsigma^n(\hat{x})=0$. Moreover, $x-\hat{x}\in G(R\langle Y_*\rangle)$, so if we set $\bar{A}^n_* := G(R\langle\{\hat{x}\}_{x\in X_*}\rangle)$, we get the required CW basis. □

3.13. **The resolution model category** $c\mathcal{S}_*$. In [DKS1, §5.8], Dwyer, Kan and Stover define a model category structure on the category $c\mathcal{S}_*$ of cosimplicial spaces (for each choice of R), which they called the "E^2-model category", though the term *resolution model category* (cf. [GH]) may perhaps be more appropriate:

A map $f:\mathbf{X}^\bullet\to\mathbf{Y}^\bullet$ of cosimplicial spaces is

- (i) a *weak equivalence* if $\pi^n H_*(f;R)$ is an isomorphism (of graded R-modules) for each $n\geq 0$;
- (ii) a *cofree map* if for each $n\geq 0$ there is a fibrant R-GEM $K^n\in\mathcal{S}_*$ and a map $X^n\to K^n$ which induces a trivial fibration $X^n\to(X^n\times_{L^n\mathbf{X}^\bullet}L^n\mathbf{Y}^\bullet)\times K^n$;
- (iii) a *fibration* if it is a retract of a cofree map;
- (iv) a *cofibration* if $f^n\bot\xi^n: X^n\amalg_{M^n\mathbf{Y}^\bullet}M^n\mathbf{X}^\bullet\to Y^n$ (cf. §3.1) is a cofibration for each $n\geq 0$, and $\pi^n f$ is a levelwise cofibration (i.e., monomorphism) of graded R-modules.

The advantage of such a model category is that it provides a way to define a *cosimplical resolution* of a simplicial set (or topological space) $\mathbf{X}\in\mathcal{S}_*$, as a fibrant replacement for the constant cosimplicial space $c(\mathbf{X})^\bullet$ – where a special case (in fact, the motivating example) is the R-resolution presented in [BK1, I, §4.1]. See also [Bl5, §2] for a slight generalization of the original construction.

4. The fundamental group and cohomology

As noted in §3.5 above, the category $c\mathcal{CA}_R$ of cosimplicial unstable R-coalgebras has a model category structure in which the objects $\mathcal{E}M^n(R,k)_{\mathcal{CA}}$ $(n\geq 0,\ k\geq 1)$ play the role of cosimplicial Eilenberg-Mac Lane objects, in the sense of representing the cohomotopy groups. Thus, if we take homotopy classes of maps between (products of) these Eilenberg-Mac Lane objects as the primary cohomotopy operations (see [Wh, V, §8]), we can endow the cohomotopy groups $\pi^*A^\bullet_*=(\pi^iA^\bullet_*)_{i=0}^\infty$ of any $A^\bullet_*\in c\mathcal{CA}_R$ with an additional structure: that of a $\mathcal{CA}_R$-*coalgebra*, that is, a graded object over $\mathcal{CA}_R$ (coabelian, in positive dimensions), endowed with an action of these primary cohomotopy operations. This concept is dual to that of a $\mathcal{K}_R$-Π-algebra, in the terminology of [BS, §3.2]. By definition, this structure is a homotopy invariant of $A^\bullet_*$.

4.1. **The coaction of the fundamental group.** In our case we shall only need the very simplest part of this structure – namely, the coaction of the fundamental group $\pi^0A^\bullet_*$ on each of the higher cohomotopy groups $\pi^nA^\bullet_*$. This is described in terms of homotopy classes of maps $\mathcal{E}M^n(R,p)_{\mathcal{CA}}\to\mathcal{E}M^0(R,k)_{\mathcal{CA}}\times\mathcal{E}M^n(R,\ell)_{\mathcal{CA}}$; but since these are cosimplicial coalgebras of finite type, it is may be easier to follow the dual description, in $s\mathcal{K}_R$, of homotopy classes of maps between simplicial suspensions $\mathcal{E}M^n(R,k)_{\mathcal{K}_R}\in s\mathcal{K}_R$ of the free unstable algebras $\mathcal{E}M(R,k)_{\mathcal{K}_R}:=H^*(\mathbf{K}(R,k);R)\in\mathcal{K}_R$.

First, if $Y^*_\bullet$ is any simplicial graded-commutative algebra over a field R, we can define the "$\star$-action" of any $a\in Y^k_0$ on $b\in Y^\ell_n$ by $a\star b:=\hat{a}\cdot b\in Y^{k+\ell}_n$, where $\hat{a}:=s_{n-1}\cdots s_0a\in Y^k_n$. If we define the *n-cycles* and *n-chains* algebras $Z_nY^*_\bullet\subset C_nY^*_\bullet$ dually to $C^n\mathbf{X}^\bullet\twoheadrightarrow B^n\mathbf{X}^\bullet$ of §3.3 (see [M, §17]), then since $\star$ commutes with the face maps, it defines a (bilinear) "action" of Y_0^* on $C_n Y_\bullet^*$ and $Z_n Y_\bullet^*$ and thus an action of $\pi_0 Y_\bullet^*$ on $\pi_n Y_\bullet^*$ for any $n \geq 1$.

Now let $X_\bullet^*$ denote the simplicial unstable algebra $\mathcal{E}M^0(R,k)_{\mathcal{K}_R} \times \mathcal{E}M^n(R,\ell)_{\mathcal{K}_R} \in s\mathcal{K}_R$, for $k,\ell,n>0$. Note that we have a short exact sequence of unstable algebras:

$$0 \to Z_n X_\bullet^* \hookrightarrow H^*(\mathbf{K}(R,k) \times \mathbf{K}(R,\ell); R) \xrightarrow{\mathbf{d}_0^\#} H^*(\mathbf{K}(R,k); R) \to 0. \tag{4.2}$$

Evidently $Z_n X_\bullet^*$ consists of elements of the form $\sum_i a_i \star b_i$ where $a_i \in X_0^*$ and $0 \neq b_i \in X_n^*$. However, if $b = b' \cdot b''$ is non-trivially decomposable in X_n^*, then $\zeta := (s_n \hat{a}) \cdot (s_0 b' \cdot s_0 b'' - s_0 b' \cdot s_1 b'') \in X_{n+1}^*$ satisfies $d_0 \zeta = \hat{a} \cdot b' \cdot b''$ and $d_j \zeta = 0$ for $1 \leq j \leq n+1$, so that $\pi_n X_\bullet^*$ is just the free $\pi_0 X_\bullet^*$-module generated by $(\mathcal{E}M(R,\ell)_{\mathcal{K}_R})_{ab}$ (see §2.6, and compare [BS, §5]), where $\pi_0 X_\bullet^* \cong \mathcal{E}M(R,k)_{\mathcal{K}_R} = H^*(\mathbf{K}(R,k); R) \in \mathcal{K}_R$.

For the dual category of cosimplicial coalgebras, we need the following

4.3. **Definition.** For a given coalgebra $J_* \in \mathcal{CA}_R$, an unstable coalgebra C_* equipped with bilinear "co-operations" $C_* \to J_* \otimes C_*$, (satisfying the universal identitites for the dual of "action" $\star : \pi_0 Y_\bullet^* \otimes \pi_n Y_\bullet^* \to \pi_n Y_\bullet^*$ defined above) will be called a *J_*-coalgebra.* The category of such will be denoted by $\mathcal{CA}_{J_*}$.

On the other hand, a *coabelian* unstable coalgebra K_* equipped with a (left) coaction map of coalgebras $\psi : K_* \to J_* \otimes K_*$, satisfying the usual identities (see [Sw, §2.1]) is called a *J_*-comodule.* We denote the category of such by J_*-$coMod$. This is an abelian category.

We say that a J_*-comodule K_* is *cofree*, with *basis* $V_* \in grR\text{-}Mod$, if $K_* = V_* \otimes_R J_*$ (as graded R-modules), and the coaction $\psi : V_* \otimes_R J_* \to (V_* \otimes_R J_*) \otimes_R J_*$ is induced by the compultiplication $J_* \to J_* \otimes_R J_*$. Similarly, a map of cofree comodules is called *cofree* if it is induced by a map of the bases.

The above discussion for the case of unstable algebras may now be summarized in:

4.4. **Proposition.** *Any cosimplicial unstable coalgebra $A_*^\bullet \in c\mathcal{CA}_R$ has a coaction of $\pi^0 A_*^\bullet \in \mathcal{CA}_R$ on $\pi^n A_*^\bullet \in (\mathcal{CA}_R)_{co\text{-}ab}$, induced by an A_*^0-coalgebra structure on A_*^n. This coaction of A_*^0 commutes with the $\mathcal{M}od$-$\mathcal{A}_p$-structure (i.e., the action of the Steenrod algebra), and respects the coface maps, and thus $\mathbf{d}^0 : B^n A_*^\bullet \to C^{n+1} A_*^\bullet$ is a map of A_*^0-coalgebras for each $n \geq 0$.*

Note that the $\pi^0 A_*^\bullet$-comodule structure on the coabelian coalgebra $\pi^n A_*^\bullet$ is just part of a bigraded cocommutative coalgebra structure on $C_*^* := \pi^* A_*^\bullet$, in which the diagonal respects the unstable R-operations. This is the cosimplicial analogue of the $\mathcal{C}$-Π-algebra-structure on the homotopy objects of a simplicial object over a category of universal algebras $\mathcal{C}$ (see [BS, §3.2]).

4.5. **Quillen cohomology.** Given an unstable coalgebra $J_* \in \mathcal{CA}_R$, and $K_* \in J_*$-$coMod$ – that is, a coabelian object $K_* \in (\mathcal{CA}_R)_{co\text{-}ab}$, with a coaction of J_* – one may define its Quillen cohomology by dualizing [Q3, §2], as follows:

Choose some cofree cosimplicial resolution $A_*^\bullet \in c\mathcal{CA}_R$ of J_*, and note that the J_*-comodule K_* is in particular an A_*^0-comodule, and each A_*^n is a coalgebra over A_*^0. Moreover, by the usual universal property we have a natural equivalence $\mathrm{Hom}_{\mathcal{CA}_{A_*^0}}(K_*, A_*^\bullet) \cong \mathrm{Hom}_{A_*^0\text{-}coMod}(K_*, (A_*^\bullet)_{co\text{-}ab'})$, where $M_{co\text{-}ab'}$ denotes the coabelianization of the A^0_*-coalgebra M (see §2.6). Thus $C^\bullet := \mathrm{Hom}_{\mathcal{CA}_{A^0_*}}(K_*, A^\bullet_*)$ is a cosimplicial abelian group, and $\pi^n C^\bullet$ is called the n-th *Quillen cohomology group* of J_* *with coefficients in* K_*, and denoted by $H^n(J_*; K_*)$ (compare [L, §3]). If J_*, K_* and $A^\bullet_*$ are of finite type, this is the vector space dual of the usual Quillen cohomology of $(J_*)^\star \in \mathcal{K}_R$ (see [Bo2, §8], and compare [Bl5, §4]).

4.6. *Remark.* As for any abelian category, given a cosimplicial object $(A^\bullet_*)_{co\text{-}ab'}$ over A^0_*-$coMod$, in each dimension $n \geq 1$ there is a direct product decomposition $(A^n_*)_{co\text{-}ab'} = C^n((A^\bullet_*)_{co\text{-}ab'}) \oplus L^n((A^\bullet_*)_{co\text{-}ab'})$ (compare [M, Cor. 22.2]). If we choose a CW basis $(\bar{A}^n_*)^\infty_{n=0}$ for $A^\bullet_*$ (§3.9), we have: $C^n(A^\bullet_*)_{co\text{-}ab'} \cong (\bar{A}^n_*)_{co\text{-}ab} \otimes A^0_*$ as unstable modules (where $(\bar{A}^n_*)_{co\text{-}ab}$ is the usual coabelianization of §2.6). Moreover, each $(A^n_*)_{co\text{-}ab'}$ is a cofree A^0_*-comodule (with a basis which may be described explicitly in terms of the CW-basis for $A^\bullet_*$ – see [Bl2, (6.3)]), and the coface maps are cofree (§4.3), so we have $C^n((A^\bullet_*)_{co\text{-}ab'}) = (C^n A^\bullet_*)_{co\text{-}ab'}$. We may therefore use the cochain complex

$$\cdots \to \mathrm{Hom}_{A^0_*\text{-}coMod}(K_*, (C^n A^\bullet_*)_{co\text{-}ab'}) \xrightarrow{\delta^n} \mathrm{Hom}_{A^0_*\text{-}coMod}(K_*, (C^{n+1} A^\bullet_*)_{co\text{-}ab'}) \to \cdots$$

(where δ^n is induced by $\mathbf{d}^0_{A^n} \circ q^n : C^n A^\bullet_* \to C^{n+1} A^\bullet_*$) to calculate $H^*(J_*; K_*)$ (compare [BK1, X, §7.1]).

4.7. **An alternative description.** Quillen's original description of the cohomology of a (simplicial) algebra included several variant approaches (cf. [Q3, §3], and by dualizing one of these, Bousfield obtained an alternative description of the cohomology of a coalgebra, as follows:

For any field R, given an unstable coalgebra $J_* \in \mathcal{CA}_R$ and a J_*-comodule $K_* \in J_*$-$coMod$, with coaction $\psi : K_* \to J_* \otimes K_*$, and a map of coalgebras $\delta : J_* \to L_*$, define a *derivation* $\varphi : K_* \to L_*$ to be a $\mathcal{M}od$-$\mathcal{A}_p$-morphism such that $\Delta_{J_*} \circ \varphi = \delta \otimes \varphi + \tau \circ (\varphi \otimes \delta) \circ \psi$ (see §2.1). Write $\mathrm{Der}_{\mathcal{CA}_R}(K_*, L_*)$ for the R-vector space of all such derivations.

For every comodule $K_* \in J_*$-$coMod$, we can think of $J_* \oplus K_*$ as a coalgebra under J_*, with diagonal $\Delta_{J_* \oplus K_*}$ defined by

$$\Delta_{J_*} + \psi + \tau \circ \psi + 0 : J_* \oplus K_* \to (J_* \otimes J_*) \oplus (J_* \otimes K_*) \oplus (K_* \otimes J_*) \oplus (K_* \otimes K_*).$$

Thus, given a map $\delta : J_* \to L_*$, we have a natural identification

$$\mathrm{Hom}_{\mathcal{CA}_R \backslash J_*}(J_* \oplus K_*, L_*) \cong \mathrm{Der}_{\mathcal{CA}_R}(K_*, L_*).$$

In fact, the functor $K_* \mapsto J_* \oplus K_*$ is left adjoint to the coabelianization functor $(\,)_{co\text{-}ab} : \mathcal{CA}_R \backslash J_* \to J_*$-$coMod$ (compare §2.6), and it induces an equivalence of categories between J_*-$coMod$ and $(\mathcal{CA}_R \backslash J_*)_{co\text{-}ab}$. Moreover, as in [Q3, §4], one has an explicit description of the functor $(\,)_{co\text{-}ab}$ in terms of a cotensor product of J_* with ΩL_* (the coalgebraic analogue of the usual Kähler module of differentials). See [Bo2, §8.5] and [Sc1, §7.7-8] for more details on this approach.

Now if $J_* \to A^\bullet_*$ is a resolution, we can show that there is a natural map

$$\mathrm{Hom}_{\mathcal{CA}_{A^0_*}}(K_*, C^n A^\bullet_*) \to \mathrm{Hom}_{\mathcal{CA}_R \backslash J_*}(J_* \oplus K_*, A^n_*),$$

which induces an isomorphism in cohomology, so Remark 4.6 above implies that the Quillen cohomology groups $H^n(J_*; K_*)$ coincide with the derived functors of $\mathrm{Der}_{\mathcal{CA}_R}(K_*, -)$ applied to J_*, which were considered by Bousfield (who showed in

[Bo2, §9] that the groups $\operatorname{Der}^{s,t}(J_*, L_*) := H^s(J_*; \tilde{H}_*(\mathbf{S}^t; \mathbb{F}_p) \otimes L_*)$ serve as the E_2-term of a certain unstable Adams spectral sequence).

To show this, use the fact that a morphism in either Hom-set must take values in a coabelian unstable coalgebra, and that the unique iterated coface map $\delta : J_* \to A^n_*$ vanishes when projected to $C^n A^\bullet_*$. We omit the details, since we shall not require this result in what follows. However, it may be observed that in the Massey-Peterson case $J_* = U(M_*)$, Bousfield's approach allows us to identify $H^n(J_*; K_*)$ with the usual $\operatorname{Ext}^n_{\mathcal{M}od\text{-}\mathcal{A}_p}(K_*, M_*,)$, so we can recover Harper's results in [Ha, Prop. 4.2] as a special case of Theorem 6.3 below.

It is possible that one could prove the results of the following sections, using Bousfield's description of cohomology as the derived functors of derivations, and dualizing Quillen's identification of these derived functors (for algebras) with suitable groups of extensions (see [Q3, §3]). However, there may be computational advantages to the explicit approach we have taken here.

5. Realizing resolutions

The key to our approach to the realization question for an unstable coalgebra K_* lies in the realization of a suitable cofree resolution of K_* – by analogy with the method used in [Bl5] for Π-algebras. In what follows $R = \mathbb{Q}$ or $\mathbb{F}_p$.

5.1. **Trying to realize a resolution.** Given an unstable coalgebra $K_* \in \mathcal{CA}_R$, choose some cosimplicial resolution $K_* \to A^\bullet_*$, with CW basis $\{\bar{A}^n_*\}_{n=0}^\infty$, as in §3.11. We would like to realize this algebraic resolution at the space level, i.e., find a cofibrant cosimplicial space $\mathbf{Q}^\bullet \in c\mathcal{S}_*$, with a CW basis $\{\bar{\mathbf{Q}}^n\}_{n=0}^\infty$, such that $H_*(\bar{\mathbf{Q}}^n; R) \cong \bar{A}^n_*$ for all $n \geq 0$, and the attaching maps $\bar{d}^0_{Q^n} : \bar{\mathbf{Q}}^n \to \bar{\mathbf{Q}}^{n+1}$ realize those for $A^\bullet_*$, so that $H_*(\mathbf{Q}^\bullet; R) \cong A^\bullet_*$.

We attempt to construct such a $\mathbf{Q}^\bullet$ (with its CW basis) by induction on the cosimplicial dimension:

The first two steps are always possible: because $\bar{A}^0_* \in \mathcal{CA}_R$ is cofree by assumption (§3.9), we can find a map $\bar{d}^0_Q : \bar{\mathbf{Q}}^0 \to \bar{\mathbf{Q}}^1$ in $\mathcal{S}_*$ such that $(\bar{d}^0_Q)_\# : H_*(\bar{\mathbf{Q}}^0; R) \to H_*(\bar{\mathbf{Q}}^1; R)$ is $\bar{d}^0_A : \bar{A}^0_* \to \bar{A}^1_*$ (§2.9). We then set $\mathbf{Q}^0 := \bar{\mathbf{Q}}^0$ and $\tilde{\mathbf{Q}}^1 := \bar{\mathbf{Q}}^1 \times L^1\mathbf{Q}^\bullet = \bar{\mathbf{Q}}^1 \times \mathbf{Q}^0$, with $\tilde{d}^0 := \bar{d}^0 \top \operatorname{Id}$ and $\tilde{d}^1 := 0 \top \operatorname{Id}$. In order to end up with a cofibrant cosimplicial space (§3.1), we now change the resulting $\tilde{\xi}^n : M^1\mathbf{Q}^\bullet \to \tilde{\mathbf{Q}}^1$ into a cofibration: $\xi^n : M^1\mathbf{Q}^\bullet \to \mathbf{Q}^1$. It will be convenient to denote $d^0 \circ p^1 : \mathbf{Q}^0 \to C^1\mathbf{Q}^\bullet$ by $\mathbf{d}^0 = \mathbf{d}^0_{Q^0} : B^0\mathbf{Q}^\bullet \to C^1\mathbf{Q}^\bullet$, to conform with the notation of Figure 1.

If we let $B^1\mathbf{Q}^\bullet$ denote the cofiber of $\mathbf{d}^0_{Q^0}$, from the long exact sequence in homology:

$$\dots H_{i+1}(B^1\mathbf{Q}^\bullet; R) \xrightarrow{\partial_1} H_i(B^0\mathbf{Q}^\bullet; R) \xrightarrow{\mathbf{d}^0} H_i(C^1\mathbf{Q}^\bullet; R) \xrightarrow{(q^1_Q)_\#} H_i(B^1\mathbf{Q}^\bullet; R) \dots$$

and Proposition 3.6 we obtain a short exact sequence of unstable coalgebras:

$$0 \to B^1(A^\bullet_*) \xrightarrow{i} H_*(B^1\mathbf{Q}^\bullet; R) \xrightarrow{\psi} \Sigma K_* \to 0,$$

where ΣK_* is the graded R-module, shifted one degree up, with the trivial coalgebra structure (and the suspended action of the Steenrod algebra, if $R = \mathbb{F}_p$).

Now assume $\mathbf{Q}^\bullet$ as required has been constructed through cosimplicial dimension n, so we have an n-cosimplicial object which (by a slight abuse of notation) we denote by $\operatorname{tr}_n \mathbf{Q}^\bullet \in c^{\langle n\rangle}\mathcal{S}_*$ (cf. §1.5), with $H_*(\operatorname{tr}_n \mathbf{Q}^\bullet; R) \cong \operatorname{tr}_n A^\bullet_*$.

By considering the cosimplicial chain complex corresponding to $R\langle \mathbf{Q}^\bullet\rangle$, one can verify (as in the proof of Proposition 3.6) that there always exists a factorization of $(q^n_Q)_\#$ as follows:

$$\begin{array}{ccccc} C^n A^\bullet_* & \xrightarrow{q^n_A} & B^n A^\bullet_* & \xrightarrow{\mathbf{d}^0_{A^n}} & C^{n+1} A^\bullet_* \\ \cong\downarrow & & \vdots\ \bar{q}^n & & \downarrow\cong \\ H_*(C^n\mathbf{Q}^\bullet; R) & \xrightarrow{(q^n_Q)_\#} & H_*(B^n\mathbf{Q}^\bullet; R) & \xrightarrow{(\mathbf{d}^0_{Q^n})_\#} & H_*(C^{n+1}\mathbf{Q}^\bullet; R) \end{array}$$

FIGURE 2

and since $A^\bullet_*$ is a resolution, $\mathbf{d}^0_{A^n}$, and thus $\bar{q}^n$, must be monic. Moreover, for any $a \in H_i(B^{n-1}\mathbf{Q}^\bullet; R)$ we have $(q^n_Q)_\#((\mathbf{d}^0_{Q^{n-1}})_\#(a)) = 0$; thus $(\mathbf{d}^0_{Q^{n-1}})_\#(a) \in H_i(C^n\mathbf{Q}^\bullet; R) = C^n A^\bullet_*$ is a cocyle for the resolution $A^\bullet_*$, so it must be in $\operatorname{Im}(\mathbf{d}^0_{A^{n-1}})$. Thus we must have

$$\operatorname{Im}(q^n_Q)_\# + \operatorname{Ker}(\mathbf{d}^0_{Q^n})_\# = H_*(B^n\mathbf{Q}^\bullet; R). \tag{5.2}$$

We therefore assume by induction that we have a short exact sequence of unstable coalgebras:

$$0 \to B^n A^\bullet_* \xrightarrow{\bar{q}^n} H_*(B^n\mathbf{Q}^\bullet; R) \xrightarrow{\psi^n} \operatorname{Coker}(\bar{q}^n) \to 0, \tag{5.3}$$

where $\operatorname{Coker}(\bar{q}^n) \cong \Sigma^n K_*$, and in fact $\psi^n|_{\operatorname{Im}(\partial_{n+1})}$ is an isomorphism onto $\operatorname{Coker}(\bar{q}^n)$, with ∂_{n+1} again the connecting homomorphism in the long exact sequence

$$\dots H_{i+1}B^{n+1}\mathbf{Q}^\bullet \xrightarrow{\partial_{n+1}} H_iB^n\mathbf{Q}^\bullet \xrightarrow{\mathbf{d}^0} H_iC^n\mathbf{Q}^\bullet \xrightarrow{(q^1)_\#} H_iB^n\mathbf{Q}^\bullet \xrightarrow{\partial_{n+1}} \dots \tag{5.4}$$

Since $\operatorname{Im}(q^n_Q)_\# = \operatorname{Im}\bar{q}^n$ in Figure 2, in fact we have a direct sum of R-modules in (5.2), and this is by definition a semi-split extension of coalgebras. This implies that

$$\partial_{n-1}|_{\operatorname{Im}(\partial_n)} \text{ is one-to-one, and surjects onto } \operatorname{Im}(\partial_{n-1}). \tag{5.5}$$

5.6. *Aside.* Observe that if we contine our construction "naively" by choosing some GEM $\bar{\mathbf{Q}}^{n+1} \in \mathcal{S}_*$ with an attaching map $\bar{d}^0_{n+1} : \mathbf{Q}^{n+1} \to \bar{\mathbf{Q}}^{n+1}$ which induces a monomorphism $H_*(B^n\mathbf{Q}^\bullet; R) \hookrightarrow H_*(\bar{\mathbf{Q}}^{n+1}; R)$ in homology, we can easily continue this process to obtain a cosimplicial space $\mathbf{Y}^\bullet$, such that $H_*(\mathbf{Y}^\bullet; R)$ is a free cosimplicial coalgebra satisfying:

$$\pi^i H_*(\mathbf{Y}^\bullet; R) \cong \begin{cases} K_* & \text{for } i = 0, \\ \Sigma^n K_* & \text{for } i = n+1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.7}$$

Such a $\mathbf{Y}^\bullet$ should be thought of as the $(n-1)$-*st Postnikov section* for the resolution $\mathbf{Q}^\bullet$. We denote a cofibrant version of it by $P^{n-1}\mathbf{Q}^\bullet$, and observe that it is unique up to homotopy equivalence (in the model category structure of §3.13). This provides a convenient homotopy-invariant version of the n-coskeleton of a cosimplicial space. See [Bl5, §3.4] for an explanation of the indexing.

In [**?**], we present an alternative approach to the dual problem of realizing (simplicial) Π-algebras, via Postnikov systems (including k-invariants) for simplicial Π-algebras and simplicial spaces. This was in fact the original program of [DKS1, DKS2]. However, because there is no satisfactory homotopy theory for cosimplicial *sets*, an analogous approach here would require developing additional machinery not presently available.

5.8. **Continuing the construction.** If we can extend our n-truncated object $\operatorname{tr}_n \mathbf{Q}^\bullet$ one more dimension, we will have a principal face map $\mathbf{d}^0_n : B^n\mathbf{Q}^\bullet \to C^{n+1}\mathbf{Q}^\bullet$, which induces a map $\lambda = (\mathbf{d}^0_n)_\# : H_*(B^n\mathbf{Q}^\bullet; R) \to H_*(C^{n+1}\mathbf{Q}^\bullet; R) \cong C^{n+1}A^\bullet_*$ (by Proposition 3.6). It turns out that such a λ is essentially all we need in order to proceed.

First note that since $\bar{A}^{n+1}_*$ is a cofree unstable coalgebra by assumption, and $\bar{q}^n : B^nA^\bullet_* \to H_*(B^n\mathbf{Q}^\bullet; R)$ in Figure 2 above is monic, $\bar{\mathbf{d}}^0_{A^n} : B^nA^\bullet_* \to \bar{A}^{n+1}_*$ extends to a map $\lambda : H_*(B^n\mathbf{Q}^\bullet; R) \to \bar{A}^{n+1}_*$ as follows:

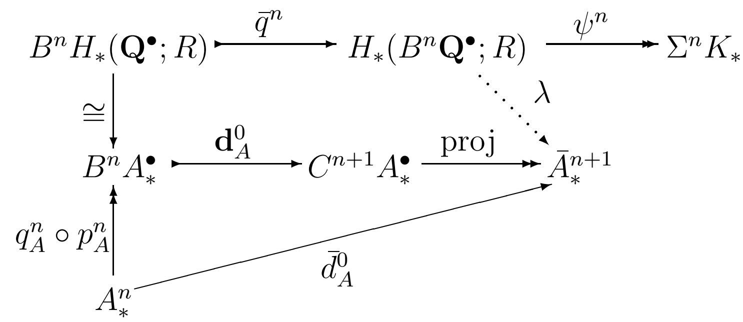

FIGURE 3

We can realize λ by a map $\ell : B^n\mathbf{Q}^\bullet \to \bar{\mathbf{Q}}^{n+1}$, for a suitable GEM $\bar{\mathbf{Q}}^{n+1}$, and thus a map $\bar{d}^0 : \mathbf{Q}^n \to \bar{\mathbf{Q}}^{n+1}$ (see Figure 1), which in turn determines an extension of $\operatorname{tr}_n \mathbf{Q}^\bullet$ to $\operatorname{tr}_{n+1} \mathbf{Q}^\bullet$; we may modify this to be cofibrant. Moreover, from the exactness of the bottom row, by (5.2), we have a unique lifting $\mu : \operatorname{Coker}(\bar{q}^n) \to B^{n+1}A^\bullet_*$ as follows:

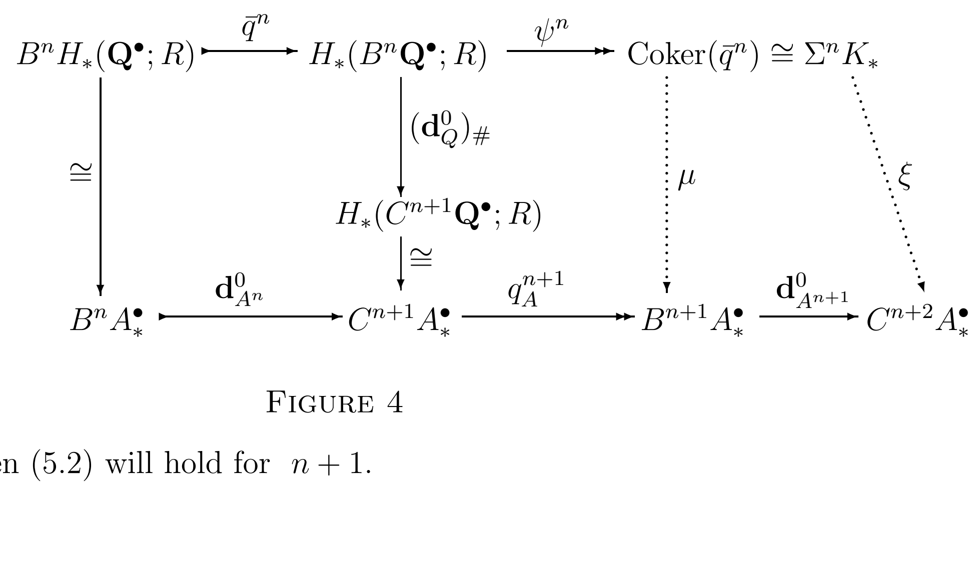

FIGURE 4

and if $\mu = 0$, then (5.2) will hold for $n + 1$.

5.9. **Definition.** The cohomology class $\chi_n \in H^{n+2}(K_*;\Sigma^n K_*)$ represented by the cocycle $\xi := \mathbf{d}^0_{A^{n+1}} \circ \mu \in \mathrm{Hom}_{\mathcal{CA}_{A^0_*}}(\Sigma^n K_*, A^\bullet_*)$ (see §4.6) is called the *characteristic class of the extension* (5.3) (compare [M1, IV, §5] and [Bl5, §4.5]).

5.10. **Theorem.** *The cohomology class $\chi_n \in H^{n+2}(K_*;\Sigma^n K_*)$ is independent of the choice of lifting λ, and $\chi_n = 0$ if and only if one can extend $P^{n-1}\mathbf{Q}^\bullet$ to an n-th Postnikov approximation $P^n\mathbf{Q}^\bullet$ of a resolution of K_*.*

Proof. Assume that we want to replace λ by a different lifting $\lambda' : H_*(B^n\mathbf{Q}^\bullet; R) \to \bar{A}^{n+1}_*$, and choose maps $\ell, \ell' : B^n\mathbf{Q}^\bullet \to \bar{\mathbf{Q}}^{n+1}$ realizing λ, λ' respectively; their respective extensions to d^0 and $(d^0)' : \mathbf{Q}^n \to \mathbf{Q}^{n+1}$ agree on $L^{n+1}\mathbf{Q}^\bullet$. We correspondingly have $\mu' : \Sigma^n K_* \to B^{n+1}A^\bullet_*$ and $\xi' := \mathbf{d}^0_{A^{n+1}} \circ \mu'$ in Figure 4.

Since $\mathbf{Q}^{n+1}$ can be any fibrant GEM realizing A^{n+1}, we may assume it is a simplicial R-module, and thus $\mathrm{Hom}_{\mathcal{S}_*}(\mathbf{Q}^n, \mathbf{Q}^{n+1})$ has a natural R-module structure. Set $h := ((d^0)' - d^0) : \mathbf{Q}^n \to \mathbf{Q}^{n+1}$. Then h induces a map $\eta : H_*(B^n\mathbf{Q}^\bullet; R) \to C^{n+1}A^\bullet_*$ whose projection onto $\bar{A}^{n+1}_*$ is $\lambda' - \lambda$. Moreover, because d^0 and $(d^0)'$ agree with $\mathbf{d}^0_{A^n}$ when pulled back to $H_*(C^n\mathbf{Q}^\bullet; R)$, we have $\eta \circ \bar{q}^n = 0$, and thus η factors through $\zeta : \Sigma^n K_* \to C^{n+1}A^\bullet_*$, and this is a map of A^0_*-coalgebras because $\Sigma^n K_*$ is a coabelian A^0_*-comodule (actually, a K_*-comodule), and ζ is induced by group operations from the A^0_*-coalgebra maps $\mathbf{d}^0$ and $(\mathbf{d}^0)'$. See Figure 5 below.

Moreover, in the abelian group structure on $\mathrm{Hom}_{K_*\text{-}coMod}(\Sigma^n K_*, -)$ we have $\xi' - \xi = \mathbf{d}^0_{A^{n+1}} \circ (\mu' - \mu) = \delta^{n+1}(\zeta)$ (see §4.6), so this is a coboundary, which proves independence of the choice of λ.

Now assume that there exists $\mathbf{Y}^\bullet \simeq P^n\mathbf{Q}^\bullet$ (§5.6) with $\mathrm{tr}_n\, \mathbf{Y}^\bullet \cong \mathrm{tr}_n\, \mathbf{Q}^\bullet$. By the discussion in §5.1, we know that (5.2) is a direct sum for $n+1$, and since $\bar{A}^{n+1}_*$ is cofree, we can choose $\lambda : H_*(B^n\mathbf{Q}^\bullet; R) \to \bar{A}^{n+1}_*$ in Figure 3 to extend $\mathbf{d}^0_A$ by zero, so $\mu = 0$ in Figure 4, and thus $\xi = 0$.

Conversely, if $\chi_n = 0$, we can represent it by a coboundary $\xi = \mathbf{d}^0_{A^{n+2}} \circ \vartheta$ for some A^0_*-coalgebra map $\vartheta : K_* \to C^{n+1}A^\bullet_*$, and thus get $\mathrm{proj}_{\bar{A}^{n+1}_*} \circ \vartheta \circ \psi^n : H_*(B^n\mathbf{Q}^\bullet; R) \to \bar{A}^{n+1}_*$, for $\mathrm{proj}_{\bar{A}^{n+1}_*} : A^{n+1}_* \to \bar{A}^{n+1}_*$ the projection. If we set $\lambda' := \lambda - \mathrm{proj}_{\bar{A}^{n+1}_*} \circ \vartheta \circ \psi^n$ (we can subtract maps, because $\bar{A}^{n+1}_*$ is a graded R-module), we have $\mathrm{Im}(q^n_Q)_\# + \mathrm{Ker}\,\lambda' = H_*(B^n\mathbf{Q}^\bullet; R)$. We can therefore choose $\bar{d}^0_{Q^{n+1}} : B^n\mathbf{Q}^\bullet \to \bar{\mathbf{Q}}^{n+1}$ realizing λ', and then $\mu' = 0$, so that $\mathrm{tr}_{n+1}\, \mathbf{Q}^\bullet$ so constructed yields $P^n\mathbf{Q}^\bullet$, as required. □

5.11. *Notation.* If we wish to emphasize the dependence on the choice of λ, we shall write $P^n\mathbf{Q}^\bullet[\lambda]$ for the extension of $P^{n-1}\mathbf{Q}^\bullet$ so constructed, and write $\chi_{n+1}(\lambda) \in H^{n+3}(K_*;\Sigma^{n+1}K_*)$ for the next cohomology class (which *does* depend on λ, in principle).

5.12. *Remark.* Note that if $\chi_n = 0$, the choice of λ determines the A^0_*-comodule structure on $\Sigma^{n+1}K_*$ via (5.3) for $n+1$. Moreover, for each $n \geq 1$, the resulting coaction $\psi_n : \Sigma^n K_* \to K_* \otimes \Sigma^n K_*$ in fact agrees with the obvious K_*-comodule structure, defined via the original comultiplication $\Delta : K_* \to K_* \otimes K_*$: that is, if $\Delta(a) = \sum_i a'_i \otimes a''_i$, and $\sigma_n : K_* \to \Sigma^n K_*$ is the re-indexing isomorphism (in $\mathcal{M}od$-$\mathcal{A}_p$), then $\psi_n(\sigma_n(a)) = \sum_i a'_i \otimes \sigma_n(a''_i)$. This follows from the description in §4.1, and the fact that the exact sequences (5.4) (and thus also (5.3)) respect the A^0_*-coalgebra structure (Proposition 4.4).

5.13. **Definition.** Note also that by standard homotopical algebra arguments the elements $\chi_n \in H^{n+2}(K_*; \Sigma^n K_*)$ do not depend on the choice of resolution $K_* \to A_*^\bullet$. If for some (and thus any) cosimplicial cofree resolution $K_* \to A_*^\bullet$, there are successive choices of liftings $(\lambda_n)_{n=0}^\infty$ in Figure 3 such that $\chi_{n+1}(\lambda_n) = 0$ for $n = 0, 1, \ldots,$ we say that we have a *coherently vanishing* sequence of characteristic classes.

Thus we may encapsulate our results so far in

5.14. **Corollary.** *Any cofree cosimplicial resolution* $A_*^\bullet$ *of an unstable coalgebra* $K_* \in \mathcal{C}A_R$ *is realizable by a cosimplicial space* $\mathbf{Q}^\bullet \in c\mathcal{S}_*$ *(with* $A_*^\bullet \cong H_*(\mathbf{Q}^\bullet; R)$*) if and only if* K_* *has a coherently vanishing sequence of characteristic classes.*

6. Realizing coalgebras

We now apply Theorem 5.10, on the realization of cosimplicial resolutions of coalgebras, to the original question, namely, that of realizing a given abstract coalgebra as the cohomology of a space. It turns out that the obstructions described in the previous section are all that is needed, at least in the simply-connected case.

6.1. **The homology spectral sequence.** In [R, §3] and [An], Rector and Anderson defined the homology spectral sequence of a cosimplicial space $\mathbf{Y}^\bullet$ (see also [Bo1, §2]). This is a second quadrant spectral sequence with

$$E^2_{p,q} \cong \pi^p H_q(\mathbf{Y}^\bullet; R) \tag{6.2}$$

abutting to $H_*(\operatorname{Tot}\mathbf{Y}^\bullet; R)$, where the *total space* $\operatorname{Tot}\mathbf{Y}^\bullet \in \mathcal{S}$ of a cosimplicial space $\mathbf{Y}^\bullet \in c\mathcal{S}$ is defined (cf. [BK1, I, §3]) to be the simplicial set $T_\bullet \in \mathcal{S}$ with $T_n = \operatorname{Hom}_{c\mathcal{S}}(\Delta[q] \times \mathbf{\Delta}^\bullet, \mathbf{Y}^\bullet)$ (see §1.4).

In general, this spectral sequence need not converge. However, under rather special conditions one does have strong convergence (see [Bo1] and [Sh]), and this yields the following:

6.3. **Theorem.** *For* $R = \mathbb{Q}$ *or* $\mathbb{F}_p$*, a simply-connected unstable* R*-coalgebra* K_* *is realizable as the homology of some simply-connected space* $\mathbf{X} \in \mathcal{T}_*$ *if and only if* K_* *has a coherently vanishing sequence of characteristic classes.*

Proof. First note that any simply-connected coalgebra over $\mathbb{Q}$ is realizable by [Q2, Thm. I] and its Corollary, so in this case the theorem merely states that one always has a coherently vanishing sequence of characteristic classes.

Given any connected space $\mathbf{X} \in \mathcal{S}_*$, the cosimplicial space $\mathbf{Y}^\bullet$ defined by $\mathbf{Y}^n = \bar{R}^{n+1}\mathbf{X}$ (where $\bar{R} : \mathcal{S}_* \to \mathcal{S}_*$ is the Bousfield-Kan monad $(\bar{R}\mathbf{X})_k := \{\sum_i r_i x_i \in R\langle X_k\rangle \mid \sum_i r_i = 1\}$ – cf. [BK1, I, §2]), is a cosimplicial resolution of $c(\mathbf{X})^\bullet$ in the sense of §3.13 – i.e., $H_*(\mathbf{Y}^\bullet; R)$ is a cosimplicial cofree resolution of $H_*(\mathbf{X}; R)$ (see [BK2, 11.5]). But then by Corollary 5.14, $H_*(\mathbf{X}; R)$ has a coherently vanishing sequence of characteristic classes.

Conversely, assume that K_* is a simply-connected unstable R-coalgebra with a coherently vanishing sequence of characteristic classes. By Corollary 5.14, *any* cosimplicial cofree resolution $K_* \to A_*^\bullet$ may be realized by a cosimplicial space $\mathbf{Q}^\bullet \in c\mathcal{S}_*$. In particular, since $K_0 = R$ and $K_1 = 0$, we may assume that the same holds for each A_*^n, so that each R-GEM $\mathbf{Q}^n$ is simply-connected. Because $A_*^\bullet$ is a resolution, $\pi^n \tilde{H}_{n+s}(\mathbf{Q}^\bullet; R) = 0$ for $n = 0$ and $s \leq 1$ or $n \geq 1$, and thus by [Bo1, Thm. 3.4] the homology spectral sequence for $\mathbf{Q}^\bullet$ converges strongly to $H_*(\operatorname{Tot}\mathbf{Q}^\bullet; R)$. Since the E^2-term of (6.2) is concentrated along the 0-line, we get

$$K_* \cong \pi^0 A_*^\bullet = E^2_{0,*} \xrightarrow{\cong} H_*(\operatorname{Tot}\mathbf{Q}^\bullet; R),$$

and this is an isomorphism of unstable coalgebras, since the edge homomorphism is induced by a topological map $\operatorname{Tot}\mathbf{Q}^\bullet \to \operatorname{Tot}_0\mathbf{Q}^\bullet = \mathbf{Q}^0$. □

6.4. **The non simply-connected case.** The simple-connectivity of K_* was only needed to guarantee convergence of the homology spectral sequence, using [Bo1, Thm. 3.4]; the (algebraic) obstruction theory described in the previous section is of course also valid in the non simply-connected case. In particular, by dualizing [Bl1, Prop. 5.1.4] we can construct a resolution with CW basis $(\bar{A}^n_*)_{n=1}^\infty$ with strictly increasing connectivity – so that in particular $\bar{A}^n_s = 0$ for $0 < s \leq n$ – and then realize $A_*^\bullet$ by a cosimplicial space $\mathbf{Q}^\bullet$, assuming that K_* has a coherently vanishing sequence of characteristic classes.

Now consider the functor $T = (T_i)_{i=0}^\infty : \mathcal{CA}_R \to \operatorname{gr} R\text{-}Mod$, defined on cofree coalgebras by $T(G_*) = V_*$ if $G_* = G(V_*)$ (we can extend this by 0-th derived functors to all of $\mathcal{CA}_R$, if we wish). If $A_*^\bullet = H_*(\mathbf{Q}^\bullet; R)$, then $TA_*^\bullet = \pi_*\mathbf{Q}^\bullet$, so $\pi^k\pi_*\mathbf{Q}^\bullet$ is just the k-th derived functor of T applied to K_*, denoted by $(L^kT)K_* \in \operatorname{gr} R\text{-}Mod$ (cf. [Bl2, §7.8]), and it makes sense to say that K_* has *projective dimension* $\leq n$ if $(L^kT)K_* = 0$ for $k > n$. For each $i \geq 0$, the functor T_i has *degree* i, in the sense dual to [Bl1, 2.3.2], so by (the dual of) [Bl1, Thm. 3.1] we have $\pi^k\pi_s\mathbf{Q}^\bullet = 0$ for $k \geq s$. Then another convergence result of Bousfield's, namely, [Bo1, Thm. 3.4], yields:

6.5. **Proposition.** *For $R = \mathbb{F}_p$, an unstable coalgebra $K_* \in \mathcal{CA}_R$ of finite projective dimension is realizable as the homology of some space $\mathbf{X} \in \mathcal{T}_*$ if and only if K_* has a coherently vanishing sequence of characteristic classes.*

However, it is not clear on the face of it whether unstable coalgebras can *ever* have non-trivial finite projective dimension (compare [LM, Thm. 4.3]).

6.6. *Remark.* As noted in the introduction, when $R = \mathbb{F}_p$, Theorem 6.3 (and perhaps also Proposition 6.5) provides a way of constructing small, even minimal, "unstable Adams resolutions" $\mathbf{Q}^\bullet$ of a given (simply-connected) space $\mathbf{X}$, which could be used in computing the Bousfield-Kan spectral sequence of [BK2] for $\pi_*(R_\infty\mathbf{X})$. In particular, when $\mathbf{X}$ is of finite type, one can choose $\mathbf{Q}^\bullet$ so that each space $\mathbf{Q}^n$ is a finite-type product of copies of $\mathbf{K}(\mathbb{F}_p, k)$ (for various k).

7. Distinguishing between realizations

Another interesting question is how one can distinguish between non-homotopy equivalent realizations $\mathbf{X}, \mathbf{Y} \in \mathcal{S}_*$ of a given unstable coalgebra K_*; we shall try to do this in terms of different realizations $\mathbf{Q}^\bullet$, $\mathbf{T}^\bullet \in c\mathcal{S}_*$ of a fixed cosimplicial cofree resolution $K_* \to A_*^\bullet$, where we assume to begin with that K_* is in fact realizable. Our goal is to find *necessary* conditions in order for two realizations $\mathbf{Q}^\bullet$ and $\mathbf{T}^\bullet$ to yield homotopy equivalent total spaces (compare [Bl5, Thm. 4.21]).

Again the key lies in the extension of coalgebras (5.3). Of course, we may assume that the characteristic class $\chi_n \in H^{n+2}(K_*; \Sigma^n K_*)$ vanishes, so that it is possible to find various splittings of (5.3) as a "semi-direct product", given by different choices of the lifting λ in Figure 3. The difference between two such semi-direct products is represented by a suitable cohomology class (compare [L, §6] and [M1, IV, §2]), constructed as follows:

7.1. **Definition.** Given two liftings $\lambda, \lambda' : H_*(B^n\mathbf{Q}^\bullet; R) \to \bar{A}_*^{n+1}$ in Figure 3 above – determining extensions of $\operatorname{tr}_n \mathbf{Q}^\bullet$ to $\operatorname{tr}_{n+1} \mathbf{Q}^\bullet)$ – as in the proof of Theorem 5.10, we may assume that the corresponding maps $\mu, \mu' : \Sigma^n K_* \to B^{n+1}A_*^\bullet$ vanish. We extend λ, λ' as in §5.8 to coface maps $d^0, d_0' : \mathbf{Q}^n \to \mathbf{Q}^{n+1}$, define $\eta : H_*(B^n\mathbf{Q}^\bullet; R) \to C^{n+1}A_*^\bullet$ with $\eta \circ \bar{q}^n = 0$, and extend to a map of A_*^0-algebras $\zeta : \Sigma^n K_* \to C^{n+1}A_*^\bullet$ (again, as in the proof of Theorem 5.10). Again $(q_{A^{n+1}}^{n+1})_\# \circ \zeta = 0$, so ζ is a cocycle in $\operatorname{Hom}_{\mathcal{CA}_{A_*^0}}(\Sigma K_*, C^*A_*^\bullet)$, representing a cohomology class $\delta_{\lambda,\lambda'} \in H^{n+1}(K_*, \Sigma^n J_*)$, which we call the *difference obstruction* for the corresponding Postnikov sections $P^n\mathbf{Q}^\bullet[\lambda]$ and $P^n\mathbf{Q}^\bullet[\lambda']$ (in the notation of §5.6).

7.2. *Remark.* Again, by standard arguments this cohomology class is independent of the specific algebraic resolution $K_* \to A_*^\bullet$ in $c\mathcal{CA}_R$. Now assume that $\mathbf{X}, \mathbf{Y} \in \mathcal{S}_*$ are two (different) realizations of K_*, with $\mathbf{Q}^\bullet$ and $\mathbf{T}^\bullet$ respectively cosimplicial spaces realizing $A_*^\bullet$ (so that $\mathbf{Q}^\bullet \simeq c(\mathbf{X})^\bullet$ and $\mathbf{T}^\bullet \simeq c(Y)^\bullet$ in the resolution model category $c\mathcal{S}_*$ of §3.13), with the same $(n-1)$-type (that is, $P^{n-1}\mathbf{Q}^\bullet \simeq P^{n-1}\mathbf{T}^\bullet$, so in particular we can assume that $\operatorname{tr}_n \mathbf{Q}^\bullet = \operatorname{tr}_n \mathbf{T}^\bullet$). Then we can choose λ and λ' so that $P^n\mathbf{Q}^\bullet[\lambda] = P^n\mathbf{Q}^\bullet$, $P^n\mathbf{Q}^\bullet[\lambda'] = P^n\mathbf{T}^\bullet$ – and thus $\delta_{\lambda,\lambda'}$ depends only on the n-type of $\mathbf{Q}^\bullet$ and $\mathbf{T}^\bullet$, respectively, so in particular only on the homotopy types of $\mathbf{X}$ and $\mathbf{Y}$.

7.3. **Theorem.** *If* $\delta_n = 0$ *in* $H^{n+1}(K_*; \Sigma^n K_*)$, *then* $P^n\mathbf{Q}^\bullet[\lambda] \simeq P^n\mathbf{Q}^\bullet[\lambda']$ *in the resolution model category structure.*

Proof. If $\delta_n = 0$, there is a map of A_*^0-comodules $\vartheta : \Sigma^n K_* \to C^n A_*^\bullet$ such that $\mathbf{d}_A^0 \circ q_A^n \circ \vartheta = \zeta$, and by the discussion in §4.6 ϑ can be lifted to a map $\theta : \Sigma^n K_* \to A_*^n$ (actually factoring through $(A_*^n)_{co\text{-}ab} \hookrightarrow A_*^n$. If we define a map of A_*^0-comodules: $\varphi := (q_Q^n)_\# \circ \vartheta \circ \psi^n$, then $\lambda' \circ \varphi : H_*(B^n\mathbf{Q}^\bullet; R) \to \bar{A}_*^{n+1}$ is just

$$\bar{d}_A^0 \circ \varphi = \operatorname{proj}_{\bar{A}} \circ \eta = \lambda - \lambda' \tag{7.4}$$

in the following diagram:

Note that because A_*^n is cofree, we can realize $\theta \circ \psi^n$ by a map $f : B^n\mathbf{Q}^\bullet \to \mathbf{Q}^n$ in $\mathcal{S}_*$, so φ is realized by $q_Q^n \circ p_Q^n \circ f : B^n\mathbf{Q}^\bullet \to B^n\mathbf{Q}^\bullet$.

We may take the simplicial GEM $\mathbf{Q}^n$ to be a simplicial R-module, with $\nu : \mathbf{Q}^n \times \mathbf{Q}^n \to \mathbf{Q}^n$ the addition map, and define $g : \mathbf{Q}^n \to \mathbf{Q}^n$ to be the composite $\nu \circ (\operatorname{Id} \top (f \circ p^n \circ q^n))$. For every $0 \leq i \leq n$ we have $g \circ d^i = d^i : \mathbf{Q}^{n-1} \to \mathbf{Q}^n$, so $p^n \circ q^n \circ g$ induces a map $h : B^n\mathbf{Q}^\bullet \to B^n\mathbf{Q}^\bullet$, with

$$h_\#(\alpha) = \alpha + \varphi(\alpha) \qquad \text{for } \alpha \in H_*(B^n\mathbf{Q}^\bullet; R), \tag{7.5}$$

and thus by (7.4) the following diagram in $\mathcal{CA}_R$ commutes:

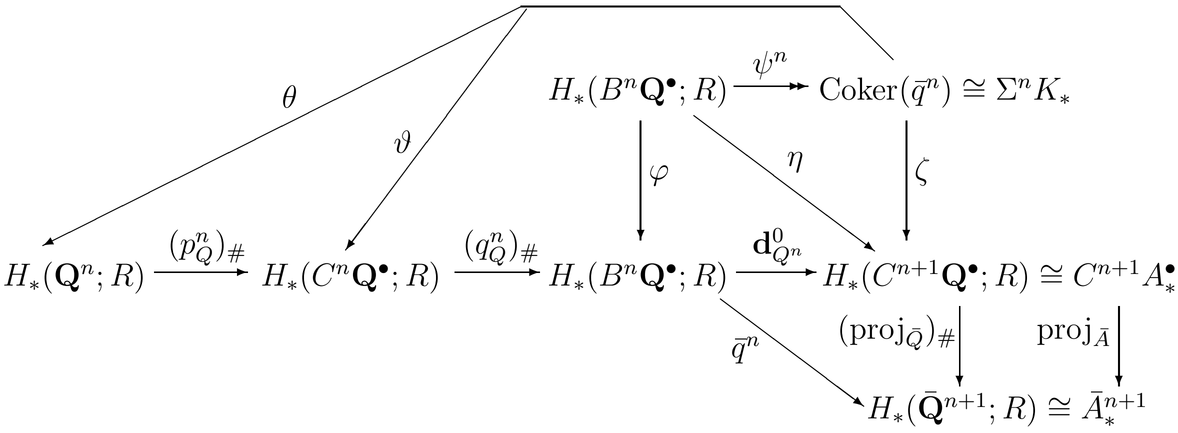

FIGURE 5

$$\begin{array}{ccccc}
B^nA^\bullet_* = B^nH_*(\mathbf{Q}^\bullet;R) & \xrightarrow{\bar{q}^n} & H_*(B^n\mathbf{Q}^\bullet;R) & \xrightarrow{\lambda} & H_*(\bar{\mathbf{Q}}^{n+1};R) = \bar{A}^{n+1}_* \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle h_\#} & & \downarrow{\scriptstyle id} \\
B^nA^\bullet_* = B^nH_*(\mathbf{Q}^\bullet;R) & \xrightarrow{\bar{q}^n} & H_*(B^n\mathbf{Q}^\bullet;R) & \xrightarrow{\lambda'} & H_*(\bar{\mathbf{Q}}^{n+1};R) = \bar{A}^{n+1}_*
\end{array}$$

which yields a map of $(n+1)$-truncated objects $\mathrm{tr}_{n+1}\,\mathbf{Q}^\bullet[\lambda] \to \mathrm{tr}_{n+1}\,\mathbf{Q}^\bullet[\lambda']$, or equivalently, a map $\rho : P^n\mathbf{Q}^\bullet[\lambda] \to P^n\mathbf{Q}^\bullet[\lambda']$, which clearly induces an isomorphism in $\pi^kH_*(-;R)$ for $k \leq n+1$.

Note that for any choice of λ we have $\pi^{n+2}P^n\mathbf{Q}^\bullet[\lambda] \cong \mathrm{Im}(\partial_{n+2}) \cong \Sigma^{n+1}K_*$, by (5.7). Since $\psi^n \circ (q^n_Q)_\# = 0$ by (5.3) and Figure 2, we have $\psi^n \circ \varphi = 0$, so by (7.5) $\psi^n \circ h_\# = \psi^n$, and since the following square commutes:

$$\begin{array}{ccc}
H_*(B^{n+1}\mathbf{Q}^\bullet[\lambda];R) & \xrightarrow{\partial_{n+1}} & H_*(B^n\mathbf{Q}^\bullet;R) \\
\downarrow{\scriptstyle \rho_\#} & & \downarrow{\scriptstyle \rho_\# = h_\#} \\
H_*(B^{n+1}\mathbf{Q}^\bullet[\lambda'];R) & \xrightarrow{\partial_{n+1}} & H_*(B^n\mathbf{Q}^\bullet;R)
\end{array}$$

and $h_\#|_{\mathrm{Im}(\partial_{n+1})}$ is an isomorphism, so is $h_\#|_{\mathrm{Im}(\partial_{n+2})}$, by (5.5). Thus $\pi^{n+2}(\rho_\#)$ is an isomorphism, too, so ρ is a weak equivalence by (5.7) for $P^n\mathbf{Q}^\bullet$. □

7.6. **Theorem.** *For $R = \mathbb{F}_p$ or $\mathbb{Q}$, assume $\mathbf{X}, \mathbf{Y} \in \mathcal{S}_*$ are two R-good realizations of a given unstable coalgebra K_* of finite type, which is either simply-connected or of finite projective dimension. If the difference obstructions for $\mathbf{X}$ and $\mathbf{Y}$ (§7.2) all vanish, then $\mathbf{X}$ and $\mathbf{Y}$ are R-equivalent (i.e., $R_\infty\mathbf{X} \simeq R_\infty\mathbf{Y}$).*

Proof. When K_* is of finite type, we can choose a cosimplicial resolution $A^\bullet_* \in c\mathcal{CA}_R$, with a CW basis in which each $\bar{A}^n_*$ (and thus each A^n_*) is of finite type. Let $\mathbf{Q}^\bullet$ and $\mathbf{T}^\bullet$ be cosimplicial spaces realizing $A^\bullet_*$, which are resolutions (in the sense of §3.13) of $\mathbf{X}$ and $\mathbf{Y}$ respectively, as in §7.2. By the Theorem 6.3 (resp., Proposition 6.5), we have $H_*(\mathrm{Tot}\,\mathbf{Q}^\bullet;R) \cong K_* \cong H_*(\mathrm{Tot}\,\mathbf{T}^\bullet;R)$.

Let $R^{\Delta}\mathbf{W}^{\bullet} \in c\mathcal{S}_*$ denote the diagonal of the bicosimplicial space $R\mathbf{W}^{\bullet\bullet}$ obtained from a given cosimplicial space $\mathbf{W}^{\bullet} \in c\mathcal{S}_*$ by applying the Bousfield-Kan R-resolution functor ([BK1, I, §4.1]) dimensionwise to $\mathbf{W}^{\bullet}$. By Theorem 7.3 there is a map of cosimplicial spaces $\rho : \mathbf{Q}^{\bullet} \to \mathbf{T}^{\bullet}$ which is a weak equivalence in $c\mathcal{S}_*$, so induces an isomorphism in the E^2-terms of the homology spectral sequences for $\mathbf{Q}^{\bullet}$ and $\mathbf{T}^{\bullet}$. Since $H_*(\mathbf{Q}^n; R)$ and $H_*(\mathbf{T}^n; R)$ are of finite type for each $n \geq 0$, by [Sh, Thm. 9.1], ρ induces a homotopy equivalence $\operatorname{Tot} R^{\Delta}\mathbf{Q}^{\bullet} \xrightarrow{\simeq} \operatorname{Tot} R^{\Delta}\mathbf{T}^{\bullet}$ (and similarly $\operatorname{Tot}\mathbf{Q}^{\bullet} \to R_{\infty}\mathbf{X}$ and $\operatorname{Tot}\mathbf{T}^{\bullet} \to R_{\infty}\mathbf{Y}$).

However, for each $n \geq 0$, the spaces $\mathbf{Q}^n$ and $\mathbf{T}^n$ are R-GEMs, so they are R-complete (cf. [BK1, V, 3.3]), and thus $\operatorname{Tot} R^{\Delta}\mathbf{Q}^{\bullet} \simeq \operatorname{Tot}\mathbf{Q}^{\bullet}$ by [Sh, Thm. 10.2], and similarly for $\mathbf{T}^{\bullet}$, so we find that $\mathbf{X}$ and $\mathbf{Y}$ are indeed R-equivalent. □

7.7. *Remark.* Shipley's theorems, in [Sh], were originally stated for $R = \mathbb{F}_p$; when $R = \mathbb{Q}$ it is no longer true that all relevant homotopy groups are finite. However, they are finite dimensional vector spaces over $\mathbb{Q}$, so [BK1, IX, §3], and the rest of Shipley's arguments, still apply.

As noted in the proof of Theorem 6.3, when $R = \mathbb{Q}$ the only problem of interest is to distinguish between different realizations of a given R-(co)algebra; the obstruction theory described here is just the vector-space dual of the theory for graded algebras over $\mathbb{Q}$ defined by Halperin and Stasheff in [HS] (see also [F]).

7.8. *Remark.* When $R = \mathbb{F}_p$, Theorem 7.6 can be thought of as providing a collection of algebraic invariants – starting with the homology coalgebra $H_*(\mathbf{X}; \mathbb{F}_p)$ – for distinguishing between p-types of spaces. As with the ordinary Postnikov systems and their k-invariants, these are not actually invariant, in the sense that distinct values (i.e., non-vanishing difference obstructions) do not guarantee distinct p-types.

This approach is the Hilton-Eckmann dual of the theory described in [Bl5] or [**?**] for distinguishing (integral) homotopy types, starting with the homotopy Π-algebra $\pi_*\mathbf{X}$, in terms of an analogous collection of cohomology classes. It is reasonable to expect a more general version of Theorem 7.6 to hold, without the assumption of finite type, and for any $R \subseteq \mathbb{Q}$; but this would require a stronger convergence result than that provided by [Sh, §9-10].

Perhaps it should be observed that many non-realization results proven in the past (see Introduction) have used higher order cohomology operations; these are implicit in the Quillen cohomology cohomology classes of Theorems 5.10 and 7.3, and were made explicit in the Π-algebra analogue in [Bl3]. We hope to return to this point in the future.

References

[A] J.F. Adams , "On the non-existence of elements of Hopf invariant one", *Ann. Math. (2)* **72** (1960), No. 1, pp. 20-104.

[AW] J.F. Adams & C.W. Wilkerson, "Finite H-spaces and algebras over the Steenrod algebra", *Ann. Math.* **111** (1980), pp. 95-143.

[Ad] J. Adem, "The iteration of the Steenrod squares in algebraic topology", *Proc. Nat. Acad. Sci. USA* **38** (1952), pp. 720-726.

[Ag] J. Aguadé, "Realizability of cohomology algebras: a survey", *Pub. Mat. Univ. Aut. Barcelona* **26** (1982), No. 2, pp. 25-68.

[ABN] J. Aguadé, C. Broto, & D. Notbohm. "Homotopy classification of some spaces with interesting cohomology and a conjecture of Cooke, Part I", *Topology* **33** (1994), No. 3, pp. 455-492.

[An] D.W. Anderson, "A generalization of the Eilenberg-Moore spectral sequence", *Bull. AMS* **78** (1972), No. 5, pp. 784-786.

[And] M. André, *Méthode Simpliciale en Algèbre Homologique et Algèbre Commutative*, Springer-Verlag *Lec. Notes Math.* **32**, Berlin-New York, 1967.

[BM] N.A. Baas & I.H. Madsen, "On the realization of certain modules over the Steenrod algebra", *Math. Scand.* **31** (1972), pp. 220-224.

[Bl1] D. Blanc, "Derived functors of graded algebras", *J. Pure Appl. Alg.* **64** (1990) No. 3, pp. 239-262.

[Bl2] D. Blanc, "New model categories from old", *J. Pure & Appl. Alg.* **109** (1996) No. 1, pp. 37-60.

[Bl3] D. Blanc, "Higher homotopy operations and the realizability of homotopy groups", *Proc. Lond. Math. Soc. (3)* **70** (1995), pp. 214-240.

[Bl4] D. Blanc, "CW simplicial resolutions of spaces, with an application to loop spaces", to appear in *Topology & Appl.*.

[Bl5] D. Blanc, "Algebraic invariants for homotopy types", *Math. Proc. Camb. Philos. Soc.* **27** (1999), No. 3, pp. 497-523.

[BDG] D. Blanc, W.G. Dwyer, & P.G. Goerss, "Cohomology invariants for simplicial spaces", preprint 1999.

[BS] D. Blanc & C.R. Stover, "A generalized Grothendieck spectral sequence", in N. Ray & G. Walker, eds., *Adams Memorial Symposium on Algebraic Topology, Vol. 1*, Lond. Math. Soc. Lec. Notes Ser. **175**, Cambridge U. Press, Cambridge, 1992, pp. 145-161.

[Bo1] A.K. Bousfield, "On the homology spectral sequence of a cosimplicial space", *Amer. J. Math.* **109** (1987), No. 2, pp. 361-394.

[Bo2] A.K. Bousfield, "Homotopy spectral sequences and obstructions", *Isr. J. Math.* **66** (1989), Nos. 1-3, pp. 54-104.

[BC] A.K. Bousfield & E.B. Curtis, "A spectral sequence for the homotopy of nice spaces", *Trans. AMS* **151** (1970), pp. 457-478.

[BK1] A.K. Bousfield & D.M. Kan, *Homotopy Limits, Completions, and Localizations*, Springer--Verlag *Lec. Notes Math.* **304**, Berlin-New York, 1972.

[BK2] A.K. Bousfield & D.M. Kan, "The homotopy spectral sequence of a space with coefficients in a ring", *Topology* **11** (1972), pp. 79-106.

[BG] E.H. Brown, Jr. & S. Gitler, "A spectrum whose cohomology is a certain cyclic module over the Steenrod algebra", *Topology* **12** (1973) No. 3, pp. 283-293.

[BP] E.H. Brown, Jr. & F.P. Peterson, "A spectrum whose Z_p cohomology is the algebra of reduced p-th powers, *Topology* **5** (1966) No. 2, pp. 149-154.

[CE] A. Clark & J.H. Ewing, "The realization of polynomial algebras as cohomology rings", *Pac. J. Math.* **50** (1974), No. 2, pp. 425-434.

[CS] G.E. Cooke & L.E. Smith, "On realizing modules over the Steenrod algebra", *J. Pure Appl. Alg.* **13** (1978), No. 1, pp. 71-100.

[Do] A. Dold, "Homology of symmetric products and other functors of complexes", *Ann. Math. (2)* **68** (1958), pp. 54-80.

[DKW] J. Duflot, N.J. Kuhn, & M.W. Winstead, "A classification of polynomial algebras over the Steenrod algebra", *Comm. Math. Helv.* **68** (1993), No. 4, pp. 622-632.

[DKS1] W.G. Dwyer, D.M. Kan, & C.R. Stover, "An E^2 model category structure for pointed simplicial spaces", *J. Pure & Appl. Alg.* **90** (1993) No. 2, pp. 137-152.

[DKS2] W.G. Dwyer, D.M. Kan, & C.R. Stover, "The bigraded homotopy groups $\pi_{i,j}X$ of a pointed simplicial space", *J. Pure Appl. Alg.* **103** (1995), No. 2, pp. 167-188.

[DMW] W.G. Dwyer, H.R. Miller, & C.C. Wilkerson, "Homotopical uniqueness of classifying spaces", *Topology* **31** (1992), No. 1, pp. 29-45.

[DW] W.G. Dwyer & C.C. Wilkerson, "A new finite loop space at the prime two", *J. AMS* **6** (1993) No. 1, pp. 37-64.

[EM] S. Eilenberg & J.C. Moore, "Homology and fibrations, I: Coalgebras, cotensor product and its derived functors", *Comm. Math. Helv.* **40** (1966), pp. 199-236.

[F] Y. Félix, *Dènombrement des types de k-homotopie: théorie de la déformation*, Mem. Soc. Math. France **3**, Paris, 1980.

[G] P.G. Goerss, "The homology of homotopy inverse limits", *J. Pure Appl. Alg.* **111** (1996), No. 1, pp. 83-122.

[GH] P.G. Goerss & M.J. Hopkins, "Resolutions in model categories", preprint 1998.

[HS] S. Halperin & J.D. Stasheff, "Obstructions to homotopy equivalences", *Adv. in Math.* **32** (1979) No. 3, pp. 233-279.

[Ha] J.R. Harper, "On the construction of mod p H-spaces", in R.J. Milgram, ed., *Algebraic and Geometric Topology, Part 2 (Stanford,CA, 1976)*, Proc. Symp. Pure Math. **32**, AMS, Providence, RI, 1978, pp. 207-214.

[Ho] H. Hopf, "Über die Topologie der Gruppen-Mannigfaltkeiten und ihre Verallgemeinerungen", *Ann. Math. (2)* **42** (1941), pp. 22-52.

[Ka] D.M. Kan, "A relation between CW-complexes and free c.s.s. groups", *Am. J. Math.* **81** (1959), pp. 512-528.

[Ku] N.J. Kuhn, "On topologically realizing modules over the Steenrod algebra", *Ann. Math., Ser. 2* **141** (1995), No. 2, pp. 321-347.

[L] K. Lee", "Cosimplicial cohomology of coalgebras", *Nagoya Math. J.* **47** (1972), pp. 199-226.

[LM] T.Y. Lin & H.R. Margolis, "Homological aspects of modules over the Steenrod algebra", *J. Pure Appl. Alg.* **9** (1976/77), No. 2, pp. 121-129.

[M1] S. Mac Lane, *Homology*, Springer-Verlag *Grund. math. Wissens.* **114**, Berlin-New York 1963.

[M2] S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag *Grad. Texts in Math.* **5**, Berlin-New York, 1971.

[M] J.P. May, *Simplicial Objects in Algebraic Topology*, U. Chicago Press, Chicago-London, 1967.

[Ma] H.R. Margolis, *Spectra and the Steenrod Algebra: Modules over the Steenrod Algebra and the Stable Homotopy Category*, North-Holland, Amsterdam-New York, 1983.

[Ml] H. Miller, "Correction to 'The Sullivan conjecture on maps from classifying spaces' " *Ann. of Math.* **121** (1985), pp. 605-609.

[Mn] J.W. Milnor, "The Steenrod algebra and its dual", *Ann. Math. (2)* **67** (1958), pp. 150-171.

[MM] J.W. Milnor & J.C. Moore, "On the structure of Hopf algebras", *Ann. Math. (2)* **81** (1965), pp. 211-264.

[Q1] D.G. Quillen, *Homotopical Algebra*, Springer-Verlag *Lec. Notes Math.* **20**, Berlin-New York, 1963.

[Q2] D.G. Quillen, "Rational homotopy theory", *Ann. Math.* **90** (1969) No. 2, pp. 205-295.

[Q3] D.G. Quillen, "On the (co-)homology of commutative rings", *Applications of Categorical Algebra*, Proc. Symp. Pure Math. **17**, AMS, Providence, RI, 1970, pp. 65-87.

[R] D.L. Rector, "Steenrod operations in the Eilenberg-Moore spectral sequence", *Comm. Math. Helv.* **45** (1970), pp. 540-552.

[Sc1] L. Schwartz, *Unstable Modules over the Steenrod Algebra and Sullivan's Fixed Point Set Conjecture*, U. Chicago Press, Chicago-London, 1994.

[Sc2] L. Schwartz, "'A propos de la conjecture de non-réalisation due à N. Kuhn", *Inv. Math.* **134** (1998), No. 1, pp. 211-227.

[Sh] B.E. Shipley, "Convergence of the homology spectral sequence of a cosimplicial space", *Amer. J. Math.* **118** (1996), No. 1, pp. 179-207.

[Sm] L.E. Smith, "On the realization and classification of symmetric algebras as cohomology rings", *Proc. AMS* **87** (1983), No. 1, pp. 144-148.

[SS] L.E. Smith & R.M. Switzer, "Realizability and nonrealizability of Dickson algebras as cohomology rings", *Proc. AMS* **89** (1983), No. 2, pp. 303-313.

[St1] N.E. Steenrod, "Cohomology operations", in J. Adem et al., eds, *Symposium internacional de topología algebraica*, UNAM/UNESCO, Mexico City, 1958, pp. 165-185.

[St2] N.E. Steenrod, "The cohomology algebra of a space", *Ens. Math.* **7** (1961), pp. 153-178.

[ST] T. Sugawara & H. Toda, "Squaring operations in truncated polynomial algebras", *Jap. J. Math.* **38** (1969), pp. 39-50.

[Sw] M.E. Sweedler, *Hopf Algebras*, W.A. Benjamin, New York, 1969.

[Ta] J. Tate, “Homology of noetherian rings and local rings”, *Ill. J. Math.* **1** (1957), pp. 14-27.

[Th] E. Thomas, “Steenrod squares and H-spaces, II”, *Ann. Math. (2)* **81** (1965), pp. 473-495.

[Wh] G.W. Whitehead, *Elements of Homotopy Theory*, Springer-Verlag *Grad. Texts Math.* **61**, Berlin-New York, 1971.

Dept. of Mathematics, University of Haifa, 31905 Haifa, Israel
E-mail address: blanc@math.haifa.ac.il