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Stems and spectral sequences

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We introduce the category $\mathfrak{P}stem[n]$ of n-stems, with a functor $\mathcal{P}[n]$ from spaces to $\mathfrak{P}stem[n]$. This can be thought of as the n-th order homotopy groups of a space. We show how to associate to each simplicial n-stem $\mathcal{Q}_\bullet$ an $(n+1)$-truncated spectral sequence. Moreover, if $\mathcal{Q}_\bullet = \mathcal{P}[n]X_\bullet$ is the Postnikov n-stem of a simplicial space $X_\bullet$, the truncated spectral sequence for $\mathcal{Q}_\bullet$ is the truncation of the usual homotopy spectral sequence of $X_\bullet$. Similar results are also proven for cosimplicial n-stems. They are helpful for computations, since n-stems in low degrees have good algebraic models.

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1 Introduction

Many of the spectral sequences of algebraic topology arise as the homotopy spectral sequence of a (co)simplicial space – including the spectral sequence of a double complex, the (stable or unstable) Adams spectral sequence, the Eilenberg-Moore spectral sequence, and so on (see § 5.14). Given a simplicial space $X_\bullet$, the E^2-term of its homotopy spectral sequence has the form $E^2_{s,t} = \pi_s \pi_t X_\bullet$, so it may be computed by applying the homotopy group functor dimensionwise to $X_\bullet$.

In this paper we show that the higher terms of this spectral sequence are obtained analogously by applying 'higher homotopy group' functors to $X_\bullet$. These functors are given explicitly in the form of certain *Postnikov stems*, defined in Section 2; the Postnikov 0-stem of a space is equivalent to its homotopy groups.

We then show how the E^r-term of the homotopy spectral sequence of a simplicial space $X_\bullet$ can be described in terms of the $(r-2)$-Postnikov stem of $X_\bullet$, for each $r \geq 2$ (see Theorem 4.13) – and similarly for the homotopy spectral sequence of a cosimplicial space $X^\bullet$ (see Theorem 5.12).

As an application for the present paper, in [5] we generalize the first author's result with Mamuka Jibladze (in [6]), which shows that the E^3-term of the stable Adams spectral

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sequence can be identified as a certain secondary derived functor Ext. We do this by showing how to define in general the *higher order derived functors* of a continuous functor $F : \mathcal{C} \to \mathcal{T}_*$, by applying F to a simplicial resolution $W_\bullet$ in $\mathcal{C}$, and taking Postnikov n-stems of $FW_\bullet$.

1.1 Notation and conventions. The category of pointed connected topological spaces will be denoted by $\mathcal{T}_*$; that of pointed sets by $\mathcal{S}et_*$; that of groups by $\mathcal{G}p$. For any category $\mathcal{C}$, $s\mathcal{C}$ denotes the category of simplicial objects over $\mathcal{C}$, and $c\mathcal{C}$ that of cosimplicial objects over $\mathcal{C}$. However, we abbreviate $s\mathcal{S}et$ to $\mathcal{S}$, $s\mathcal{S}et_*$ to $\mathcal{S}_*$, and $s\mathcal{G}p$ to $\mathcal{G}$. For any small indexing category I, the category of functors $I \to \mathcal{C}$ is denoted by $\mathcal{C}^I$.

1.2 Acknowledgements We wishe to thank the referee for his or her careful reading of the paper and helpful comments on it.

2 Postnikov stems

The Postnikov system of a topological space (or simplicial set) X is the tower of fibrations:

$$\ldots \to P^{n+1}X \xrightarrow{p^{n+1}} P^nX \xrightarrow{p^n} P^{n-1}X \ldots P^1X \xrightarrow{p^1} P^0X \,, \tag{2.1}$$

equipped with maps $q^n : X \to P^nX$ (with $p^n \circ q^n = q^{n-1}$), which induce isomorphisms on homotopy groups in degrees $\leq n$. Here P^nX is n-coconnected (that is, $\pi_i P^nX = 0$ for $i > n$) and $\pi_i p^n$ is an isomorphism for $i < n$. The fiber of the map $p^n : P^nX \to P^{n-1}X$ is the Eilenberg-Mac Lane space $K(\pi_nX, n)$, so the fibers are determined up to homotopy by π_*X. Thus a generalization of the homotopy groups of X is provided by the following notion:

2.2 Definition For any $n \geq 0$, a *Postnikov n-stem* in $\mathcal{T}_*$ is a tower:

$$\mathcal{Q} := \Big(\ldots \to Q_{k+1} \xrightarrow{q_{k+1}} Q_k \xrightarrow{q_k} Q_{k-1} \ldots Q_0\Big) \tag{2.3}$$

in $\mathcal{T}_*^{(\mathbb{N},\leq)}$, in which Q_k is $(k-1)$-connected and $(n+k)$-coconnected (so that $\pi_i(Q_k) = 0$ for $i < k$ or $i > n+k$) and $\pi_i(q_k)$ is an isomorphism for $k \leq i < n+k$. Here $(\mathbb{N}, \leq)$ is the usual linearly ordered category of the natural numbers. The space Q_k is called the k-th *n-window* of $\mathcal{Q}$.

Such an n-stem is thus a collection of overlapping $(k-1)$-connected $n+k$-types, which may be depicted for $n=2$ as follows:

$$
\begin{array}{ccccccc}
\dots & * & * & * & & & \\
 & & * & * & * & & \\
 & & & * & * & * & \\
 & & & & * & * & * \quad \dots
\end{array}
$$

where each row exhibits the $n+1$ non-trivial homotopy groups (denoted by $*$) of one n-window, and all those in the i-th column (corresponding to π_i) are isomorphic.

We denote by $\mathcal{P}\mathit{stem}[n]$ the full subcategory of Postnikov n-stems in the functor category $\mathcal{T}_*^{(\mathbb{N},\leq)}$ (with model category structure on the latter as in [21, 11.6]). Thus the morphisms in $\mathcal{P}\mathit{stem}[n]$ are given by strictly commuting maps of towers, and $f\colon \mathcal{Q} \to \mathcal{Q}'$ is a weak equivalence (respectively, a fibration) if each $f_k\colon Q_k \to Q'_k$ is such. This lets us define the homotopy category of Postnikov n-stems, $\operatorname{ho}\mathcal{P}\mathit{stem}[n]$, as a full sub-category of $\operatorname{ho}\mathcal{T}_*^{(\mathbb{N},\leq)}$.

The category $\mathcal{P}\mathit{stem}[n]$ is pointed, has products, and is equipped with canonical functors

(2.4)
$$
\begin{tikzcd}
 & \mathcal{T}_* \arrow[dr, "\mathcal{P}[n]"'] \arrow[drr, bend left=10, "\mathcal{P}[n-1]"] \arrow[drrrr, bend left=15, "\mathcal{P}[0]"] & & & & \\
\dots & & \mathcal{P}\mathit{stem}[n] \arrow[r, "\overline{\mathcal{P}}{[n-1]}"'] & \mathcal{P}\mathit{stem}[n-1] \arrow[r, "\overline{\mathcal{P}}{[n-2]}"'] & \cdots \arrow[r, "\overline{\mathcal{P}}{[0]}"'] & \mathcal{P}\mathit{stem}[0]
\end{tikzcd}
$$

which preserve products and weak equivalences.

2.5 Remark The sequence of functors (2.4) is described by a commuting diagram, in which we may take all maps to be fibrations:

$$
\begin{tikzcd}
 & \arrow[d, dotted, no head] & \arrow[d, dotted, no head] & \arrow[d, dotted, no head] & & \\
\cdots \arrow[r] & Q^{n+k+1}_{k+1} \arrow[r, "q^n_{k+1}"] \arrow[d, "p^n_{k+1}"'] \arrow[ur] & Q^{n+k}_{k} \arrow[r, "q^n_{k}"] \arrow[d, "p^n_{k}"'] \arrow[ur] & Q^{n+k-1}_{k-1} \arrow[r] \arrow[d, "p^n_{k-1}"] \arrow[ur] & \cdots \arrow[r] & Q^n_0 \arrow[d, "p^n_0"] \\
\cdots \arrow[r] & Q^{n+k}_{k+1} \arrow[r, "q^{n-1}_{k+1}"] \arrow[d, "p^{n-1}_{k+1}"'] \arrow[ur, "r^{n-1}_{k+1}"] & Q^{n+k-1}_{k} \arrow[r, "q^{n-1}_{k}"] \arrow[d, "p^{n-1}_{k}"'] \arrow[ur, "r^{n-1}_{k}"] & Q^{n+k-2}_{k-1} \arrow[r] \arrow[d, "p^{n-1}_{k-1}"] \arrow[ur] & \cdots \arrow[r] & Q^{n-1}_0 \arrow[d, "p^n_0"] \\
\cdots \arrow[r] & Q^{n+k-1}_{k+1} \arrow[r, "q^{n-2}_{k+1}"] \arrow[d, dotted, no head] \arrow[ur, "r^{n-2}_{k+1}"] & Q^{n+k-2}_{k} \arrow[r, "q^{n-2}_{k}"] \arrow[d, dotted, no head] \arrow[ur, "r^{n-2}_{k}"] & Q^{n+k-3}_{k-1} \arrow[r] \arrow[d, dotted, no head] \arrow[ur] & \cdots \arrow[r] & Q^{n-2}_0 \arrow[d, dotted, no head] \\
\cdots \arrow[r] & Q^{k+1}_{k+1} \arrow[r, "q^0_{k+1}"] & Q^{k}_{k} \arrow[r, "q^0_{k}"] & Q^{k-1}_{k-1} \arrow[r] & \cdots \arrow[r] & Q^0_0
\end{tikzcd}
\tag{2.6}
$$

Here $\pi_i Q^n_k = 0$ for $i < k$ or $i > n$, and all maps induce isomorphisms in π_i whenever possible. Thus:

(a) The k-th column (from the right) is the Postnikov tower for $Q_k := \lim_n Q^n_k$.

(b) The diagonals are the dual Postnikov system of connected covers for Q^j_0.

(c) The n-th row (from the bottom) is a Postnikov n-stem.

(d) In particular, each space in the 0-stem (the bottom row) is an Eilenberg-Mac Lane space, and the maps q^0_k are nullhomotopic. Thus the homotopy type of the bottom line in $\operatorname{ho}\mathcal{P}stem[0]$ is determined by the collection of homotopy groups $\{\pi_k Q^k_k\}_{k=0}^{\infty}$.

2.7 Definition The motivating example of a Postnikov n-stem is a *realizable* one, associated to a space $X \in \mathcal{T}_*$, and denoted by $\mathcal{P}[n]X$, with $(P[n]X)_k := P^{n+k}X\langle k\rangle$. As usual, $Y\langle k\rangle$ denotes the $(k-1)$-connected cover of a space $Y \in \mathcal{T}_*$. Each fibration $q_k\colon (P[n]X)_k \to (P[n]X)_{k-1}$ fits into a commuting triangle of fibrations:

$$
\begin{tikzcd}
P^{n+k+1}X\langle k+1\rangle \arrow[rr, "q"] \arrow[dr, "p"'] & & P^{n+k}X\langle k\rangle \\
 & P^{n+k}X\langle k+1\rangle \arrow[ur, "r"'] &
\end{tikzcd}
\tag{2.8}
$$

in which the maps p and r are the fibration of (2.1) and the covering map, respectively. See [4, §10.5] for a natural context in which non-realizable Postnikov n-stems arise.

2.9 Examples of stems. The functor $\mathcal{P}[0]_*\colon \mathcal{T}_* \to \operatorname{ho}\mathfrak{P}\mathit{stem}[0]$ induced by $\mathcal{P}[0]$ is equivalent to the homotopy group functor: in fact, the homotopy groups of a space define a functor $\pi_*\colon \mathcal{T}_* \to \mathcal{K}$ into the product category $\mathcal{K} := \prod_{i=0}^{\infty} \mathcal{K}_i$, where $\mathcal{K}_0 = \mathcal{S}et_*$, $\mathcal{K}_1 = \mathcal{G}p$, and $\mathcal{K}_i = \mathcal{A}b\mathcal{G}p$, for $i \geq 2$. Moreover, there is an equivalence of categories $\vartheta\colon \mathcal{K} \equiv \operatorname{ho}\mathfrak{P}\mathit{stem}[0]$, such that the functor $\mathcal{P}[0]_*$ is equivalent to the composite functor $\vartheta \circ \pi_*\colon \mathcal{T}_* \to \mathcal{K}$.

Similarly, the functor $\mathcal{T}_* \longrightarrow \operatorname{ho}\mathfrak{P}\mathit{stem}[1]$ induced by $\mathcal{P}[1]$ is equivalent to the secondary homotopy group functor of [7, §4], in the sense that each secondary homotopy group $\pi_{n,*}X$ completely determines the n-th 1-window of X. Using the results on secondary homotopy groups in [7], one obtains a homotopy category of algebraic 1-stems which is equivalent to $\operatorname{ho}\mathfrak{P}\mathit{stem}[1]$.

A category of algebraic models for 2-stems is only partially known. The homotopy classification of $(k-1)$-conected $(k+2)$-types is described for all k in [3]; this theory can be used to classify homotopy types of Postnikov 2-stems.

3 The spectral sequence of a simplicial space

We begin with the construction of the homotopy spectral sequence for a simplicial space (cf. [22], [15, Theorem B.5], and [16, X,§6]), using the version given by Dwyer, Kan, and Stover in [20, §8] (see also [13, §2,5], [12], and [19, §3.6]). For this purpose, we require some explicit constructions for the E^2-model category of simplicial spaces.

3.1 Definition Given a simplicial object $X_\bullet \in s\mathcal{C}$, over a complete pointed category $\mathcal{C}$, for each $n \geq 1$ define its n-*cycles* object to be

$$Z_n X_\bullet \;:=\; \{x \in X_n \mid d_i x = * \text{ for } i = 0, \ldots, n\}\;.$$

Similarly, the the n-*chains* object for $X_\bullet$ is

$$C_n X_\bullet \;:=\; \{x \in X_n \mid d_i x = * \text{ for } i = 1, \ldots, n\}$$

Set $Z_0 X_\bullet := X_0$. We denote the map $d_0|_{C_n X_\bullet}\colon C_n X_\bullet \to Z_{n-1} X_\bullet$ by $\mathbf{d}_0^{X_n}$.

3.2 Notation For any non-negatively graded object T_*, we write ΩT_* for the graded object with $(\Omega T_*)_j := T_{j+1}$ for all $j \geq 0$. The notation is motivated by the natural isomorphism of graded groups $\pi_* \Omega X \cong \Omega(\pi_* X)$ for $X \in \mathcal{T}_*$.

3.3 Definition Now assume that $\mathcal{C}$ is a pointed model category of spaces, such as $\mathcal{T}_*$ or $\mathcal{G}$, and $X_\bullet$ is a Reedy fibrant simplicial object over $\mathcal{C}$ – that is, for each $n \geq 1$, the universal face map $\delta_n \colon X_n \to M_n X_\bullet$ into the n-th matching object of $X_\bullet$ is a fibration (see [21, 15.3]). The map $\mathbf{d}_0 = \mathbf{d}_0^{X_n}$ then fits into a fibration sequence in $\mathcal{C}$:

(3.4)
$$\cdots \Omega Z_n X_\bullet \to Z_{n+1} X_\bullet \xrightarrow{j_{n+1}^{X_\bullet}} C_{n+1} X_\bullet \xrightarrow{\mathbf{d}_0^{X_{n+1}}} Z_n X_\bullet$$

(see [20, Prop. 5.7]).

For each $n \geq 0$, the n-th *natural homotopy group* of the simplicial space $X_\bullet$, denoted by $\pi_n^\natural X_\bullet = \pi_{n,*}^\natural X_\bullet$, the cokernel of the map $(\mathbf{d}_0^{X_{n+1}})_\#$ (induced on homotopy groups by $\mathbf{d}_0^{X_{n+1}}$). Note that the cokernel of a maps of groups or pointed sets is generally just a pointed set.

We thus have an exact sequence of graded groups:

(3.5)
$$\pi_* C_{n+1} X_\bullet \xrightarrow{(\mathbf{d}_0^{X_{n+1}})_\#} \pi_* Z_n X_\bullet \xrightarrow{\hat{\vartheta}_n} \pi_{n,*}^\natural X_\bullet \to 0 \,.$$

Together the groups $(\pi_{n,k}^\natural X_\bullet)_{n,k=0}^\infty$ constitute the *bigraded homotopy groups* of [20, §5.1].

3.6 Construction of the spiral sequence. Applying the functor π_* to the fibration sequence (3.4) yields a long exact sequence, with connecting homomorphism $\partial_\# \colon \Omega \pi_* Z_n X_\bullet = \pi_* \Omega Z_n X_\bullet \to \pi_* Z_{n+1} X_\bullet$. Note that the inclusion $\iota \colon C_n X_\bullet \hookrightarrow X_n$ induces an isomorphism $\iota_\star \colon \pi_* C_n X_\bullet \cong C_n(\pi_* X_\bullet)$ for each $n \geq 0$ (see [10, Prop. 2.7]). From (3.5) we see that:

$$\Omega \pi_n^\natural X_\bullet = \Omega\, \mathrm{Coker}\, (\mathbf{d}_0^{X_{n+1}})_\# \cong \mathrm{Im}\ \partial_\# \cong \mathrm{Ker}\, (j_{n+1}^{X_\bullet})_\# \subseteq \pi_* Z_{n+1} X_\bullet \ ,$$

so we obtain a commutative diagram with exact rows and columns:

(3.7)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathrm{Ker}\,(j_n)_* \arrow[r, hook] \arrow[d, hook] & B_{n+1} X_\bullet \arrow[r, two heads, "(j_n)_*"] \arrow[d, hook] & B_{n+1} \pi_* X_\bullet \arrow[r] \arrow[d, hook] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega \pi_{n-1}^\natural X_\bullet \arrow[r, hook, "\ell_{n-1}"] \arrow[d] \arrow[dr, "s_n"] & \pi_* Z_n X_\bullet \arrow[r, "(j_n^{X_\bullet})_\#"] \arrow[d, "\hat{\vartheta}_n"] & Z_n \pi_* X_\bullet \arrow[r, two heads] \arrow[d, "\vartheta_n"] & \mathrm{Coker}\ h_n \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathrm{Ker}\ h_n \arrow[r, hook] \arrow[d] & \pi_n^\natural X_\bullet \arrow[r, "h_n"] \arrow[d] & \pi_n \pi_* X_\bullet \arrow[r, two heads] \arrow[d] & \mathrm{Coker}\ h_n \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

in which $B_{n+1}X_\bullet := \operatorname{Im}(\mathbf{d}_0^{X_{n+2}})_\# \subseteq \pi_* Z_n X_\bullet$ and $B_{n+1}\pi_* X_{n+2} := \operatorname{Im}\mathbf{d}_0^{\pi_* X_{n+2}}$ are the respective boundary objects. Note that the map $(j_n^{X_\bullet})_\# \colon \pi_* Z_n X_\bullet \to \pi_* C_n X_\bullet$ induced by the inclusion $j_n^{X_\bullet}$ of (3.4) above in fact factors through $Z_n \pi_* X_\bullet$, as indicated in the middle row of (3.7).

This defines the map of graded groups $h_n \colon \pi_n^\natural X_\bullet \to \pi_n(\pi_* X_\bullet)$. Note that for $n = 0$ the map $\hat{\iota}_\star$ is an isomorphism, so h_0 is, too. The map $s_n \colon \Omega \pi_{n-1}^\natural X_\bullet \to \pi_n^\natural X_\bullet$ is the composite of the inclusion $\ell_{n-1} \colon \operatorname{Ker}(j_n^{X_\bullet})_\# \hookrightarrow \pi_* Z_n X_\bullet$ with the quotient map $\hat{\vartheta}_n \colon \pi_* Z_n X_\bullet \to \pi_n^\natural X_\bullet$ of (3.5), using the natural identification of $\Omega \pi_n^\natural X_\bullet$ with $\operatorname{Ker}(j_{n+1}^{X_\bullet})_\#$.

The map $\partial_{n+2} \colon \pi_{n+2}\pi_* X_\bullet \to \Omega \pi_n^\natural X_\bullet$ is induced by the composite

$$Z_{n+2}\pi_* X_\bullet \subseteq C_{n+2}\pi_* X_\bullet \cong \pi_* C_{n+2} X_\bullet \xrightarrow{(\mathbf{d}_0^{X_{n+2}})_\#} \pi_* Z_{n+1} X_\bullet \,, \tag{3.8}$$

which actually lands in $\operatorname{Ker}(j_{n+1}^{X_\bullet})_\#$ by the exactness of the long exact sequence for the fibration (3.4).

These maps s_n, h_n, and ∂_n fit into a *spiral long exact sequence:*

$$\begin{aligned} \ldots \to \Omega \pi_{n-1}^\natural X_\bullet \xrightarrow{s_n} \pi_n^\natural X_\bullet \xrightarrow{h_n} \pi_n \pi_* X_\bullet \xrightarrow{\partial_n} \Omega \pi_{n-2}^\natural X_\bullet \\ \xrightarrow{s_{n-1}} \pi_{n-1}^\natural X_\bullet \to \ldots \to \pi_0^\natural X_\bullet \xrightarrow{\cong} \pi_0 \pi_* X_\bullet \end{aligned} \tag{3.9}$$

(cf. [20, 8.1]).

3.10 The spectral sequence of a simplicial space. For any simplicial space $X_\bullet \in s\mathcal{T}_*$ (or bisimplicial set), Bousfield and Friedlander showed that there is a first-quadrant spectral sequence of the form

$$E^2_{s,t} = \pi_s \pi_t X_\bullet \Rightarrow \pi_{s+t}\|X_\bullet\| \,, \tag{3.11}$$

where $\|X_\bullet\| \in \mathcal{T}_*$ is the realization (or the diagonal, in the case of $X_\bullet \in s\mathcal{S}_*$). The spectral sequence is always defined, but $X_\bullet$ must satisfy certain “Kan conditions” to guaranteee *convergence* – see [15, Theorem B.5].

In [20, §8.4], Dwyer, Kan and Stover showed that (3.11) coincides up to sign, from the E^2-term on, with the spectral sequence associated to the exact couple of (3.4), which we call the *spiral spectral sequence* for $X_\bullet$.

If we assume that each X_n is connected, by taking loops (or applying Kan’s functor G, if $X_\bullet \in s\mathcal{S}_*$), we may replace $X_\bullet$ by a bisimplicial group $GX_\bullet \in s\mathcal{G}$, and then (3.11) becomes the spectral sequence of [22].

4 Simplicial stems and truncated spectral sequences

As noted in §2.9, the E^2-term of any of the above equivalent spectral sequences for a simplicial space $X_\bullet$ is determined explicitly by the simplicial 0-stem of $X_\bullet$.

Our goal is to extend this description to the higher terms of the spectral sequence. For this purpose, fix $n \geq 0$, and consider a simplicial Postnikov n-stem $\mathcal{Q}_\bullet$ (which need not be realizable as $\mathcal{P}[n]X_\bullet$ for some simplicial space $X_\bullet$). This is equivalent to having a collection of simplicial spaces $\mathcal{Q}_\bullet^{n+k}\langle k\rangle$ for each $k \geq 0$, equipped with maps as in (2.3), with $\pi_i \mathcal{Q}_\bullet^{n+k}\langle k\rangle = 0$ for $i < k$ or $i > n+k$.

We assume that $\mathcal{Q}_\bullet$ is *Reedy fibrant* in the sense that for each $k \geq 0$, the simplicial space $\mathcal{Q}_\bullet^{n+k}\langle k\rangle$ is Reedy fibrant. In this case, the "n-stem version" of the spiral long exact sequence is defined as follows: for each $t, i, k \geq 0$, set $\pi_{t,i}^{\natural\,(k,n)}\mathcal{Q}_\bullet := \pi_{t,i+k}^{\natural}\mathcal{Q}_\bullet^{n+k}\langle k\rangle$ and

$$(4.1)\qquad \pi_i^{(k,n)}\mathcal{Q}_\bullet \;:=\; \pi_{i+k}\mathcal{Q}_\bullet^{n+k}\langle k\rangle \;=\; \begin{cases} \pi_{i+k}\mathcal{Q}_\bullet & \text{if } 0 \leq i \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Note that the $(i+k)$-th homotopy group $\pi_{i+k}\mathcal{Q}_\bullet$ of a Postnikov n-stem $\mathcal{Q}_\bullet$ is well-defined, and coincides with $\pi_{i+k}X_\bullet$ for $0 \leq i \leq n$ when $\mathcal{Q}_\bullet = \mathcal{P}[n]X_\bullet$.

4.2 Definition The collection of long exact sequences (3.9) for $\mathcal{Q}_\bullet^{n+k}\langle k\rangle$ (indexed by $k \geq 0$):

$$(4.3)\qquad \ldots\,\Omega\pi_{t-1,*}^{\natural\,(k,n)}\mathcal{Q}_\bullet \xrightarrow{s_t^{(k,n)}} \pi_{t,*}^{\natural\,(k,n)}\mathcal{Q}_\bullet \xrightarrow{h_t^{(k,n)}} \pi_t\pi_*^{(k,n)}\mathcal{Q}_\bullet \xrightarrow{\partial_t^{(k,n)}} \Omega\pi_{t-2,*}^{\natural\,(k,n)}\mathcal{Q}_\bullet \ldots\,,$$

together with the maps between adjacent k-windows induced by the map q in (2.6), will be called the *spiral n-system* of $\mathcal{Q}_\bullet$. When $\mathcal{Q}_\bullet = \mathcal{P}[n]X_\bullet$, we will refer to this simply as the spiral n-system of $X_\bullet$.

4.4 Remark Using the exactness of (4.3), definition (4.1) implies that:

$$(4.5)\qquad \pi_{t,i}^{\natural\,(k,n)}\mathcal{Q}_\bullet = \pi_{t,i}^{\natural}\mathcal{Q}_\bullet^{n+k}\langle k\rangle \;=\; 0 \quad \text{for } i > n\,,$$

by induction on $t \geq 0$. Note, however, that while the groups $\pi_i^{(k,n)}\mathcal{Q}_\bullet$ are explicitly described by (4.1), the dependence of $\pi_{t,i}^{\natural\,(k,n)}\mathcal{Q}_\bullet$ on k and n requires more care.

4.6 The E^2-term of the spectral sequence. The spiral 0-system of a simplicial Postnikov 0-stem $\mathcal{Q}_\bullet$ reduces to a series of isomorphisms $h_t\colon \pi_{t,*}^{\natural\,(k,0)}\mathcal{Q}_\bullet \cong \pi_t\pi_*^{(k,0)}\mathcal{Q}_\bullet$ (for each $k \geq 0$). When $\mathcal{Q}_\bullet = \mathcal{P}[0]X_\bullet$ is the Postnikov 0-stem of a simplicial space $X_\bullet$, this allows us to identify the $E^2_{t,k}$-term of the spiral spectral sequence for $X_\bullet$, which is:

$$\pi_t\pi_k X_\bullet \;=\; \pi_t\pi_k P^{0+k}X_\bullet\langle k\rangle \;=\; \pi_t\pi_k(P[0]X_\bullet)_k \;=\; \pi_t\pi_*^{(k,0)}\mathcal{P}[0]X_\bullet \;=\; \pi_t\pi_*^{(k,0)}\mathcal{Q}_\bullet,$$

with $\pi^{\natural\,(k,0)}_{t,*}\mathcal{Q}_\bullet = \pi^{\natural\,(k,0)}_{t,*}\mathcal{P}[0]X_\bullet$.

The first interesting case is the spiral 1-system, for which we have:

4.7 Proposition *The E^3-term of the spiral spectral sequence for a simplicial space $X_\bullet$ is determined by the spiral 1-system of $X_\bullet$. In fact, $d^2_{t,k}$ may be identified with $\partial_t^{(k,1)}\colon \pi_t\pi_k X_\bullet \to \Omega\pi^{\natural\,(k,1)}_{t-2,0}X_\bullet$, while $E^3_{t,k}$ is the image of the composite map*

(4.8)
$$\pi^{\natural\,(k,1)}_{t,0}X_\bullet \xrightarrow{h_t^{(k,1)}} \pi_t\pi_k X_\bullet \;\cong\; \pi_t\pi_1^{(k-1,1)}X_\bullet \xleftarrow[\cong]{h_t^{(k-1,1)}} \pi^{\natural\,(k-1,1)}_{t,1}X_\bullet \xrightarrow{s_{t+1}^{(k-1,1)}} \pi^{\natural\,(k-1,1)}_{t+1,0}X_\bullet\,.$$

Observe that (4.8) involves maps from different windows of the spiral 1-system, implicitly identified using the isomorphisms induced by the map q in (2.6).

Proof Because $n=1$ throughout, we abbreviate $\pi^{\natural\,(k,1)}_{t,i}\mathcal{Q}_\bullet$ to $\pi^{\natural\,(k)}_{t,i}\mathcal{Q}_\bullet$, and $\pi_i^{(k,1)}\mathcal{Q}_\bullet$ to $\pi_i^{(k)}\mathcal{Q}_\bullet$, observing that $\pi_i^{(k)}\mathcal{Q}_\bullet$ is simply $\pi_{i+k}X_\bullet$ for $i=0,1$, and zero otherwise, since $\mathcal{Q}_\bullet = \mathcal{P}[1]X_\bullet$. Thus the spiral 1-system (4.3) is non-trivial for each $t\geq 1$ in (internal) degrees $i=0,1$ only, and we can write it in two rows:

$$\begin{array}{ccccccccccc}
 & 0 & \longrightarrow & \pi^{\natural\,(k)}_{t,1}\mathcal{Q}_\bullet & \xrightarrow{\cong} & \pi_t\pi_1^{(k)}\mathcal{Q}_\bullet & \longrightarrow 0 & \longrightarrow & \pi^{\natural\,(k)}_{t-1,1}\mathcal{Q}_\bullet & \xrightarrow{\cong} & \pi_{t-1}\pi_1^{(k)}\mathcal{Q}_\bullet \\
\Omega\pi^{\natural\,(k)}_{t-1,0}\mathcal{Q}_\bullet & \xrightarrow{s_t} & \pi^{\natural\,(k)}_{t,0}\mathcal{Q}_\bullet & \xrightarrow{h_t} & \pi_t\pi_0^{(k)}\mathcal{Q}_\bullet & \xrightarrow{\partial_t} & \Omega\pi^{\natural\,(k)}_{t-2,0}\mathcal{Q}_\bullet & \xrightarrow{s_{t-1}} & \pi^{\natural\,(k)}_{t-1,0}\mathcal{Q}_\bullet & \xrightarrow{h_{t-1}} & \pi_{t-1}\pi_0^{(k)}\mathcal{Q}_\bullet
\end{array}$$

Since $\mathcal{Q}_\bullet := \mathcal{P}[1]X_\bullet$ is the simplicial Postnikov 1-stem of $X_\bullet$, we actually have a collection of two-row long exact sequences, one for each k-window of $\mathcal{P}[1]X_\bullet$.

For each such k-window $\mathcal{P}_k[1]X_\bullet$, we can use the top row to identify

$$\Omega\pi^{\natural\,(k)}_{t,0}\mathcal{Q}_\bullet \;=\; \Omega\pi^{\natural}_{t,0}\mathcal{P}_k[1]X_\bullet \;=\; \pi^{\natural}_{t,1}\mathcal{P}_k[1]X_\bullet \;=\; \pi^{\natural\,(k)}_{t,1}\mathcal{Q}_\bullet$$

with $\pi_t\pi_1^{(k)}\mathcal{Q}_\bullet = \pi_t\pi_t^{(1)}\mathcal{P}_k[1]X_\bullet = \pi_t\pi_{k+1}X_\bullet$, so the bottom row reduces to:

$$\begin{tikzcd}
\pi_{t-1}\pi_{k+1}X_\bullet \arrow[rr, "s_t^{(k,1)}"] \arrow[dr] & & \pi^{\natural\,(k)}_{t,0}\mathcal{Q}_\bullet \arrow[rr, "h_t^{(k,1)}"] \arrow[dr] & & \pi_t\pi_k X_\bullet \arrow[rr, "\partial_t^{(k,1)}"] \arrow[dr] & & \pi_{t-2}\pi_{k+1}X_\bullet \\
 & \mathrm{Im}\,(s_t^{(k,1)}) \arrow[ur] & & \begin{array}{c}\mathrm{Im}\,(h_t^{(k,1)}) \\ = \\ \mathrm{Ker}\,(\partial_t^{(k,1)})\end{array} \arrow[ur] & & \mathrm{Im}\,(\partial_t^{(k,1)}) \arrow[ur] &
\end{tikzcd}$$

Note that the following part of the E^1-term of the exact couple for the fibration sequence $C_{n+1}P^1\Omega^i X_\bullet \to Z_n P^1\Omega^i X_\bullet$, (as in (3.4)):

$$\begin{tikzcd}
\pi_1 Z_{t-1}P^1\Omega^k X_\bullet \arrow[r, "(j_{t-1})_\#"] \arrow[dddd, "\partial_*"] & \pi_1 C_{t-1}P^1\Omega^k X_\bullet \arrow[r, "(d_0^{t-1})_\#"] & \pi_1 Z_{t-2}P^1\Omega^k X_\bullet \arrow[r, "(j_{t-2})_\#"] \arrow[d] \arrow[dr] \arrow[dddd, bend right=60, "\partial_*"'] & \pi_1 C_{t-2}X_\bullet \to \dots \\
 & & \Omega\pi^{\natural\,(k)}_{t-2,0}X_\bullet = \pi^{\natural\,(k)}_{t-2,1}X_\bullet \arrow[dr, "h^{(k+1,1)}_{t-2,1}", "\cong"'] \arrow[ddd, hook] & Z_{t-2}\pi_1 P^1\Omega^k X_\bullet \arrow[u, hook, "\mathrm{inc}"] \arrow[d, "\vartheta_{t-2}"] \\
 & & & \pi_{t-2}\pi_{k+1}X_\bullet \\
 & & & \\
\pi_0 Z_t P^1\Omega^k X_\bullet \arrow[r, "(j_t)_\#"] \arrow[d, "\hat{\vartheta}_t"] \arrow[dr] & \pi_0 C_t P^1\Omega^k X_\bullet \arrow[r, "(d_0^t)_\#"] & \pi_0 Z_{t-1}P^1\Omega^k X_\bullet \arrow[r, "(j_{t-1})_\#"] \arrow[d, "\hat{\vartheta}_{t-1}"] \arrow[dr] & \pi_0 C_{t-1}P^1\Omega^k X_\bullet \to \dots \\
\pi^{\natural\,(k)}_{t,0}X_\bullet \arrow[dr, "h_t^{(k,1)}"] & Z_t\pi_k X_\bullet \arrow[u, hook, "\mathrm{inc}"] \arrow[d, "\vartheta_t"] & \pi^{\natural\,(k)}_{t-1,0}X_\bullet \arrow[dr, "h^{(k,1)}_{t-1}"] & Z_{t-1}\pi_k X_\bullet \arrow[u, hook, "\mathrm{inc}"] \arrow[d, "\vartheta_{t-1}"] \\
 & \pi_t\pi_k X_\bullet \arrow[uuuuur, "\partial^{(k,1)}_{t,0}"] & & \pi_{t-1}\pi_k X_\bullet
\end{tikzcd}$$

is naturally isomorphic to the exact couple for $C_{n+1}\Omega^k X_\bullet \to Z_n\Omega^k X_\bullet$, since C_{n+1} and Z_n are limits, so they commute with P^1, and then $\pi_1 P^1 Z_{t-1}\Omega^k X_\bullet \cong \pi_1 Z_{t-1}\Omega^k X_\bullet$, and so on. This does not imply, of course, that $\pi^{\natural\,(k)}_{t,1}X_\bullet \cong \pi^{\natural}_{t,k+1}X_\bullet$.

We therefore see from (3.7) and (3.8) that the differential $d^2_{t,k}\colon E^2_{t,k} \to E^2_{t-2,k+1}$ may be identified with:

(4.9)

$$\pi_t\pi_k X_\bullet \cong \pi_t\pi_0^{(k,1)}X_\bullet \xrightarrow{\partial^{(k,1)}_{t,0}} \Omega\pi^{\natural\,(k)}_{t-2,0}X_\bullet = \pi^{\natural\,(k)}_{t-2,1}X_\bullet \overset{h_t}{\cong} \pi_{t-2}\pi_1^{(k,1)}X_\bullet \cong \pi_{t-2}\pi_{k+1}X_\bullet$$

Now by definition, $E^3_{t,k}$ fits into a commutative diagram:

(4.10)
$$\begin{tikzcd}
E^2_{t+2,k-1} \arrow[r, "d^2_{t+2,k-1}"] \arrow[d, "r"'] & E^2_{t,k} \arrow[r, two heads, "q"] & \mathrm{Coker}\,(d^2_{t+2,k-1}) \\
\mathrm{Im}\,(d^2_{t+2,k-1}) \arrow[r, hook, "\ell"] & \mathrm{Ker}\,(d^2_{t,k}) \arrow[r, two heads, "s"] \arrow[u, hook, "j"'] & E^3_{t,k} \arrow[u, hook, "\kappa"']
\end{tikzcd}$$

with exact rows, ℓ j and κ monic, and thus $E^3_{t,k} \cong \mathrm{Im}\,(q\circ j)$.

From the exactness of (4.3) we see that:

$$\mathrm{Coker}\,(d^2_{t+2,k-1}) = \mathrm{Coker}\,(\partial^{(k-1,1)}_{t+2}) = \mathrm{Im}\,(s^{(k-1,1)}_{t+1})$$

and

$$\mathrm{Ker}\,(d^2_{t,k}) = \mathrm{Ker}\,(\partial^{(k,1)}_{t}) = \mathrm{Im}\,(h^{(k,1)}_{t})\ ,$$

so $E^3_{t,k} = \mathrm{Im}\,(q \circ j)$ is indeed the image of the map in (4.8). □

4.11 Definition An r-*truncated spectral sequence* is one defined up to and including the E^r-term, together with the differential $d^n\colon E^r_{t,i} \to E^r_{t-r-1,t+r}$, but without requiring that $d^r \circ d^r = 0$ (so the E^{r+1}-term is defined in terms of the r-truncated spectral sequence only if $d^r d^r = 0$).

The main example is the n-truncation of an (ordinary) spectral sequence (such as that of a simplicial space). In this case we do have $d^r \circ d^r = 0$, of course.

4.12 Corollary *Any Reedy fibrant simplicial Postnikov* 1*-stem has a well-defined* 2*-truncated spiral spectral sequence. Moreover, if* $\mathcal{Q}_\bullet = \mathcal{P}[1]X_\bullet$ *for some simplicial space* $X_\bullet$*, this* 2*-truncated spectral sequence coincides with the* 2*-truncation of the Bousfield-Friedlander spectral sequence for* $X_\bullet$*.*

In general, we have a less explicit description of the higher terms in the spiral spectral sequence:

4.13 Theorem *For each* $r \geq 0$*, the* E^{r+2}*-term of the spiral spectral sequence for a simplicial space* $X_\bullet$ *is determined by the spiral* r*-system of* $X_\bullet$*. Moreover, for any* $\alpha \in E^{r+1}_{t,i}$*, we have* $d^{r+1}_{t,i}(\alpha) = \beta \in E^{r+1}_{t-r-1,i+r}$ *if and only if* α *and* β *have representatives* $\overline{a} \in \pi_t\pi_i X_\bullet$ *and* $\overline{b} \in \pi_{t-r-1}\pi_{i+r}X_\bullet$*, respectively, such that:*

$$(s^{(i,r)}_{t-2,1}) \circ (s^{(i,r)}_{t-3,2}) \circ \cdots \circ (s^{(i,r)}_{t-r,r-1}) \circ (h^{(i,r)}_{t-r-1,r})^{-1}(\overline{b}) \;=\; \partial^{(i,r)}_{t,0}(\overline{a}) \tag{4.14}$$

Proof We naturally identify $\pi^{\natural\,(i,r)}_{t,k}X_\bullet$ with $\pi^{\natural\,(i,r-s)}_{t,k+s}X_\bullet$ for $k \geq s$, and similarly for the maps in (4.3), so the spiral $(r-1)$-system embeds in the spiral r-system (with an index shift).

Again we write out the E^1-term of the spiral exact couple:

$$
\begin{tikzcd}[column sep=small]
 & \pi_r C_{t-r} P^r \Omega^i X_\bullet \arrow[r, "(d_0^{t-r})_\#"] & \pi_r Z_{t-r-1} P^r \Omega^i X_\bullet \arrow[r, "(j_{t-r-1})_\#"] \arrow[d, two heads, "\hat{\vartheta}_{t-r-1}"] \arrow[dr, "(j^{X_\bullet}_{t-r-1})_\#"'] & \pi_r C_{t-r-1} P^r \Omega^i X_\bullet & \\
 & & \Omega\pi^{\natural\,(i,r)}_{t-r-1,r-1} X_\bullet = \pi^{\natural\,(i,r)}_{t-r-1,r} X_\bullet \arrow[dr, "h^{(i,r)}_{t-r-1,r}", "\cong"'] \arrow[dd, hook, "\ell_{t-r-1,r}"] & Z_{t-r-1}\pi_{i+r} X_\bullet \arrow[u, hook, "\mathrm{inc}"'] \arrow[d, two heads, "\vartheta_{t-r-1}"] & \\
 & & & \pi_{t-r-1}\pi_{i+r} X_\bullet & \\
 & \pi_{r-1} C_{t-r+1} P^r \Omega^i X_\bullet \arrow[r, "(d_0^{t-r+1})_\#"] & \pi_{r-1} Z_{t-r} P^r \Omega^i X_\bullet \arrow[r, "(j_{t-r})_\#"] & \pi_{r-1} C_{t-r} P^r \Omega^i X_\bullet & \\
 & \vdots & \vdots & \vdots & \\
 & \pi_2 C_{t-2} P^r \Omega^i X_\bullet \arrow[r, "(d_0^{t-2})_\#"] & \pi_2 Z_{t-3} P^r \Omega^i X_\bullet \arrow[r, "(j_{t-3})_\#"] \arrow[d, two heads, "\hat{\vartheta}_{t-3}"] \arrow[dr, "(j^{X_\bullet}_{t-3})_\#"'] & \pi_2 C_{t-3} P^r \Omega^i X_\bullet & \\
 & & \Omega\pi^{\natural\,(i,r)}_{t-3,1} X_\bullet = \pi^{\natural\,(i,r)}_{t-3,2} X_\bullet \arrow[dr, "h^{(i,r)}_{t-3,2}"] \arrow[dd, hook, "\ell_{t-3,2}"] \arrow[ddd, bend right=60, "s^{(i,r)}_{t-3,1}"'] & Z_{t-3}\pi_{i+2} X_\bullet \arrow[u, hook, "\mathrm{inc}"'] \arrow[d, two heads, "\vartheta_{t-3}"] & \\
 & & & \pi_{t-3}\pi_{i+2} X_\bullet & \\
 & \pi_1 C_{t-1} P^r \Omega^i X_\bullet \arrow[r, "(d_0^{t-1})_\#"] & \pi_1 Z_{t-2} P^r \Omega^i X_\bullet \arrow[r, "(j_{t-2})_\#"] \arrow[d, two heads, "\hat{\vartheta}_{t-2}"] \arrow[dr, "(j^{X_\bullet}_{t-2})_\#"'] & \pi_1 C_{t-2} P^r \Omega^i X_\bullet & \\
 & & \Omega\pi^{\natural\,(i,r)}_{t-2,0} X_\bullet = \pi^{\natural\,(i,r)}_{t-2,1} X_\bullet \arrow[dr, "h^{(i,r)}_{t-2,1}"] \arrow[dd, hook, "\ell_{t-2,1}"] & Z_{t-2}\pi_{i+1} X_\bullet \arrow[u, hook, "\mathrm{inc}"'] \arrow[d, two heads, "\vartheta_{t-2}"] & \\
 & & & \pi_{t-2}\pi_{i+1} X_\bullet & \\
\pi_0 Z_t P^r \Omega^i X_\bullet \arrow[r, "(j_t)_\#"] \arrow[d, two heads, "\hat{\vartheta}_t"] \arrow[dr, "(j^{X_\bullet}_t)_\#"'] & \pi_0 C_t P^r \Omega^i X_\bullet \arrow[r, "(d_0^t)_\#"] & \pi_0 Z_{t-1} P^r \Omega^i X_\bullet \arrow[r, "(j_{t-1})_\#"] & \pi_0 C_{t-1} P^r \Omega^i X_\bullet \arrow[r] & \ldots \\
\pi^{\natural\,(i)}_{t,0} X_\bullet \arrow[dr, "h^{(i,r)}_t"] & Z_t\pi_i X_\bullet \arrow[u, hook, "\mathrm{inc}"'] \arrow[d, two heads, "\vartheta_t"'] & & & \\
 & \pi_t\pi_i X_\bullet \arrow[uuuur, "\partial^{(i,r)}_{t,0}"] & & &
\end{tikzcd}
$$

The differential $d^{r+1}_{t,i}\colon E^{r+1}_{t,i} \to E^{r+1}_{t-r-1,i+r}$ may then be described as a "relation" (cf. [18, § 3.1]) in the usual way:

Given a class $\alpha \in E^{r+1}_{t,i}$, choose a representative for it $a \in E^1_{t,i} = \pi_0 C_t P^r \Omega^i X_\bullet$. Since it is a cycle for $d^1_{t,i} = (j_{t-1})_\# \circ (d^t_0)_\#$, it lies in $Z_t\pi_i X_\bullet$ and thus represents an element $\bar{a} \in \pi_t\pi_i X_\bullet = E^2_{t,i}$. From the exactness of the middle row of (3.7) we see that $(d^t_0)_\#(a) \in \operatorname{Ker}((j_{t-1})_\#) = \Omega\pi^{\natural\,(i,r)}_{t-2,0}X_\bullet$, and in fact $(d^t_0)_\#(a)$ represents $\partial^{(i,r)}_{t,0}(\bar{a})$. Since $\hat{\vartheta}_{t-2}$ is surjective, we can choose $e_{t-2} \in \pi_1 Z_{t-2}P^r\Omega^i X_\bullet$ mapping to $(d^t_0)_\#(a)$. Because $d^2_{t,i}(\bar{a}) = h^{(i,r)}_{t-2,1} \circ \partial^{(i,r)}_{t,0}(\bar{a})$, as in the proof of Proposition 4.7 (though $h^{(i,r)}_{t-2,1}$ need no longer be an isomorphism!), we see that it is represented by $(j_{t-2})_*(e_{t-2})$. If $r = 1$, we are done. Otherwise, we know that $d^2_{t,i}(\bar{a}) = 0$, so we can choose e_{t-2} so that $(j_{t-2})_*(e_{t-2}) = 0$, using exactness of the third column of of (3.7). Again this implies that $e_{t-2} \in \operatorname{Ker}((j_{t-2})_\#) = \Omega\pi^{\natural\,(i,r)}_{t-3,1}X_\bullet$, and $d^3_{t,i}(\langle a\rangle)$ is represented by $h^{(i,r)}_{t-3,2}(e_{t-2})$. Moreover, we see from (3.7) that $s^{(i,r)}_{t-3,1}(e_{t-2}) = \partial^{(i,r)}_{t,0}(\bar{a})$, using the identification $\Omega\pi^{\natural\,(i,r)}_{t-2,0}X_\bullet = \pi^{\natural\,(i,r)}_{t-2,1}X_\bullet$.

Choosing a lift to $e_{t-3} \in \pi_2 Z_{t-3}P^r\Omega^i X_\bullet$, we may assume that $(j_{t-3})_*(e_{t-3}) = 0$, so $e_{t-3} \in \Omega\pi^{\natural\,(i,r)}_{t-4,2}X_\bullet$ and $s^{(i,r)}_{t-4,2}(e_{t-3}) = e_{t-2}$. Continuing in this way, we finally reach $e_{t-r-1} \in \Omega\pi^{\natural\,(i,r)}_{t-r-1,r-1}X_\bullet$ with $s^{(i,r)}_{t-r-2,r}(e_{t-r-1}) = e_{t-r}$, and so on, and see that $d^{r+1}_{t,i}(\langle a\rangle)$ is represented by $h^{(i,r)}_{t-r-1,r}(e_{t-r-1})$. Since (as in the proof of Proposition 4.7) $h^{(i,r)}_{t-r-1,r}$ is an isomorphism, we deduce that $d^{r+1}_{t,i}(\alpha)$ is as in (4.14). □

4.15 Remark From the exactness of (4.3) we have $\operatorname{Im}(\partial^{(i,r)}_{t,0}) = \operatorname{Ker}(s^{(i,r)}_{t-1,0})$, so the image of $d^{r+1}_{t,i}$ as described in (4.14) is $\operatorname{Ker}(\sigma^{r+1}_{t,i})$, where $\sigma^{r+1}_{t,i} := (s^{(i,r)}_{t-1,0}) \circ (s^{(i,r)}_{t-2,1}) \circ (s^{(i,r)}_{t-3,2}) \circ \cdots \circ (s^{(i,r)}_{t-r,r-1})$. Therefore, $E^{r+1}_{t+r-1,i+r}$ embeds naturally in $\operatorname{Im}(\sigma^{r+1}_{t,i})$.

4.16 Corollary *Every Reedy fibrant simplicial Postnikov r-stem has a well-defined $(r+1)$-truncated spiral spectral sequence. If $\mathcal{Q}_\bullet = \mathcal{P}[r]X_\bullet$ for some simplicial space $X_\bullet$, this truncated spectral sequence coincides with the $(r+1)$-truncation of the Bousfield-Friedlander spectral sequence for $X_\bullet$.*

Thus the bigraded homomorophism

$$d^{r+1} \circ d^{r+1} \colon E^r_{t,i} \to E^{r+1}_{t-2r-2,i+2r} \qquad (t \geq 2r+2, i \geq 0)$$

serves as the first obstruction to the realizablity of the simplicial Postnikov r-stem $\mathcal{Q}_\bullet$ by a simplicial space $X_\bullet$.

5 A cosimplicial version

There are actually four variants of the above spectral sequence which we might consider, for a simplicial or cosimplicial object over simplicial or cosimplicial sets. The case of bicosimplicial sets is in principle strictly dual to that of bisimplicial sets, but because the category of cosimplicial *sets* has no (known) useful model category structure, we must restrict to bicosimplicial abelian groups – or equivalently, ordinary double complexes. Thus the main new case of interest is that of cosimplicial simplicial sets, or *cosimplicial spaces*.

5.1 The spectral sequence of a cosimplicial space. If $X^\bullet \in c\mathcal{S}_*$ is a fibrant cosimplicial pointed space with total space $\operatorname{Tot} X^\bullet$, there are various constructions for the homotopy spectral sequence of $X^\bullet$:

(a) Using the tower of fibrations for $(\operatorname{Tot}_n X^\bullet)_{n=0}^\infty$ (cf. [16, X,§ 6]).

(b) Using "relations" on the normalized cochains $N^n\pi_t X^\bullet := \pi_t X^n \cap \operatorname{Ker}(s^0) \cap \ldots \cap \operatorname{Ker}(s^{n-1})$ (cf. [18, § 7]).

(c) Using a cofibration sequence dualizing (3.4) (cf. [23, § 3]).

Bousfield and Kan showed that the result is essentially unique (see [18]). Since the main ingredient needed for to define the spiral exact couple is the diagram (3.7), we use the first approach:

5.2 Definition For any Reedy fibrant cosimplicial pointed space $X^\bullet \in c\mathcal{S}_*$, consider the fibration sequence

$$F_n X^\bullet \xrightarrow{j_n} \operatorname{Tot}_n X^\bullet \xrightarrow{p_n} \operatorname{Tot}_{n-1} X^\bullet , \tag{5.3}$$

where $\operatorname{Tot}_n X^\bullet := \operatorname{map}_{c\mathcal{S}_*}(\mathrm{sk}_n , X^\bullet)$ and the fibration p_n is induced by the inclusion of cosimplicial spaces $\mathrm{sk}_{n-1} \hookrightarrow \mathrm{sk}_n$.

The cokernel of $(j_n)_\# \colon \pi_* F_n X^\bullet \hookrightarrow \pi_* \operatorname{Tot}_n X^\bullet$ is called the n-th *natural (graded) cohomotopy group* of $X^\bullet$, and denoted by $\pi^n_{\natural *} X^\bullet$.

5.4 Remark We may identify $F_n X^\bullet$ with the looped normalized cochain object $\Omega^n N^n X^\bullet$, where

$$N^n X^\bullet := X^n \cap \operatorname{Ker}(s^0) \cap \ldots \cap \operatorname{Ker}(s^{n-1}) , \tag{5.5}$$

and $\pi_* N^n X^\bullet$ with $N^n \pi_* X^\bullet$ (see [16, X, Proposition 6.3]).

Moreover, the composite

$$\pi_{*+1}\Omega^n N^n X^\bullet \cong \pi_{*+1} F_n X^\bullet \xrightarrow{(j_n)_\#} \pi_{*+1} \operatorname{Tot}_n X^\bullet \xrightarrow{\partial_n} \pi_* F_{n+1} X^\bullet \cong \pi_* \Omega^{n+1} N^{n+1} X^\bullet$$

(where ∂_n is the connecting homomotphism for the (5.3)), may then be identified with the differential

$$\delta^n := \sum_{i=0}^{n} (-1)^i d^i \; : \; N^n \pi_* X^\bullet \to N^{n+1} \pi_* X^\bullet \; , \tag{5.6}$$

for the normalized cochain complex $N^* \pi_* X^\bullet$, so that

$$\mathrm{Ker}\,(\delta^n) / \,\mathrm{Coker}\,(\delta^{n+1}) \cong \pi^n \pi_* X^\bullet \tag{5.7}$$

(cf. [16, X, § 7.2]).

5.8 Proposition *For any pointed cosimplicial space* $X^\bullet$ *there is a natural* spiral *long exact sequence:*

$$\begin{aligned} \ldots \to \Omega \pi^{n-1}_{\natural *} X^\bullet \xrightarrow{s^n} \pi^n_{\natural *} X^\bullet \xrightarrow{h^n} \pi^n \pi_* X^\bullet \xrightarrow{\partial^n} \Omega \pi^{n-2}_{\natural *} X^\bullet \\ \xrightarrow{s^{n-1}} \pi^{n-1}_{\natural *} X^\bullet \to \ldots \to \pi^0_{\natural *} X^\bullet \xrightarrow{\cong} \pi^0 \pi_* X^\bullet \end{aligned} \tag{5.9}$$

Proof By choosing a fibrant replacement in the model category of cosimplicial simplicial sets defined in [16, X, § 5], if necessary, we may assume that $X^\bullet$ is Reedy fibrant. We then obtain a commutative diagram as in (3.7) with exact rows and columns:

(5.10)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathrm{Ker}\,(j_n)_* \arrow[r, hook] \arrow[d, hook] & B^{n+1} X^\bullet \arrow[r, two heads, "(j_n)_*"] \arrow[d, hook] & B^{n+1} \pi_* X^\bullet \arrow[r] \arrow[d, hook] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega \pi^{n-1}_{\natural *} X^\bullet \arrow[r, hook, "\ell_{n-1}"] \arrow[d] \arrow[dr, "s_n"] & \pi_* \,\mathrm{Tot}_n X^\bullet \arrow[r, "(j_n^{X^\bullet})_\#"] \arrow[d, "\hat{\vartheta}_n"] & Z^n \pi_* X^\bullet \arrow[r, two heads] \arrow[d, "\vartheta_n"] & \mathrm{Coker}\ h^n \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathrm{Ker}\ h^n \arrow[r, hook] \arrow[d] & \pi^n_{\natural *} X^\bullet \arrow[r, "h_n"] \arrow[d] & \pi^n \pi_* X^\bullet \arrow[r, two heads] \arrow[d] & \mathrm{Coker}\ h^n \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

in which $B^{n+1} X^\bullet := \mathrm{Im}\,(j_{n+1})_\# \subseteq \pi_* \,\mathrm{Tot}_n X^\bullet$ and $B^{n+1} \pi_* X^\bullet := \mathrm{Im}\,(\delta^{n+1}) = \mathrm{Im}\,(\partial_{n+1} \circ (j_{n+1})_\#)$ are the respective coboundary objects.

The construction of the maps h^n, s^n, and ∂^n, and the proof of the exactness of (5.9), are then precisely as in § 3.6. □

5.11 Definition The *spiral n-system* of a pointed cosimplicial space $X^\bullet \in c\mathcal{S}_*$ is defined to be the collection of long exact sequences (5.9) for the Postnikov n-stem functor $\mathcal{P}[n]$ applied to $X^\bullet$, one for each k-window of $\mathcal{P}[n]X^\bullet$.

As in Definition 4.2, this may actually be defined for a cosimplicial Postnikov n-stem $\mathcal{P}^\bullet$, not necessarily realizable as $\mathcal{P}^\bullet = \mathcal{P}[n]X^\bullet$.

By construction, the homotopy spectral sequence of a (fibrant) cosimplicial space $X^\bullet$, obtained as in (5.1), is associated to the spiral exact couple (5.9). The proofs of Proposition 4.7 and Theorem 4.13 use only the description of the spiral exact couple for $X_\bullet$ derived from (5.10), so by using (5.10) instead we can prove their analogues in the cosimplicial case, and show:

5.12 Theorem *The E_{r+2}-term of the homotopy spectral sequence for a cosimplicial space $X^\bullet$ is determined by the spiral r-system of $X^\bullet$.*

An analogue of Corollary 4.16 also holds, as well as:

5.13 Proposition *The differential $d_2^{t,i}\colon E_2^{t,i} \to E_2^{t+2,i+1}$ may be identified with $\partial^t_{(i,1)}\colon \pi^t\pi_i X^\bullet \to \Omega\pi^{t+2,0}_{\natural\,(i)}X^\bullet$.*

5.14 Examples As noted in the introduction, many commonly used spectral sequences arise as the spiral spectral sequence of an appropriate (co)simplicial space, so Theorems 4.13 and 5.12 allow us to extract their E^r- or E_r-terms from the appropriate spiral systems. For instance:

(a) Segal's homology spectral sequence (cf. [24]), the van Kampen spectral sequence (cf. [25]), and the Hurewicz spectral sequence (cf. [9]) are constructed using bisimplicial sets.

(b) The unstable Adams spectral sequences of [14, 17] and [8, §4], Rector's version of the Eilenberg-Moore spectral sequence (cf. [23]), and Anderson's generalization of the latter (cf. [2]) are all associated to cosimplicial spaces.

(c) The usual construction of the stable Adams spectral sequence for $\pi_*^s X \otimes \mathbb{Z}/p$ (cf. [1, §3]) uses a tower of (co)fibrations, rather than a cosimplicial space, but when X is finite dimensional, it agrees in a range with the unstable version for $\Sigma^N X$, so Theorem 5.12 applies stably, too.

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