HIGHER HOMOTOPY OPERATIONS AND THE REALIZABILITY OF HOMOTOPY GROUPS

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Abstract. We describe an obstruction theory for the realization of a Π-algebra – that is, a graded group G_* with a prescribed action of the primary homotopy operations – as the homotopy groups of some space. The obstructions consist of higher homotopy operations, for which we provide an explicit definition in terms of certain sequences of polyhedra. There is a similar theory for realizing morphisms between Π-algebras, and thus in particular for distinguishing different realizations of a fixed Π-algebra.

As an application we show that, for all primes p, the Π-algebra $\pi_*\mathbf{S}^r \otimes \mathbb{Z}/p$ cannot be realized.

1. introduction

Given a sequence of groups $G_* = \{G_k\}_{k=1}^{\infty}$ (abelian for $k > 1$), we ask if it can be realized as the homotopy groups of some space: $G_* \cong \pi_*\mathbf{X}$. This question has a long history, going back to J.H.C. Whitehead (see [47]; also [18], [19], [20], [28], [30], [40]). Of course, some additional structure must be imposed on G_*, since any such sequence of groups is realizable by a product of Eilenberg-Mac Lane spaces: thus Whitehead showed that any prescribed $\mathbb{Z}[G_1]$-module structure (=π_1-action) on the higher G_k's is realizable.

More generally, given a space $\mathbf{X}$, its homotopy groups $\pi_*\mathbf{X}$ support various homotopy operations (cf. [46, XI, §1]) – i.e., they constitute a Π-*algebra*: a graded group with an action of the primary homotopy operations (see §2.4 below). Our problem thus has two parts:

The first is algebraic – choosing a suitable Π-algebra structure for the groups G_*. This question may be highly non-trivial, given the complicated structure of the collection of all primary homotopy operations (which involve, inter alia, all the unstable homotopy groups of spheres, Hilton-Hopf invariants, etc.). We shall not be concerned with this algebraic problem here (but see remark 8.4 below).

The second is topological: we assume that we are given a full Π-algebra structure on G_*, and try to find an (or all) $\mathbf{X}$'s with $\pi_*\mathbf{X} \cong \mathbf{G}_*$ *as* Π-*algebras*. This is the question we wish to address; so we may may formulate our

Basic question: which Π-algebras are realizable, and in how many different ways?

To answer this question, we give an explicit definition of n-th order *higher homotopy operations* as subsets of certain track groups $[\Sigma^{n-1}\mathbf{X}, \mathbf{Y}]$ $(n \geq 2)$, depending only on homotopy classes of maps (see §5 below); the subset in question will be non-empty only if certain lower order operations vanish. The definition involves a sequence of convex "face-map" polyhedra, also known as *permutohedra*. We then have:

Theorem A: *Given a* Π-*algebra* G_*, *there is a sequence of higher homotopy operations (defined in sections 5 and 6), depending only on maps between wedges of spheres, and taking value in homotopy groups of spheres, such that* $G_* \cong \pi_*\mathbf{X}$ *for some space* $\mathbf{X}$ *if and only if all the operations vanish coherently.*

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Thus the obstruction theory we define for realizing Π-algebras is just a generalization (and formalization) of the well known fact that Toda brackets are an obstruction to attaching cells in constructing CW complexes.

The second part of the question is a special case of an analogous question on the realizability of a given morphism between two (realizable) Π-algebras, to which we give a similar answer in

Theorem B: *Given an (abstract) morphism* $\phi : \pi_*\mathbf{X} \to \pi_*\mathbf{Y}$ *of* Π*-algebras, there is a sequence of higher homotopy operations (defined in section 7), taking value in the homotopy groups of* $\mathbf{Y}$*, such that* $\phi = \pi_*(f)$ *for some map* $f : \mathbf{X} \to \mathbf{Y}$ *if and only if all the operations vanish.*

The two theorems above appear to be only of theoretical interest, since it is very difficult to calculate any specific secondary operations, not to speak of *all* higher order operations. However, one can in fact make use of the many calculations of Toda brackets in the literature to deduce the non-realizability of certain Π-algebras. As an example, we prove the following extension to all primes of Proposition 3.3 of [4]:

Theorem C: *For any prime p, the* Π*-algebra* $\pi_*\mathbf{S}^r \otimes \mathbb{Z}/|$ *cannot be realized for* $r \geq 4(p-1)$ *(for* $r \geq 6$, *if* $p = 2$*).*

(For a discussion of the precise meaning of the expression "the Π-algebra $\pi_*\mathbf{S}^r \otimes \mathbb{Z}/|$", see remark 8.4 below).

1.1. **notation and conventions.** $\mathbb{N}$ denotes as usual the natural numbers, and $\mathbb{R}$ the reals. $\mathcal{T}_*$ will denote the category of pointed connected CW complexes with base-point preserving maps, and by a *space* we shall always mean an object in $\mathcal{T}_*$, which will be denoted by a boldface letter: $\mathbf{X}$, $\mathbf{S}^n$. (This is no restriction on the realizability of Π-algebras, by [46, V, Thm 3.2]). In particular, $\mathbf{\Delta}[n]$ denotes the standard topological n-simplex in $\mathbb{R}^{\ltimes+\not\Bbbk}$.

The homotopy category of such spaces is denoted by $ho\mathcal{T}_*$; diagrams taking value in $ho\mathcal{T}_*$ will be distinguished by a preceeding superscript h − for example, ${}^h\mathbf{X}_\bullet \in \mathbf{ho}\mathcal{T}_*^{\Delta^{\mathbf{op}}}$ (see below).

The category of Π-algebras will be denoted Π-*Alg*.

1.2. **organization:** In section 2 we give some background on Π-algebras and their simplicial resolutions, and relate these to the realization problem for Π-algebras (§2.9). In section 3 we recall some facts on rectifying homotopy-commutative diagrams into strictly commutative ones. In section 4 we describe certain polyhedra, which are used in section 5 to define higher homotopy operations (§5.4). After further details on adding degeneracies (section 6), we summarize the relation to the realization problem in Theorem A (=Theorem 6.12).

The question of realizing morphisms of Π-algebras is dealt with in section 7, yielding Theorem B (=Theorem 7.15). Finally, in section 8 we apply the theory to deduce Theorem C (=Theorem 8.1).

1.3. **acknowledgents.** I wish to thank Gil Kalai for introducing me to permutohedra, and Emmanuel Dror-Farjoun, Jeff Smith and Bill Dwyer for several useful conversations. I would also like to thank Jim Stasheff for pointing out some connections with work in other fields.

It should be noted that Bill Dwyer, Dan Kan, and Chris Stover have developed a different approach to the realizability of Π-algebras, based on certain "Postnikov approximations" (cf. [12]).

2. Simplicial Π-algebras

As noted above, in order to give content to the question of realizing a given graded group G_* as $\pi_*\mathbf{X}$ for some space $\mathbf{X}$, we must impose some additional structure on it − namely, that of a Π-algebra. First, some definitions and notation for simplicial objects:

2.1. **simplicial objects.** We let Δ denote the category of ordered sequences $mathbfn = \langle 0, 1, \dots, n\rangle$ $(n \in \mathbb{N})$, with order-preserving maps, and Δ_∂ the subcategory having the same objects, but allowing only 1-1 morphisms (so in particular, morphisms from $mathbfn$ to $mathbfm$ exist only for $n \leq m$). $\Delta^{\mathbf{op}}$, $\Delta_\partial^{\mathbf{op}}$ are the opposite categories.

As usual (cf. [26, §2]), a *simplicial object* over any category $\mathcal{C}$ is a functor $X : \Delta^{\mathbf{op}} \to \mathcal{C}$ − i.e., a sequence of objects $\{X_n\}_{n=0}^{\infty}$ in $\mathcal{C}$, equipped with *face maps* $d_i : X_n \to X_{n-1}$ and *degeneracies*

$s_j : X_n \to X_{n+1}$, satisfying the simplicial identities ([26, §1.1]):

$$\begin{array}{llll} (i) & d_i \circ d_j = & d_{j-1} \circ d_i & \text{for } i < j \\ (ii) & d_i \circ s_j = & \begin{cases} s_{j-1} \circ d_i & \text{if } i < j \\ id & \text{if } i = j, j+1 \\ s_j \circ d_{i-1} & \text{if } i > j+1 \end{cases} \\ (iii) & s_j \circ s_i = & s_i \circ s_{j-1} & \text{for } j > i \end{array} \tag{2.2}$$

We let $\mathcal{C}^{\Delta^{\mathrm{op}}}$ denote the category of simplicial objects over $\mathcal{C}$.

Definition 2.3. An *n-simplicial object* over $\mathcal{C}$ is a sequence of objects $\{X_k\}_{k=0}^n$, with face and degeneracy maps as above in dimensions $\leq n$. If we denote by $\mathcal{C}^{\Delta^{\mathrm{op}}_{\mathbf{n}}}$ the category of such objects, there is an obvious truncation functor $\tau_n : \mathcal{C}^{\Delta^{\mathrm{op}}} \to \mathcal{C}^{\Delta^{\mathrm{op}}_{\mathbf{n}}}$.

We shall assume all our simplicial spaces – i.e., objects in $\mathcal{T}_*^{\Delta^{\mathrm{op}}}$ – are *proper*, in the sense that the degeneracy maps are inclusions of subcomplexes (so in particular cofibrations).

We let $\mathcal{C}^{\Delta^{\mathrm{op}}_{\partial}}$ denote the category of simplicial objects without the degeneracies, which we shall call Δ-*simplicial objects* and denote by $X_\bullet^{[\partial]}$, (with each $X_n^{[\partial]} \in \mathcal{C}$). When $\mathcal{C} = Set$, these have been called Δ-sets, *ss*-sets, or *restricted* simplicial sets (see [33, 21]). Similarly $\mathcal{C}^{\Delta^{\mathrm{op}}_{\partial,\mathbf{n}}}$ will be the category of *n*-Δ-*simplicial objects*. We have a forgetful functor $U_{\mathcal{C}} : \mathcal{C}^{\Delta^{\mathrm{op}}} \to \mathcal{C}^{\Delta^{\mathrm{op}}_{\partial}}$ (="omit degeneracies").

Definition 2.4. Recall that a Π-*algebra* is a graded group $G_* = \{G_k\}_{k=1}^\infty$ (abelian in degrees > 1), together with an action of the primary homotopy operations (=Whitehead products, as in [46, X, §7], and compositions) on it, satisfying the usual identities. See [2, §3] or [3, §2.1] for a more explicit description.

The *free* Π-algebras are those isomorphic to $\pi_* \mathbf{W}$, for some (possibly infinite) wedge of spheres $\mathbf{W}$: More precisely, let T be a graded set $\{T_j\}_{j=1}^\infty$, and let $\mathbf{W} = \bigvee_{\mathbf{j=1}}^\infty \bigvee_{\mathbf{x \in T_j}} \mathbf{S^j_x}$, where each $\mathbf{S}^j_x$ is a j-sphere. Then we say that $\pi_* \mathbf{W}$ is the *free* Π-*algebra generated by* T. We shall consider each element $x \in T_j$ to be an element of $\pi_* \mathbf{W}$, by identifying it with that generator of $\pi_j \mathbf{W}$ which represents the inclusion $\mathbf{S}^j_x \hookrightarrow \mathbf{W}$.

Remark 2.5. If we let Π denote the homotopy category of wedges of spheres, and $\mathcal{F} \subset \Pi$-Alg the full subcategory of free Π-algebras, then the functor $\pi_* : \Pi \to \mathcal{F}$ is an equivalence of categories.

Next, we must consider resolutions of a given Π-algebra; for more details, see [31, II, §4] or [5, §2]. For this we recall some well-known definitions:

Definition 2.6. A simplicial Π-algebra $A_\bullet$ is called *free* if for each $n \geq 0$ there is a graded set $T^n \subseteq A_n$ such that A_n is the free Π-algebra generated by T^n, and each degeneracy map $s_j : A_n \to A_{n+1}$ takes T^n to T^{n+1}.

Definition 2.7. We define a *free simplicial resolution* of a Π-algebra G_* to be a free simplicial Π-algebra $A_\bullet$, together with an augmentation $\varepsilon : A_0 \to G_*$ with

$$\varepsilon \circ d_0 = \varepsilon \circ d_1, \tag{2.8}$$

such that if for each $n \geq 0$ the underlying graded group of A_n is denoted $\{(A_n)_k\}_{k=1}^\infty$ then:

(a) the homotopy groups of the simplicial group $(A_\bullet)_k$ vanish in dimensions $n \geq 1$;
(b) the augmentation induces an isomorphism $\pi_0((A_\bullet)_k) \cong G_k$.

Such resolutions always exist, for any Π-algebra G_* – see [31, II, §4], or the explicit construction in [2, §4.3].

2.9. **realizing Π-algebras.** Now given any Π-algebra G_*, one could try to realize it as follows:

First, choose any free simplicial resolution $A_\bullet \to G_*$; since each A_n is a free Π-algebra, one can choose a wedge of spheres $\mathbf{W_n}$ such that $\pi_* \mathbf{W_n} \cong \mathbf{A_n}$, and thus obtain a simplicial space up to homotopy ${}^h\mathbf{W}_\bullet \in \mathbf{ho}\mathcal{T}_*^{\Delta^{\mathrm{op}}}$ by remark 2.5.

If ${}^h\mathbf{W}_\bullet$ can be rectified (§3.1) into a (strict) simplicial space $\mathbf{W}_\bullet$, its *realization* (or homotopy direct limit), the space $\mathbf{X} = \|\mathbf{W}_\bullet\|$, is constructed by making identifications in $\coprod_{n=0}^\infty \mathbf{W_n} \times \mathbf{\Delta[n]}$

according to the face and degeneracy maps of $\mathbf{W}_\bullet$ (cf. [35, §1]). There is a first quadrant spectral sequence with

$$E^2_{s,t} = \pi_s(\pi_t \mathbf{W}_\bullet) \Rightarrow \pi_{\mathbf{s+t}}\|\mathbf{W}_\bullet\| \tag{2.10}$$

(see [7, Thm B.5], [32]). Since $A_\bullet \to G_*$ is a resolution, we find $\pi_*\mathbf{X} = \pi_*\|\mathbf{W}_\bullet\| \cong \mathbf{G}_*$ as Π-algebras, so G_* in fact can be realized. (Clearly $\pi_*\mathbf{X} \cong \mathbf{G}_*$ as graded groups; and the natural map $\mathbf{W_0} \to \|\mathbf{W}_\bullet\|$ (cf. [8, XII, 2.3]) induces the Π-algebra morphism).

Remark 2.11. Let us suppose we only want to realize a given Π-algebra G_* in a range − that is, find a space $\mathbf{X}$ such that $\pi_i\mathbf{X} \cong \mathbf{G_i}$ for $k \leq i \leq \ell$, with the prescribed action of the relevant homotopy operations. (We can always assume that G_*, as well as $\mathbf{X}$, is $(k-1)$-connected).

In this case we can choose a free Π-algebra resolution $A_\bullet \to G_*$ (in which we may take each $A_n \cong \pi_*(\bigvee_\alpha \mathbf{S}^{r_\alpha})$ to be $(k-1)$-connected), and define a new free simplicial Π-algebra $B_\bullet$ by setting

$$B_n \;=\; \pi_*(\bigvee_{r_\alpha \leq \ell} \mathbf{S}^{r_\alpha}).$$

If we can find a simplicial space $\mathbf{W}_\bullet$ realizing $B_\bullet$, and set $\mathbf{X} = \|\mathbf{W}_\bullet\|$, then (2.10) implies that $\pi_i\mathbf{X} \cong \mathbf{G_i}$ for $i \leq \ell$ (though we have no control over the higher homotopy groups).

3. Rectifying homotopy-commutative diagrams

It is well known that diagrams in the homotopy category can be changed into strict diagrams of spaces if and only if the diagram can be made ∞-homotopy commutative (see, e.g., [6, Cor. 4.21 & Thm. 4.49]). We shall use the specific version of this statement described by Dwyer, Kan, and Smith in [11]; for this we need some definitions:

3.1. **rectifying a simplical space up to homotopy.** Let us assume we are given a simplical space up to homotopy ${}^h\mathbf{Y}_\bullet$ − that is, an object in $ho\mathcal{T}_*^{\Delta^{\mathbf{op}}}$, for which the simplicial identities (2.2) hold only up to homotopy − and would like to *rectify* it: i.e., to replace it by a (strict) simplicial space $\mathbf{Y}_\bullet \in \mathcal{T}_*^{\Delta^{\mathbf{op}}}$. For our purposes we need not worry over the precise definition of "replacing ${}^h\mathbf{Y}_\bullet$" (see [11, §2.2]); all we require is that upon applying the functor $\pi_*(-)$ to ${}^h\mathbf{Y}_\bullet$ and $\mathbf{Y}_\bullet$ we obtain isomorphic simplicial Π-algebras. (This is also refered to as *realizing* the diagram ${}^h\mathbf{Y}_\bullet$ in $\mathcal{T}_*$, but we already have too many kinds of realization in this paper). As a first approximation we shall start with rectification of the underlying Δ-simplicial space $U_{ho\mathcal{T}_*}({}^h\mathbf{Y}_\bullet)$.

Definition 3.2. In [9], Cordier and Porter define a simplicial category (cf. [31, II, §1]) $F_*\Delta^{\mathbf{op}}_\partial$, whose objects are the sequences $mathbfn \in \Delta$ (or $n \in \mathbb{N}$), and with the function complex $F_*\Delta^{\mathbf{op}}_\partial(\mathbf{mathbfn}, \mathbf{mathbfm})$ for each $n > m$, is the simplicial set whose k-simplices are:

$$(F_*\Delta^{\mathbf{op}}_\partial(\mathbf{mathbfn}, \mathbf{mathbfm}))_\mathbf{k} \quad = \quad \{\text{sequences of face maps from } mathbfn \text{ to } mathbfm\text{, arranged in arbitrary nestings of brackets, } (k+1)\text{-deep}\}$$

The face maps are d_i="remove i-th level brackets"; the degeneracy maps are s_j="repeat j-th level brackets". See [11, §1.4(iv)] for details.

Example 3.3. For example, the 1-simplices of the simplical set $F_*\Delta^{\mathbf{op}}_\partial(\mathbf{2}, \mathbf{0})$ consist of:

$$\{[[d_0][d_1]],\ [[d_0][d_0]],\ [[d_0d_1]],\ [[d_1][d_1]],\ [[d_0][d_2]],\ [[d_0d_2]],\ [[d_1][d_2]],\ [[d_1][d_1]],\ [[d_1d_2]]\ \}$$

where $[[d_0d_1]] = s_0([d_0d_1])$, e.g., $d_0([[d_0][d_0]]) = [d_0d_1]$, and $d_1([[d_0][d_0]]) = [d_0][d_0]$.

Definition 3.4. Given a Δ-simplicial space up to homotopy ${}^h\mathbf{Y}^{[\partial]}_\bullet \in \mathbf{ho}\mathcal{T}_*^{\Delta^{\mathbf{op}}_\partial}$, let $\mathcal{T}^\#_*$ denote the simplicial category having objects $mathbfn \in \mathbb{N}$, and function complexes

$$\underline{Hom}_{\mathcal{T}^\#_*}(mathbfn, mathbfm) = map(\mathbf{Y_n}, \mathbf{Y_m}).$$

(Recall from [26, §6.4] that for any two spaces $\mathbf{X}, \mathbf{Y}$, the *function complex* $map(\mathbf{X}, \mathbf{Y})$ is the simplicial set whose k-simplices are $map(\mathbf{X}, \mathbf{Y})_\mathbf{k} \stackrel{\mathbf{Def}}{=} \mathbf{Hom}(\mathbf{X} \times \mathbf{\Delta}[\mathbf{k}], \mathbf{Y})$.)

An ∞*-homotopy commutative* Δ*-simplicial space* over ${}^h\mathbf{Y}^{[\partial]}_\bullet$ is a simplicial functor $Y_\infty : F_*\Delta^{\mathbf{op}}_\partial \to \mathcal{T}^\#_*$, such that for any $\alpha \in F_*\Delta^{\mathbf{op}}_\partial(\mathbf{mathbfn}, \mathbf{mathbfm})$, the homotopy class of $Y_\infty(\alpha)$ is that prescribed by ${}^h\mathbf{Y}^{[\partial]}_\bullet$. Thus Y_∞ assigns a map $\mathbf{Y_n} \to \mathbf{Y_m}$ to each 0-simplex of $F_*\Delta^{\mathbf{op}}_\partial(\mathbf{mathbfn}, \mathbf{mathbfm})$, a homotopy between the appropriate maps to each 1-simplex, and higher homotopies to the higher dimensional simplices.

Theorem 3.5. [11, Thm 2.4]. *Any Δ-simplicial space up to homotopy ${}^{h}\mathbf{Y}^{[\partial]}_{\bullet} \in \mathbf{ho}\mathcal{T}_{*}{}^{\Delta^{\mathbf{op}}}$ can be rectified (in the sense of §3.1) if and only if there is a ∞-homotopy commutative Δ-simplicial space over ${}^{h}\mathbf{Y}_{\bullet}$.*

4. Polyhedra

There is a smaller combinatorial version of the category $F_{*}\Delta^{\mathbf{op}}_{\partial}$ defined above which can be used to describe the ∞-homotopy commutative Δ-simplicial spaces, by means of certain *face-map polyhedra* $P_n(\delta)$.

Since all the polyhedra we shall be dealing with here are (combinatorially isomorphic to) polytopes in $\mathbb{R}^{\ltimes}$, in this and the following sections we shall use the same notation for an abstract (combinatorial) polyhedron P and an (unspecified) geometric realization of P as such a polytope.

Definition 4.1. For each $0 < k \leq n$, let

$$D(k,n) \stackrel{Def}{=} \{0,1,\ldots,k\} \times \{0,1,\ldots,k,k+1\} \times \ldots \times \{0,1,\ldots,n\}$$

where we think of $(i_k,\ldots,i_n)$ as corresponding to the composition of face maps

$$d_{i_k} \circ \ldots \circ d_{i_n} : mathbfn \to mathbfk - mathbf1 \qquad \text{in} \quad \Delta^{\mathbf{op}}_{\partial}.$$

There is an equivalence relation $\sim$ on $D(k,n)$, generated by

$$(i_k,\ldots,i_j,i_{j+1},\ldots,i_n) \sim (i_k,\ldots,i_{j+1}-1,i_j,\ldots,i_n) \qquad \text{if} \quad i_j < i_{j+1} \tag{4.2}$$

(that is, $(i_k,\ldots,i_n) \sim (j_k,\ldots,j_n)$ if the corresponding morphisms in $\Delta^{\mathbf{op}}_{\partial}$ are equal: $d_{i_k}d_{i_{k+1}}\ldots d_{i_n} = d_{j_k}d_{j_{k+1}}\ldots d_{j_n}$ – cf. (2.2)(i)). We call $n-k$ the *length* of δ, and denote it by $|\delta|$. We call an equivalence class $\gamma \in D(j,m)/\sim$ a *subclass* of $\delta \in D(k,n)/\sim$, written $\gamma \subseteq \delta$, if $k \leq j \leq m \leq n$ and δ has some representative $(i_k,\ldots,i_j,\ldots,i_m,\ldots i_n)$ such that $\gamma = [(i_j,\ldots,i_m)]$.

Definition 4.3. To every equivalence class $\delta = [(i_k,\ldots,i_n)] \in D(k,n)/\sim$ we associate an $(n-k)$-dimensional abstract polyhedral complex $P_{n-k}(\delta)$, defined by induction on $d = |\delta| = n-k$, starting with $P_0([i_n]) =$ a point.

For each representative $(i_k,\ldots,i_n) \in D(k,n)$ of δ, and each partition of $(i_k,\ldots,i_n)$ into r consecutive blocks

$$\langle\, i_k,\ldots i_{\ell_1} \mid i_{\ell_1+1},\ldots i_{\ell_2} \mid \ldots | i_{\ell_{r-1}+1},\ldots i_n \,\rangle$$

of size $s_1 = (\ell_1 - k)$, $\ldots$, $s_r = (n - \ell_{r-1} + 1)$ respectively, $P_{n-k}(\delta)$ will have an $(s_1-1)\cdot(s_2-1)\cdot\ldots\cdot(s_r-1))$-dimensional sub-polyhedron

$$P_{s_1-1}([i_k,\ldots i_{\ell_1}]) \times P_{s_2-1}([i_{\ell_1+1},\ldots i_{\ell_2}]) \times \ldots \times P_{s_r-1}([i_{\ell_{r-1}+1},\ldots i_n]). \tag{4.4}$$

We denote by $P_n(\delta)^{(k)}$ the union of all sub-polyhedra of $P_n(\delta)$ of dimension $\leq k$.

In particular, if $r = n$ we see that $P_{n-k}(\delta)$ has $(n-k+1)!$ vertices, corresponding to the different ways of decomposing the composite face map $\delta : mathbfn \to mathbfk$. All $P_d(\delta)$'s of the same dimension d are isomorphic.

Example 4.5. The 1-face-map polyhedron $P_1(\delta)$ is isomorphic to a 1-simplex $\mathbf{\Delta}[1]$, and the 2-face-map polyhedron $P_2(\delta)$ is a hexagon, as in Figure 1:

The 3-face-map polyhedron $P_3(\delta)$ is isomorphic to a truncated cuboctohedron, whose facets consist of 8 hexagons $P_2(\gamma)$ and 6 squares $P_1(\beta) \times P_1(\beta')$. An example appears in Figure 2:

Definition 4.6. Given $n+1$ real numbers $a_0 < a_1 < \ldots < a_n$, the corresponding *permutohedron* $Pe_n = Pe_n(a_0,\ldots,a_n)$ is defined to be the convex hull in $\mathbb{R}^{\ltimes+\Vdash}$ of the $(n+1)!$ points $(\sigma(a_0),\sigma(a_2),\ldots,\sigma(a_n)) \in \mathbb{R}^{\ltimes+\Vdash}$, indexed by permutations $\sigma \in \Sigma_{n+1}$. (The concept goes back to Schoute at the beginning of the century; cf. [34, 15]). It is an n-dimensional polyhedron, and different choices of $a_0,\ldots,a_n$ yield combinatorially isomorphic polyhedra.

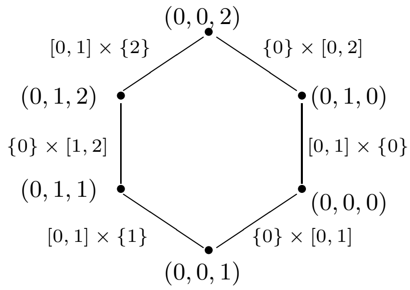


FIGURE 1. $P_2(\langle 0,1,2\rangle)$

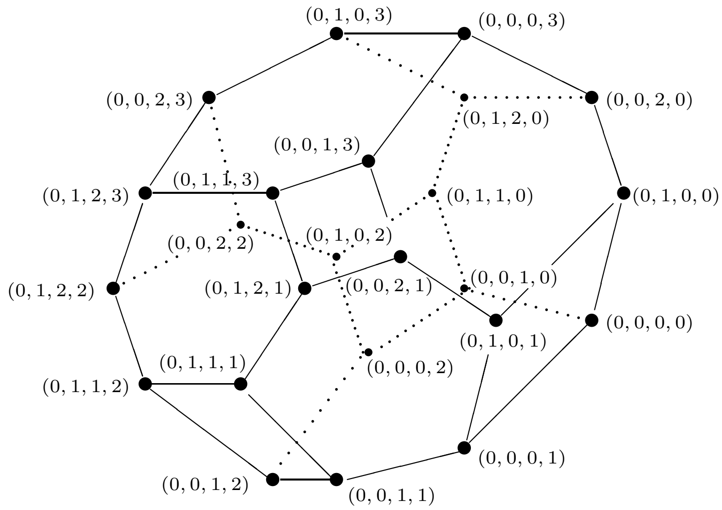


FIGURE 2. The 3-face-map polyhedron $P_3(\langle 0,1,2,3\rangle)$

Such polyhedra have figured in algebraic topology (under various names) in the context of iterated loop spaces, in the work of Milgram [27, §4], Stasheff [39, §11], Baues [1, III, (4.5)], and others. They also appear in category theory (e.g., [41]).

Lemma 4.7. *For any* $\delta \in D(k,k+n)/\sim$ *and* $a_0 < \dots < a_n$, *the polyhedra* $P_n(\delta)$ *and* $Pe_n(a_0,\dots,a_n)$ *are (combinatorially) isomorphic.*

Proof. Without loss of generality we may assume $\delta = [(0,1,\dots,n)]$, and define a 1-1 mapping Ψ between sequences $(i_0,\dots,i_n) \in D(0,n)$ representing δ (and thus corresponding to the vertices of $P_n(\delta)$) and permutations $\sigma \in \Sigma_{n+1}$ (corresponding to the vertices of Pe_n), as follows:

Set $\Psi(0,1,\dots,n) = (a_0,a_1,\dots,a_n) \in \Sigma_{n+1}$. Now any other $(i_1,\dots,i_{n+1}) \in D(1,n+1)$ equivalent under $\sim$ to $(0,1,\dots,n)$ may be obtained from it by a sequence of applications of the rule (4.2), and thus we may define Ψ inductively by the requirement that if $i_j < i_{j+1}$ then

$$\Psi(i_k,\dots,i_j,i_{j+1},\dots,i_n) = (j,j+1)\circ\Psi(i_k,\dots,i_{j+1}-1,i_j,\dots,i_n).$$

where $(j,j+1)\in\Sigma_{n+1}$ denotes the transposition. Note that vertices of $P_n(\delta)$ are connected by an edge if and only if they are indexed by $(i_k,\dots,i_j,i_{j+1},\dots,i_n)$ and $(i_k,\dots,i_{j+1}-1,i_j,\dots,i_n)$ respectively $(i_j<i_{j+1})$. Moreover, [13, Thm 1] shows that the corresponding vertices (under Ψ) are precisely those connected by edges in Pe_n.

Now recall from [13, Thm 2] that Pe_n may be described as the set of points $\bar{x} = (x_0, \dots, x_n) \in \mathbb{R}^{\ltimes+\nVdash}$ satisfying

$$\sum_{i \in J_k} x_i \quad \geq \quad \sum_{i=0}^{k-1} a_i \qquad \text{for every } J_k \subset \{0, 1, \dots, n\}, \quad |J_k| = k \leq n \tag{4.8}$$

$$\sum_{i=0}^{n} x_i \quad = \quad \sum_{i=0}^{n} a_i \;^{1} \tag{4.9}$$

Moreover, any codimension m sub-polyhedron P of Pe_n is defined by requiring equality in m of the inequalities in (4.8), corresponding to any choice of m subsets of the form $J_{k_1} \subset J_{k_2} \subset \dots \subset J_{k_m}$ (ibid.).

Setting $J'_1 = J_{k_1}$ and $J'_j = J_{k_j} - J_{k_{j-1}} = \{i_0, \dots i_\ell\}$ (with $\ell = k_j - k_{j-1} - 1$), with $y_0 = x_{i_0}, \dots, y_\ell = x_{i_\ell}$ and $b_0 = a_{k_{j-1}+1}, \dots, b_\ell = a_{k_j}$, one has $b_0 < b_1 < \dots b_\ell$ and

$$\sum_{i=0}^{\ell} y_i = \sum_{i=0}^{\ell} b_i \qquad \text{and} \qquad \sum_{i \in K_r} y_i \geq \sum_{i=0}^{r-1} b_i \quad \text{for every } K_r \subset \{0, 1, \dots, \ell\}, \; |J_r| = r \leq \ell.$$

Thus the sub-polyhedron P is isomorphic to a product of permutohedra

$$\prod Pe_{\ell-1}(b_0, \dots, b_\ell),$$

and a simple counting argument shows that these correspond to the codimension m sub-polyhedra of $P_n(\delta)$, so by induction (starting with $d = 1$) one finds that $P_n(\delta) \cong Pe_n$. □

Corollary 4.10. *(cf.* [27, Lemma 4.2]*)* $P_n(\delta)^{(n-1)}$ *is homeomorphic to* $\mathbf{S}^{n-1}$. □

4.10. **the connection to $F_*\Delta_\partial^{\mathbf{op}}$.** The description of ∞-homotopy commutative diagrams using $F_*\Delta_\partial^{\mathbf{op}}$ (§3.4) could also be stated in terms of appropriate polyhedra – in fact, if we disregard the degeneracies (as we may), the simplicial sets $F_*\Delta_\partial^{\mathbf{op}}(\mathbf{mathbfn}, \mathbf{mathbfm})$ are simplicial complexes, whose components correspond to the various $P_{n-m}(\delta)$ with $\delta \in D(m, n)/\sim$. It is not hard to see that the component corresponding to $P_{n-m}(\delta)$ is a "barycentric subdivision" of the face-map polyhedron, in which we have added a barycenter to each sub-polyhedron, corresponding to the map $\delta : mathbfn \to mathbfm$ (as a result, the products of polyhedra (4.4) are also subdivided).

For example, the component in $F_*\Delta_\partial^{\mathbf{op}}(\mathbf{3}, \mathbf{0})$ of $\delta = [(0, 1, 2)]$ is the subdivided hexagon of Figure 3 below.

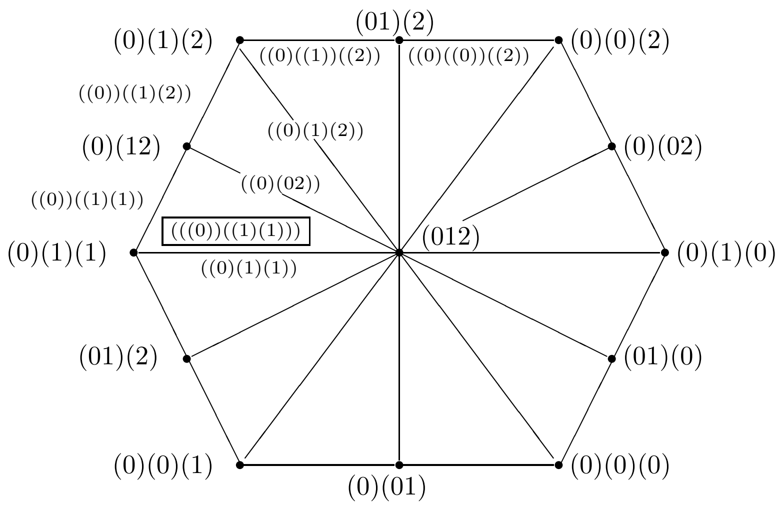


FIGURE 3. $F_*\Delta_\partial^{\mathbf{op}}(\mathbf{3}, \mathbf{0})$ (partial description)

[1] This condition is omitted in [13]

The reason for the subdivision is that definition 3.4 requires the choice of representatives not only for each arrow in Δ_∂ (i.e., each individual face map), but also for all possible composites. However, it is evidently more convenient to work with the more economical face-map polyhedra – as seen from a comparison of Figures 1 and 3.

Example 4.12. To illustrate this, let ${}^h\mathbf{Y}_\bullet^{[\partial]}$ be a Δ-simplicial space up to homotopy. In the face-map polyhedron approach to rectifying ${}^h\mathbf{Y}_\bullet^{[\partial]}$, initially we would choose representatives only for each "indecomposable" face map $d_i : \mathbf{Y_n} \to \mathbf{Y_{n-1}}$. On the other hand, the $F_*\Delta_\partial^{\mathbf{op}}$ approach of §3.4 requires also explicit choices for each composite face map – such as $d_1 d_2 : \mathbf{Y_n} \to \mathbf{Y_{n-2}}$ (which is the same as $d_1 d_1$).

However, once we choose a homotopy between the composites: $H : d_1 \circ d_1 \sim d_1 \circ d_2$ – i.e., a map $H : P_1(0,1) \times \mathbf{Y_n} \to \mathbf{Y_{n-2}}$ – we are designating the restriction of H to the barycenter of $P_1(0,1) \cong \mathbf{\Delta}[1]$ as our representative for the composite $d_1 d_2 = d_1 d_1$. Similarly, in Figure 3 the restriction of the higher homotopy $G : P_2(0,1,2) \ltimes \mathbf{Y_n} \to \mathbf{Y_{n-3}}$ to the barycenter of $P_2(0,1,2)$ will be the representative we select for the composite face map $d_1 d_1 d_2$.

5. Higher homotopy operations

We now wish to use the above face-map polyhedra to give a general definition of higher homotopy operations (for a given Δ-simplicial space up to homotopy):

First recall that if $\mathbf{X}, \mathbf{Y} \in \mathcal{T}_*$, their *half-smash* is $\mathbf{X} \ltimes \mathbf{Y} \stackrel{\mathbf{Def}}{=} (\mathbf{X} \times \mathbf{Y})/(\mathbf{X} \times \{*\})$; if $\mathbf{X}$ is a suspension, there is a (non-canonical) homotopy equivalence

$$\mathbf{X} \ltimes \mathbf{Y} \simeq \mathbf{X} \wedge \mathbf{Y} \vee \mathbf{X}.$$

Definition 5.1. Let ${}^h\mathbf{Y}_\bullet^{[\partial]} \in \mathbf{ho}\mathcal{T}_*^{\Delta_\partial^{\mathbf{op}}}$ be a Δ-simplicial space up to homotopy, and $\mathcal{C}$ be a set of equivalence classes $\delta \in D(n-k, n)/\sim$ (for various n, k) which is closed under taking subclasses (i.e., $\gamma \subseteq \delta \in \mathcal{C} \Rightarrow \gamma \in \mathcal{C}$). A *compatible collection for* $\mathcal{C}$ and ${}^h\mathbf{Y}_\bullet^{[\partial]}$ is a set $\{g^\delta\}_{\delta \in \mathcal{C}}$ of maps $g^\delta : P_{n-k}(\delta) \ltimes \mathbf{Y_n} \to \mathbf{Y_{k-1}}$ for each $\delta = [(i_k, \ldots, i_n)] \in \mathcal{C}$, satisfying the following condition:

If $\langle\ i_k, \ldots i_{\ell_1} \mid i_{\ell_1+1}, \ldots i_{\ell_2} \mid \ \ldots | i_{\ell_{r-1}+1}, \ldots i_n\ \rangle$ is a partition of some sequence representing δ into r blocks, with $\gamma_1 = [(i_k, \ldots i_{\ell_1})], \ldots,\ \gamma_r = [(i_{\ell_{r-1}+1}, \ldots i_n)]$, and we set $P = P_{\ell_1-k-1}(\gamma_1) \times P_{\ell_2-\ell_1}(\gamma_2) \ldots P_{n-\ell_{r-1}}(\gamma_r)$, then we require that $g^\delta|_{P \ltimes Yn}$ be the composite of the corresponding g^{γ_i}'s in the sense that

$$g^\delta(x_1, \ldots, x_r, y) = g^{\gamma_1}(x_1, g^{\gamma_2}(x_2, \ldots, g^{\gamma_r}(x_r, y) \ldots)) \tag{5.2}$$

for $x_i \in P_{\ell_i - \ell_{i-1}}(\gamma_i)$ and $y \in \mathbf{Y_n}$.

We further require that if $\delta = [i_j]$ is of length 0, then g^δ must be in the prescribed homotopy class of $[d_{i_j}] \in [\mathbf{Y_{j+1}},\ \mathbf{Y_j}]$. Thus in particular, for each vertex v of $P_k(\delta)$ indexed by $(i_{n-k}, \ldots, i_n)$, the map $g^\delta|_{\{v\} \times \mathbf{Y_n}}$ represents the class $[d_{i_{n-k}} \circ \ldots \circ d_{i_n}]$.

We shall usually be interested in such collections only up to a suitable homotopy relation: If $\{g^\delta\}_{\delta \in \mathcal{C}}$ and $\{\bar{g}^\delta\}_{\delta \in \mathcal{C}}$ are two compatible collections for $\mathcal{C}$, a *compatible homotopy* between them is a collection of homotopies $\{H^\delta : g^\delta \sim \bar{g}^\delta\}_{\delta \in \mathcal{C}}$ such that for each $0 \leq t \leq 1$, the maps $\{H_t^\delta\}_{\delta \in \mathcal{C}}$ constitute a compatible collection for $\mathcal{C}$.

Definition 5.3. For any (fixed) class $\delta = [(i_{n-k}, \ldots, i_n)] \in D(k, n)/\sim$, let $\mathcal{C}(\delta)$ denote the collection of all proper subclasses of δ. Then any compatible collection $\{g^\gamma\}_{\gamma \subset \delta}$ for $\mathcal{C}(\delta)$ induces a map $f = f^\delta : P_k(\delta)^{(k-1)} \ltimes \mathbf{Y_n} \to \mathbf{Y_{n-k-1}}$ (since all the faces of $P_k(\delta)^{(k-1)}$ are products of $P_j(\gamma)$'s for $\gamma \in \mathcal{C}(\delta)$, and condition (5.2) guarantees that the g^γ's agree on intersections).

Note that compatibly homotopic collections induce homotopic maps: $f \sim \bar{f}$.

Definition 5.4. Given ${}^h\mathbf{Y}_\bullet^{[\partial]}$ as above, for each $k \geq 2$ and $\delta \in D(n-k, n)$ the *k-th order homotopy operation* associated to ${}^h\mathbf{Y}_\bullet$ and δ is a subset $\langle\!\langle \delta \rangle\!\rangle$ of the track group $[\Sigma^{k-1}\mathbf{Y_n}, \mathbf{Y_{n-k-1}}]$, defined as follows:

Let $S \subseteq [P_k(\delta)^{(k-1)} \ltimes \mathbf{Y_n},\ \mathbf{Y_{n-k-1}}]$ be the set of homotopy classes of maps $f = f^\delta : P_k(\delta)^{(k-1)} \ltimes \mathbf{Y_n} \to \mathbf{Y_{n-k-1}}$ which are induced as above by some compatible collection $\{g^\gamma\}_{\gamma \subset \delta}$ for $\mathcal{C}(\delta)$.

Now choose a splitting

$$P_k(\delta)^{(k-1)} \ltimes \mathbf{Y_n} \cong \mathbf{S^{k-1}} \ltimes \mathbf{Y_n} \simeq \mathbf{S^{k-1}} \wedge \mathbf{Y_n} \vee \mathbf{Y_n} \tag{5.5}$$

and let $\langle\!\langle\delta\rangle\!\rangle \subseteq [\Sigma^{k-1}\mathbf{Y_n},\ \mathbf{Y_{n-k-1}}]$ be the image under the resulting projection of the subset $S \subseteq [P_k(\delta)^{(k-1)} \ltimes \mathbf{Y_n},\ \mathbf{Y_{n-k-1}}]$

Note that the projection of a class $[f] \in S$ on the other summand $[\mathbf{Y_n}, \mathbf{Y_{n-k-1}}]$ coming from the splitting (5.5) is of no interest, since it is just the class of δ (considered as a composite face map in ${}^h\mathbf{Y}_\bullet$).

5.6. **coherent vanishing.** It is clearly a necessary condition in order for the subset $\langle\!\langle\delta\rangle\!\rangle$ to be non-empty that all the lower order operations (for subclasses $\gamma \subset \delta$) *vanish* – i.e., contain the null class – because otherwise the various $g^\gamma : P_j(\gamma)^{(j-1)} \ltimes \mathbf{Y_m} \to \mathbf{Y_{m-j-1}}$ cannot even extend over the interior of $P_j(\gamma)$. A sufficent condition is that they do so *coherently*, in the following sense:

Definition 5.7. We assume that for each of the maximal proper subclasses $\gamma \subset \delta$ (with $|\gamma| = |\delta| - 1$) we have a compatible collection $\{g^{\beta,\gamma}\}_{\beta\subset\gamma}$ for $\mathcal{C}(\gamma)$ (§5.1), such that the induced map $f^\gamma : P_j(\gamma)^{(j-1)} \ltimes \mathbf{Y_m} \to \mathbf{Y_{m-j-1}}$ (§5.3) extends to a map $g^\gamma : P_j(\gamma) \ltimes \mathbf{Y_m} \to \mathbf{Y_{m-j-1}}$ (i.e., the higher homotopy operation associated to γ vanishes). We say that the collections $\{\{g^{\beta,\gamma}\}_{\beta\subset\gamma}\}_{\gamma\in\mathcal{C}(\delta)}$ are *ℓ-coherent* if $g^{\beta,\gamma} = g^{\beta,\gamma'}$ whenever β is a proper subclass of both γ and γ' of length $|\beta| \leq \ell$.

If the collections $\{g^{\beta,\gamma}\}_{\beta\subset\gamma}$ are $(|\delta| - 1)$-coherent, they in fact fit together to form a compatible collection for $\mathcal{C}(\delta)$, and thus induce an element of $\langle\!\langle\delta\rangle\!\rangle$.

Remark 5.8. The condition for 0-coherence just says that we have choosen (arbitrary) fixed representatives $f^{i_k} : \mathbf{Y_k} \to \mathbf{Y_{k-1}}$ in the prescribed homotopy classes of the face map d_{i_k}. Note that the resulting elements of $\langle\!\langle\delta\rangle\!\rangle$ are independent of these choices, since any homotopy $H^{i_k} : f^{i_k} \sim \bar{f}^{i_k}$ between two such choices can be used radially on small balls around each vertex of the $P(\gamma)$'s to define a homotopy between corresponding representatives of members of $\langle\!\langle\delta\rangle\!\rangle$.

More generally, let $\{g^\gamma\}_{\gamma\in\mathcal{C}}$ and $\{\bar{g}^\gamma\}_{\gamma\in\mathcal{C}}$ be two $(\ell-1)$-coherent compatible collections for $\mathcal{C}$. Assume that for some $\gamma_0 \in D(m-\ell, m)/\sim$ we have a homotopy $H : \bar{g}^{\gamma_0} \sim g^{\gamma_0}$ (rel $P_\ell(\gamma_0)^{(\ell-1)} \ltimes \mathbf{Y_m}$). Then we can (repeatedly) use the homotopy extension property for the DNR's $P_k(\gamma)^{(k-1)} \subset P_k(\gamma)$ (cf. [46, I, (1.9), §5]) to produce a compatible collection $\{\hat{g}^\gamma\}_{\gamma\in\mathcal{C}}$ such that $\hat{g}^\gamma = \bar{g}^\gamma$ for $|\gamma| \leq \ell$, $\gamma \neq \gamma_0$, and $\hat{g}^{\gamma_0} = g^{\gamma_0}$, as well as a compatible homotopy $\{H^\gamma : \bar{g}^\gamma \sim \hat{g}^\gamma\}_{\gamma\in\mathcal{C}}$ extending H.

5.9. **obstructions to coherence.** Note that any $\delta \in D(k,n)/\sim$ has a canonical representative $(i_k, i_{k+1}, \dots, i_n) \in D(k,n)$ with $i_k < i_{k+1} < \dots < i_n$. This allows us to define a linear ordering $\prec$ on all $\delta \in \bigcup_{n,k} D(k,n)/\sim$ – first by $|\delta| = (n-k)$, then by k, and then by left lexicographic ordering (say) on the canonical representatives.

Given $(\ell-1)$-coherent compatible collections $\{\{g^{\beta,\gamma}\}_{\beta\subset\gamma}\}_\gamma$ as above, we have a sequence of obstructions to ℓ-coherence, defined as follows: order the maximal proper subclasses of δ under $\prec$

$$\gamma_1 \prec \gamma_2 \prec \dots \gamma_s \prec \gamma_{s+1} \prec \dots \prec \gamma_r,$$

and assume by induction on $s \geq 1$ that we have ℓ-coherence for $\{\{g^{\beta,\gamma_i}\}_{\beta\subset\gamma_i}\}_{i=1}^s$. Next, consider all the classes β of length ℓ such that both $\beta \subset \gamma_{s+1}$ and $\beta \subset \gamma_{k_i}$ for some $1 \leq k_i \leq s$, and order them: $\beta_1 \prec \beta_2 \prec \dots \beta_t$. Assume by a further induction on $m \geq 0$ that $\gamma_1, \dots \gamma_s, \gamma_{s+1}$ satisfy the ℓ-coherence condition with respect to $\{\beta_1, \dots \beta_m\}$.

Now let $\beta = \beta_{m+1} \in D(m-\ell, m)/\sim$, and $g^\beta = g^{\beta,\gamma_i}$ (by assumption, in does not matter which $1 \leq i \leq s$ we choose), $\bar{g}^\beta = g^{\beta,\gamma_{s+1}}$. Since we have $(\ell-1)$-coherence, $\bar{g}^\beta|_{P_\ell(\beta)^{(\ell-1)}\ltimes\mathbf{Y_m}} = g^\beta|_{P_\ell(\beta)^{(\ell-1)}\ltimes\mathbf{Y_m}}$, and since $P_\ell(\beta)^{(\ell-1)} \cong \mathbf{S}^{\ell-1}$ and $P_\ell(\beta) \cong C\mathbf{S}^{\ell-1}$ (Lemma 4.7), g^β and $\bar{g}^\beta$ together define an map $\varphi_{s+1} : \Sigma\mathbf{Y_m} \to \mathbf{Y}_{\mathbf{m}-\ell-\mathbf{1}}$, which is nullhomotopic if and only if $g^\beta \sim \bar{g}^\beta$ rel $P_\ell(\beta)^{(\ell-1)} \ltimes \mathbf{Y_m}$.

But by remark 5.8, in this case we have a compatible homotopy of $\{g^{\beta,\gamma_{s+1}}\}_{\beta\subset\gamma_{s+1}}$ with a compatible collection $\{\hat{g}^\beta\}_{\beta\subset\gamma_{s+1}}$. If we now replace the given collection $\{g^{\beta,\gamma_{s+1}}\}_{\beta\subset\gamma_{s+1}}$ by $\{\hat{g}^\beta\}_{\beta\subset\gamma_{s+1}}$, we find that the new $\{g^{\beta,\gamma_1}\}, \dots, \{g^{\beta,\gamma_{s+1}}\}$ satisfy the ℓ-coherence condition with respect to $\{\beta_1, \dots \beta_{m+1}\}$ – completing the inner induction. When $m = t$ we have obtained ℓ-coherence for the new $\{\{g^{\beta,\gamma_i}\}_{\beta\subset\gamma_i}\}_{i=1}^{s+1}$.

Thus we have a sequence of obstruction classes in various track groups $[\varphi_i] \in [\Sigma\mathbf{Y_{m_i}}, \mathbf{Y}_{\mathbf{m_i}-\ell-\mathbf{1}}]$ $(i = 1, \dots, t)$, each defined only after we have chosen a nullhomotopy for the previous one, whose vanishing guarantees the ℓ-coherence of new collections compatibly homotopic to the given collections $\{\{g^{\beta,\gamma}\}_{\beta\subset\gamma}\}_{\gamma})$.

5.10. **the relation to Toda brackets.** We have given a new definition of second order homotopy operations here, and one would of course like to know how these are related to the usual Toda brackets:

Let there be given homotopy classes $\mathbf{Y_3}\xrightarrow{\gamma}\mathbf{Y_2}\xrightarrow{\beta}\mathbf{Y_1}\xrightarrow{\alpha}\mathbf{Y_0}$. Choosing representatives a, b, c for α, β, γ respectively, as well as nullhomotopies $F: a\circ b\simeq *$, $G: b\circ c\simeq *$, one defines a map $f:\Sigma\mathbf{Y_3}\to\mathbf{Y_0}$ as the "sum" of $a\circ G$ and $F\circ C(c)$ (as maps of $: C\mathbf{Y_3}\to\mathbf{Y_0}$). Varying the choices yields other representatives of the double coset

$$\langle\alpha,\beta,\gamma\rangle \;\in\; [\Sigma\mathbf{Y_3},\mathbf{Y_0}]\;/\;(\;(\Sigma\gamma)^{\#}[\Sigma\mathbf{Y_2},\mathbf{Y_0}]\;+\;\alpha_{\#}[\Sigma\mathbf{Y_3},\mathbf{Y_1}]\;) \tag{5.11}$$

(see [43, Lemma 1.1]). We assume for simplicity that all spaces in question are suspensions. The corresponding 3-Δ-simplicial space up to homotopy ${}^{h}\mathbf{Y}_{\bullet}^{[\mathbf{3},\partial]}\in\mathbf{ho}\mathcal{T}_{*}{}^{\Delta^{\mathbf{op}}_{\partial,\mathbf{3}}}$ has the d_0's equal to α, β, γ respectively, and all other face maps equal to the null class.

If we now use the choices of a, b, c, F, G made above, (and the null map to represent the null class, or homotopies between null maps), we obtain maps of the boundaries of the four 2-face-map polyhedra $P_2(\delta)^{(1)}$ with $\delta=\langle 0,1,2\rangle$, $\langle 0,1,3\rangle$, $\langle 0,2,3\rangle$, or $\langle 1,2,3\rangle$. It is easy to see all but the first will be trivial, while if H_* denotes the constant homotopy between null maps then the map of $P_2(\langle 0,1,2\rangle)^{(1)}\ltimes\mathbf{Y_3}$ into $\mathbf{Y_0}$ – representing the usual construction of the Toda bracket $\langle\alpha,\beta,\gamma\rangle$ – is described by the diagram in Figure 4.

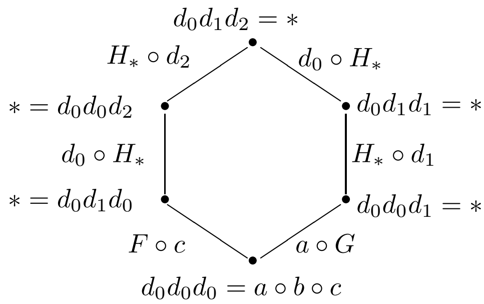


FIGURE 4. The Toda bracket

However, the construction of the map $P_2(\langle 0,1,2\rangle)^{(1)}\ltimes\mathbf{Y_3}\to\mathbf{Y_0}$ is "more homotopy-theoretic" in that it does not require specific representatives for any maps, including the null class (unlike the usual Toda bracket); as a result, we might expect it to have greater indeterminacy. This does not happen, in fact: by a slight variant of our previous notation, let $\langle\!\langle\alpha,\beta,\gamma\rangle\!\rangle\subseteq[\Sigma\mathbf{Y_3},\mathbf{Y_0}]$ denote the coset representing the secondary homotopy operation determined by ${}^{h}\mathbf{Y}_{\bullet}^{[\mathbf{3},\partial]}$. Then

Lemma 5.12. *For* $\mathbf{Y_3}\xrightarrow{\gamma}\mathbf{Y_2}\xrightarrow{\beta}\mathbf{Y_1}\xrightarrow{\alpha}\mathbf{Y_0}$ *as above we have*

$$\langle\alpha,\beta,\gamma\rangle \;=\; \langle\!\langle\alpha,\beta,\gamma\rangle\!\rangle\subseteq\;[\Sigma\mathbf{Y_3},\mathbf{Y_0}].$$

Proof. We showed above how the representative of the usual Toda bracket $\langle\alpha,\beta,\gamma\rangle$ determined by a, b, c, and appropriate nullhomotopies yields an element of $\langle\!\langle\alpha,\beta,\gamma\rangle\!\rangle$.

In general, to obtain a representative of $\langle\!\langle\alpha,\beta,\gamma\rangle\!\rangle$ one must choose maps $f_i:\mathbf{Y_1}\to\mathbf{Y_0}$ $(i=0,1)$, $g_i:\mathbf{Y_2}\to\mathbf{Y_1}$ $(i=0,1,2)$, and $h_i:\mathbf{Y_3}\to\mathbf{Y_2}$ $(i=0,\dots,3)$ to represent all face maps in ${}^{h}\mathbf{Y_\bullet}\in\mathbf{ho}\mathcal{T}_{*}{}^{\Delta^{\mathbf{op}}_{\partial,\mathbf{3}}}$, with all but $f_0\simeq a$, $g_0\simeq b$ and $h_0\simeq c$ being nullhomotopic (but not necessarily null!). In addition, we must choose homotopies $F: f_0\circ g_1\simeq f_0\circ g_0$, $G: g_0\circ h_1\simeq g_0\circ h_0$, $G': g_0\circ h_2\simeq g_1\circ h_0$, $G'': g_1\circ h_2\simeq g_1\circ h_1$, and so on. This yields a map $\mathbf{Y_3}\ltimes\mathbf{P_2}(\langle\mathbf{0},\mathbf{1},\mathbf{2}\rangle)^{(\mathbf{1})}\to\mathbf{Y_0}$ represented by the diagram:

To reduce this more general construction to the usual Toda bracket (as represented by Figure 4 above), we must choose nullhomotopies $H: f_1\simeq *$, $K: g_1\simeq *$, $L: h_1\simeq *$, and $M: h_2\simeq *$, to obtain a diagram as in Figure 6.

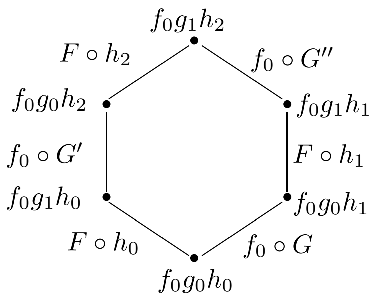


FIGURE 5. Generalized Toda brackets

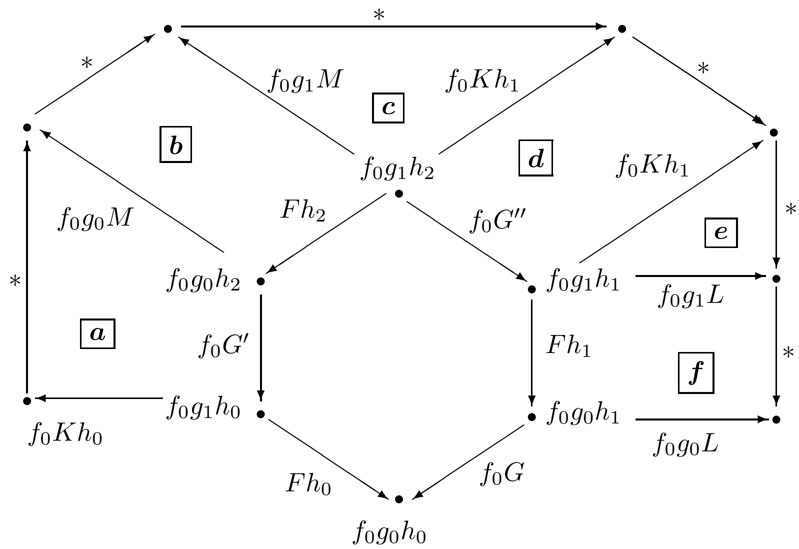


FIGURE 6

Our general secondary operation as described in Figure 5 yields an ordinary Toda bracket only if the 2-simplices marked $\boldsymbol{a} \dots \boldsymbol{f}$ in the diagram (sometimes shown as squares for typographical convenience) can be filled in by maps $\boldsymbol{\Delta}[2] \times \mathbf{Y_3} \to \mathbf{Y_0}$ extending the indicated maps on the boundaries.

For region $\boldsymbol{b}$ one may define $H : \mathbf{Y_3} \times \mathbf{I} \times \mathbf{I} \to \mathbf{Y_0}$ by $H(y, s, t) = F(M(y, s), t)$, and similarly for regions $\boldsymbol{c}, \boldsymbol{e}, \boldsymbol{f}$. On the other hand, in order to fill region $\boldsymbol{a}$, one obtains a certain non-standard "Toda bracket" $\varphi \in [\Sigma\mathbf{Y_3}, \mathbf{Y_1}]$ (which is the sum of $g_0 \circ M : C\mathbf{Y_3} \to \mathbf{Y_1}$, $G' : \mathbf{Y_3} \times \mathbf{I} \to \mathbf{Y_1}$, and $K \circ Ch_0 : C\mathbf{Y_3} \to \mathbf{Y_1}$), and $\boldsymbol{a}$ can be filled in if and only if $f_0 \circ \varphi \simeq *$. Similarly for region $\boldsymbol{d}$. Thus the indeterminacy of $\langle\!\langle \alpha, \beta, \gamma \rangle\!\rangle$ is contained in $\langle \alpha, \beta, \gamma \rangle + \alpha_*[\Sigma\mathbf{Y_3}, \mathbf{Y_1}] \subseteq \langle \alpha, \beta, \gamma \rangle$, by (5.11). □

One can similarly relate other constructions of secondary homotopy operations, such as those of Hardie (cf. [16, 17]) or Spanier [38], to the definition above; likewise for higher order operations (see [14], [45], and compare [37]).

Similar higher order operations (in an algebraic context) have also been considered by Kapranov and Voevodsky (e.g., [22, 23, 24]), in the connection with Mac Lane's categorical theory of coherence (see [25]).

6. ADDING DEGENERACIES

Let there be given a full simplicial space up to homotopy ${}^h\mathbf{W}_\bullet \in \mathbf{ho}\mathcal{T}_*^{\Delta^{\mathrm{op}}}$, and assume we have rectified the underlying Δ-simplicial space, so that $\mathbf{W}_\bullet^{[\partial]} \in \mathcal{T}_*^{\Delta_\partial^{\mathrm{op}}}$. We wish to add degeneracy maps $s_j : \mathbf{W_n} \to \mathbf{W_{n+1}}$ so as to obtain a full rectification of ${}^h\mathbf{W}_\bullet$.

6.1. **relative rectification.** Now Dwyer, Kan and Smith have a relative version of the theory refered to in §3, in which they assume given a category $\mathcal{D}$, a subcategory $\mathcal{C} \subset \mathcal{D}$ (with $Obj(\mathcal{C}) = Obj(\mathcal{D})$), and a diagram $J : \mathcal{C} \to \mathcal{T}_*$ which extends to a "diagram up to homotopy" $K : \mathcal{D} \to ho\mathcal{T}_*$.

Analogously to the $F_*\Delta^{\mathbf{op}}_{\partial}$ of §3.2, one can define a category $F^{\mathcal{C}}_*\mathcal{D}$ and thus a concept of a ∞-homotopy commutative $(\mathcal{D},\mathcal{C})$-diagram over (J,K). As in section 5, the existence of such a diagram turns out to be equivalent in our context to the coherent vanishing of a certain sequence of higher homotopy operations. In our case it is most convenient to assume by induction that we have succeeded in adding (strict) degeneracies to $\mathbf{W}^{[\partial]}_{\bullet}$ through simplicial dimension n, and now wish to choose representatives for the maps $[s_j] : \mathbf{W_n} \to \mathbf{W_{n+1}}$ in such a way as to satisfy the identities (2.2)(ii) for $d_i : \mathbf{W_{n+1}} \to \mathbf{W_{n+1}}$ and (2.2)(iii) for $s_i : \mathbf{W_{n-1}} \to \mathbf{W_n}$ (and the new s_j's).

Thus $\mathcal{D} = \Delta^{\mathbf{op}}_{\mathbf{n+1}}$ is the $(n+1)$-truncated subcategory of $\Delta^{\mathbf{op}}$, and $\mathcal{C} = \Delta^{\mathbf{op}}_{\mathbf{n}} \cup \Delta^{\mathbf{op}}_{\partial,\mathbf{n+1}}$ is the subcategory of $\mathcal{D}$ consisting of face-maps through simplicial dimension $n+1$, and degeneracies through simplicial dimension n. The additional morphisms in $\mathcal{D}$ are of course just the degeneracies $s_j : mathbfn \to mathbfn{+}mathbf1$.

6.2. **simplicial-morphism polyhedra.** We now define a collection of polyhedra which serve to construct the obstruction to this relative rectification problem:

Definition 6.3. For each $k,m \geq 0$, we denote by $E(k,m)$ the set of all sequences of face and degeneracy maps in $\Delta^{\mathbf{op}}$ which compose to yield a simplicial morphism $\phi : mathbfk \to mathbfm$. Thus a typical element may be represented by a sequence of sequences $\langle I^1, J^1, I^2, J^2, \dots, I^r, J^r\rangle$ where $I^a = (i^a_1, i^a_2, \dots, i^a_{t_a})$, $J^a = (j^a_1, j^a_2, \dots, j^a_{u_a})$ with $0 \leq i^1_1 \leq k$, $0 \leq i^1_2 \leq k-1, \dots$ and $0 \leq j^1_1 \leq k - t_1, \dots$. We write $|I^a| = t_a$, $|J^a| = u_a$ for the lengths of the sequences. I^1 and J^r are allowed to be empty.

This corresponds to the compositition of simplicial morphisms $s_{J^r} \circ d_{I^r} \circ \dots \circ s_{J^1} \circ d_{I^1}$, where $d_{I^a} = d_{i^a_{t_a}} \circ \dots d_{i^a_1}$ and and $s_{J^a} = s_{j^a_{u_a}} \circ \dots s_{j^a_1}$. By convention, the empty sequence $\langle\rangle$ corresponds to the identity map $id : mathbfk \to mathbfk = mathbfm$.

We denote by $E(k,m;n) \subset E(k,m)$ the set of such sequences which do not factor through $mathbfn{+}mathbf1$.

Example 6.4. $E(0,0) = \{d_\emptyset,\ d_0 \circ s_0,\ d_1 \circ s_0,\ d_0 \circ s_0 \circ d_0 \circ s_0, \dots\ d_0 \circ d_0 \circ s_0 \circ s_0, \dots\}$, while $E(0,0;1) = \{d_\emptyset,\ d_0 \circ s_0,\ d_1 \circ s_0,\ d_0 \circ s_0 \circ d_0 \circ s_0,\ d_1 \circ s_0 \circ d_0 \circ s_0,\ \dots\}$.

There is an equivalence relation $\sim$ on $E(k,m)$ (or $E(k,m;n)$) generated by the simplicial identies (2.2), and $E(k,m)/\sim$ is just $Hom_{\Delta^{\mathbf{op}}}(mathbfk, mathbfm)$ (and if $n \geq k,m$, this is the same as $E(k,m;n)/\sim$). Every class ϕ in $E(k,m)/\sim$ has a canonical representative of the form $\phi = s_J \circ d_I$ with $I = i_1 > i_2 > \dots > i_s$ and $J = j_1 < j_2 < \dots < j_t$.

Definition 6.5. The *simplicial-morphism polyhedra* $F(\phi,n)$ are defined analogously to the face-map polyhedra of §4.3: that is, for each simplicial morphism $\phi : mathbfk \to mathbfm$ in $\Delta^{\mathbf{op}}$ (i.e., $\phi \in E(k,m)/\sim$), with $k,n \leq n+1$, there is a polyhedron $F(\phi,n)$ whose vertices are indexed by $E(k,m;n)$. However, since these polyhedra are infinite-dimensional, in order to obtain higher order operations as in §5 we must filter them by finite-dimensional polyhedra $F_d(\phi,n) \subseteq F_{d+1}(\phi,n) \subseteq \dots F(\phi,n)$, defined as follows:

The vertices of $F_d(\phi,n)$ are indexed by those sequences in $E(k,m;n)$ of length $p \leq d$:

$$\Lambda = \langle \lambda_1, \lambda_2, \dots, \lambda_p\rangle = \langle s_{i^r_{u_r}}, \dots, s_{j^r_r},\ d_{i^r_{t_r}}, \dots, d_{i^r_1},\ \dots, s_{j^1_{u_1}}, \dots, s_{j^1_1},\ d_{i^1_{s_1}}, \dots, d_{i^1_1}\rangle$$

(that is, each λ_ℓ is either an s_j or a d_i) such that $\lambda_1 \circ \dots \circ \lambda_p = \phi$.

More generally, for each such sequence $\Lambda = \langle \lambda_1, \lambda_2, \dots, \lambda_p\rangle$ representing ϕ in $\Delta^{\mathbf{op}}$, with $p \leq d$, and each partition of Λ into r consecutive blocks:

$$\langle\ \lambda_1, \dots \lambda_{\ell_1}\ |\ \lambda_{\ell_1+1}, \dots \lambda_{\ell_2}\ |\ \dots |\lambda_{\ell_{r-1}+1}, \dots \lambda_n\ \rangle$$

of size $e_1 = (\ell_1 - k),\ \dots,\ e_r = (n - \ell_{r-1} + 1)$ respectively, $P_{n-k}(\delta)$ will have a sub-polyhedron

$$F_{e_1}(\lambda_k \circ \dots \circ \lambda_{\ell_1}, n) \times F_{e_2}(\lambda_{\ell_1+1} \circ \dots \circ \lambda_{\ell_2}, n) \times \dots \times F_{e_r}(\lambda_{\ell_{r-1}+1} \circ, \dots \circ \lambda_p, n]).$$

(The vertices are obtained when all $e_i = 1$).

Unfortunately the $F(\phi, n)$'s and $F_d(\phi, n)$'s do not have any simple complete combinatorial description like that of the face-map polyhedra (Lemma 4.7). In particular, as the following example shows, they are not generally convex polytopes, but only the union of such. Nevertheless, it is evident that one may describe an algorithm for constructing each $F_d(\phi, n)$, based on combinatorial information only.

Example 6.6. For example, for $\phi = d_0 : mathbf{1} \to \mathbf{0}$, $F_1(\phi, 1) = F_2(\phi, 1)$ consists of a single vertex (d_0), while $F_3(\phi, 1)$ appears in Figure 7.

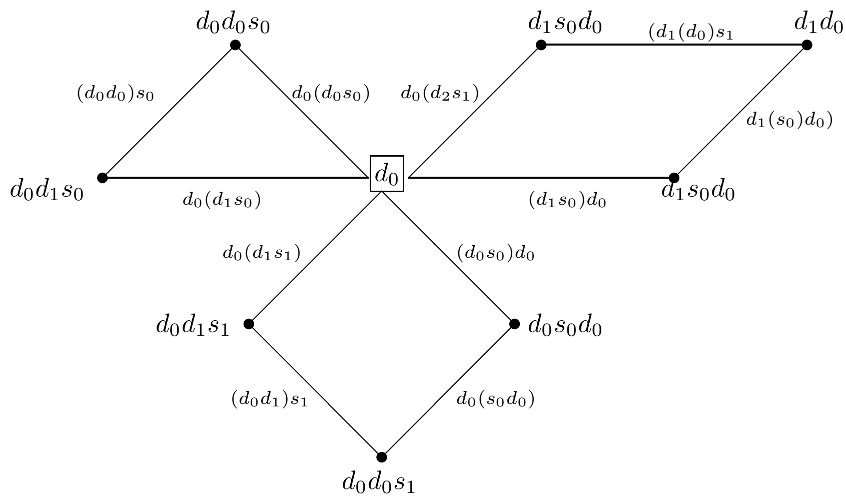


FIGURE 7. $F_3(\phi, 1)$

$F_5(\phi, 1) = F_6(\phi, 1)$ has $1 + 8 + 64 = 73$ vertices.

For $\psi = d_0 : \mathbf{2} \to mathbf{1}$ we find $F_3(\psi, 2)$ resembles $F_3(\phi, 1)$ above, but with 15 vertices, arranged in 4 quadrangles and one triangle (with a common vertice).

6.7. **obstructions to adding degeneracies.** Let there be given a (strict) Δ-simplicial space $\mathbf{W}_\bullet^{[\partial]}$, for which we have specified homotopy classes $[\bar{s}_j] \in [\mathbf{W_n}, \mathbf{W_{n+1}}]$ for the degeneracies, extending $\mathbf{W}_\bullet^{[\partial]}$ to a full simplical space up to homotopy. (Actually, in the light of [36, Proposition A.1(iv)] for our purposes we do not care which classes $[s_j]$ are chosen; but in the context of §2.9 these classes are in fact given to us).

We assume by induction that we have already chosen representatives for the degeneracies on $\mathbf{W_m}$ for $m < n$ in such a way that $\tau_n\mathbf{W}_\bullet$ is a (strict) n-simplicial space, and we want to choose representatives $s_j : \mathbf{W_n} \to \mathbf{W_{n+1}}$ making $\tau_{n+1}\mathbf{W}_\bullet$ into a strict $(n+1)$-simplicial space $-$ all this without changing the d_i's.

Now the analogue of section 5 allows us to define higher homotopy operations based on the simplicial-morphism polyhedra, which will serve as obstructions to this relative rectification problem:

As in §5.9 there is a linear ordering $\prec$ on all $\phi \in \bigcup_{k,m\leq n} E(k,m)/\sim$ $-$ first by $|k-m|$, then by k, and then by left lexicographic ordering (say) on the canonical representative $s_J \circ d_I$ for ϕ.

For each n, the obstructions to "rectifying" the additional degeneracies $s_j : \mathbf{W_n} \to \mathbf{W_{n+1}}$ are constructed by a triple induction: first on the simplicial morphisms $\phi : mathbf{k} \to mathbf{m}$ $(k, m \leq n)$, ordered by $\prec$; then on the filtration dimension d on $F_d(\phi; n)$; and finally, on the dimension of the faces of $F_d(\phi; n)$.

Of course, we will need to ensure coherence between the choices of maps and homotopies made at the various stages, in a manner analogous to §5.6-5.9. In fact, because the faces of $F_d(\phi; n)$ are not (homeomorphic to) cells, in order to obtain higher homotopy operations taking value in subsets of homotopy groups it is necessary first to subdivide each face into sub-polyhedra which are combinatorially equivalent to cells (e.g., convex polyhedra), define the higher operations on these (suitably ordered), and then use coherence obstructions – also taking value in homotopy groups – to get coherent vanishing as in 5.9.

We do not attempt to provide full details here, both because they are much more complicated than the description in sections 4 & 5 (and in particular, lack an explicit combinatorial description), and because they will not be needed for applications in the stable range by Lemma 6.10 below (and possibly not at all – see Conjecture 6.9 below).

Summary 6.8. Given a simplicial space up to homotopy ${}^{h}\mathbf{W}_{\bullet}$, the obstructions to rectifying it have been defined in two stages:

First, we have the sequence of higher homotopy operations of §5.4, whose coherent vanishings (§5.6) are the condition for successively rectifying the n-Δ-simplicial spaces up to homotopy $\tau_n U_{ho\mathcal{T}_*}({}^{h}\mathbf{W}_{\bullet})$.

Then, for each possible choice of a Δ-simplicial space-rectification, we have the sequence of obstructions of §6.7 to adding degeneracies. If these vanish (coherently), this particular Δ-simplicial space-rectification can be extended to a rectification of the full simplicial space.

One could evidently combine the obstructions to rectification of ${}^{h}\mathbf{W}_{\bullet}$ into a single sequence, rather than first rectifying the underlying Δ-simplicial space. (At the n-th stage we would be trying to extend the given rectification of $\tau_n^h\mathbf{W}_{\bullet}$ to simplicial dimension $(n+1)$, including both the face maps $d_i : \mathbf{W_{n+1}} \to \mathbf{W_n}$ and the degeneracies $s_j : \mathbf{W_n} \to \mathbf{W_{n+1}}$). Our motivation for dealing with the Δ-simplicial space rectification first (in sections 4 & 5) is the following

Conjecture 6.9. *If ${}^{h}\mathbf{Y}_{\bullet} \in \mathbf{ho}\mathcal{T}_*^{\Delta^{\mathbf{op}}}$ is a simplicial space up to homotopy, and $\mathbf{Y}_{\bullet}^{[\partial]} \in \mathcal{T}_*^{\Delta_\partial^{\mathbf{op}}}$ is a Δ-simplicial space rectifying $U_{ho\mathcal{T}_*}({}^{h}\mathbf{Y}_{\bullet})$, then ${}^{h}\mathbf{Y}_{\bullet}$ itself may be rectified.*

A weaker form of the conjecture would state that if ${}^{h}\mathbf{Y}_{\bullet}$ is a "resolution up to homotopy" of a space $\mathbf{X}$ in the sense of §2.9, and $\mathbf{Y}_{\bullet}^{[\partial]}$ is a Δ-simplicial space rectification, then the homotopy colimit of $\mathbf{Y}_{\bullet}^{[\partial]}$ (see §7.1) is weakly equivalent to $\mathbf{X}$.

There is some evidence for the conjecture, both algebraic and other (cf. [36, Proposition A.1(iv)]). A discrete version of this is known to hold (see [21] and [33, Theorem 5.7]). In a future paper we intend to show that the weakened conjecture is true if the Π-algebra resolution of §2.9 is of a suitable form (that described in [2, §4.3]).

In addition, we have:

Lemma 6.10. *In order to realize a Π-algebra G_* in the stable range (see §2.11), it suffices to realize the underlying Δ-simplicial Π-algebra $U_{\Pi\text{-}Alg}A_{\bullet}$ of a resolution $A_{\bullet} \to G_*$.*

Proof. In the stable range a Π-algebra is just a (truncated) module over the stable homotopy ring $\pi_*^S\mathbf{S}^0$, and such modules form an abelian category. Thus a simplicial resolution $A_{\bullet} \to G_*$, suitably truncated, is equivalent to an ordinary chain-complex resolution (cf. [10, §1]), which may be obtained from $A_{\bullet}$ by taking the quotient of each A_n by the (free) submodule generated by the image of the degeneracies. Once this has been realized by a Δ-simplicial space $\hat{\mathbf{W}}_{\bullet}$ as in §2.11, a full simplicial space $\mathbf{W}_{\bullet}$ may be obtained from it by "adding all degeneracies" (see [2, §4.5.1] and [26, p. 95(i)]). □

As noted above, by Theorem 3.5 the coherent vanishing of the higher homotopy operations defined in §5.4 & 6.7 is a sufficient condition for rectification of the simplicial space up to homotopy ${}^{h}\mathbf{W}_{\bullet}$ of §2.9, and thus for realizing the Π-algebra G_*. Conversely, Dwyer, Kan, and Stover have shown;

Proposition 6.11. [12] *If $A_{\bullet} \to G_*$ is a free simplicial resolution of the Π-algebra G_*, and $G_* \cong \pi_*\mathbf{X}$ for some space $\mathbf{X}$, then $A_{\bullet}$ may be realized by a (strict) simplicial space.*

Thus the non-realizability of one resolution $A_{\bullet} \to G_*$ will imply the non-realizability of the Π-algebra G_*. We can sum up the results of this section in

Theorem 6.12. *Given a Π-algebra G_*, there is a space $\mathbf{X}$ such that $G_* \cong \pi_*\mathbf{X}$ if and only if all the sequence of higher homotopy operations defined in §5.4 & 6.7 (which depend only on maps between wedges of spheres, and take value in homotopy groups of spheres) vanish coherently.*

7. Realization of morphisms

We next consider the question of realizing a given morphism $\phi : G_* \to H_*$ between two Π-algebras – assuming, of course, that G_* and H_* are realizable. Thus we have $G_* \cong \pi_* \mathbf{X}$ and $H_* \cong \pi_* \mathbf{Y}$, and we want a map $f : \mathbf{X} \to \mathbf{Y}$ such that $\pi_*(f) = \phi$.

7.1. **realizing ϕ.** As in §2.9, choose any two free simplicial resolutions $A_\bullet \to G_*$ and $B_\bullet \to H_*$ which are realizable by (strict) augmented simplicial spaces $\mathbf{V}_\bullet \xrightarrow{\eta} \mathbf{X}$ and $\mathbf{W}_\bullet \xrightarrow{\varepsilon} \mathbf{Y}$. By the universal property of free resolutions (cf. [31, I,§1]), ϕ induces a morphism of simplicial Π-algebras $\tilde{\phi} : A_\bullet \to B_\bullet$ (unique up to simplicial homotopy) – and thus a simplicial morphism "up to homotopy" $[f_\bullet] : \mathbf{V}_\bullet \to \mathbf{W}_\bullet$ in $ho\mathcal{T}_*{}^{\Delta_\partial^{\mathbf{op}}}$ by remark 2.5.

The homotopy colimit, or *realization*, of a Δ-simplicial space $\mathbf{Z}_\bullet^{[\partial]}$ is constructed analogously to the realization of a simplicial space (§2.9) – without making the identifications imposed by the degeneracies; it too is denoted by $\|\mathbf{Z}_\bullet^{[\partial]}\|$. See [8, XII, §2]. By [36, Proposition A.1(iv)], for $\mathbf{Z}_\bullet^{[\partial]} = \mathbf{V}_\bullet^{[\partial]}$ (the underlying Δ-simplicial space of $\mathbf{V}_\bullet$), this space is homotopy equivalent to the realization $\|\mathbf{V}_\bullet\|$ of the full simplicial space – and thus to $\mathbf{X}$.

Thus if $[f_\bullet]$ can be represented by a (strict) map of Δ-simplicial spaces $f_\bullet : \mathbf{V}_\bullet^{[\partial]} \to \mathbf{W}_\bullet^{[\partial]}$, then its realization $\|f_\bullet\| : \|\mathbf{V}_\bullet^{[\partial]}\| \to \|\mathbf{W}_\bullet^{[\partial]}\|$ yields the required map $f : \mathbf{X} \to \mathbf{Y}$ with $\pi_*(f) = \phi$, since $\|\mathbf{V}_\bullet^{[\partial]}\| \to \mathbf{X}$ and $\|\mathbf{W}_\bullet^{[\partial]}\| \to \mathbf{Y}$ are (weak) homotopy equivalences.

In fact, we do not need to rectify $[f_\bullet]$ itself: Note that 2.2(i) and (2.8) imply that all composite face maps $d_{i_1} \circ d_{i_2} \circ \dots d_{i_n} : \mathbf{W_n} \to \mathbf{W_0}$ are equal after composing with $\varepsilon : \mathbf{W_0} \to \mathbf{Y}$; call the resulting map $\lambda_n : \mathbf{W_n} \to \mathbf{Y}$. Thus if we can produce maps $h_n : \mathbf{V_n} \to \mathbf{Y}$ with $[h_n] = [\lambda_n] \circ [f_n]$ such that

$$h_{n-1} \circ d_i = h_n \qquad \text{for all} \quad 0 \le i \le n, \tag{7.2}$$

then by the functoriality of the homotopy colimit we obtain a map $g : \|\mathbf{V}_\bullet^{[\partial]}\| \to \mathbf{Y}$ with $\pi_* g = \phi$.

7.3. **rectifying g.** We now wish to apply the theory for relative rectification refered to in §6.1: here $\mathcal{C} = \Delta_\partial^{\mathbf{op}}$, $\mathcal{D} = \Delta_\partial^{\mathbf{op}} \coprod \{*\}$ (where $\{*\}$ is the category with one object and its identity morphism), and the additional morphisms in $\mathcal{D}$ consist of a single $\gamma_n \in \mathcal{D}(mathbfn, *)$ for each $n \ge 0$, with $\gamma_{n-1} \circ d_i = \gamma_n$. Thus the diagram $K : \mathcal{D} \to ho\mathcal{T}_*$ extending J consists of a collection of homotopy classes $[h_n] \in [\mathbf{V_n}, \mathbf{Y}]$ satisfying (7.2) up to homotopy.

(Note that once we know such classes $[h_n]$ exist – which in our context is guaranteed by the existence of the Δ-simplicial map up to homotopy $[f_\bullet]$ – they are completely determined by $[h_0]$ and (7.2)).

In this case the existence of a ∞-homotopy commutative $(\mathcal{D}, \mathcal{C})$-diagram over (J, K) (cf. §6.1) turns out to be equivalent to the coherent vanishing of a sequence of certain higher homotopy operations, which can be explicitly described in this case in much simpler (and more explicit) terms than those of §6.7:

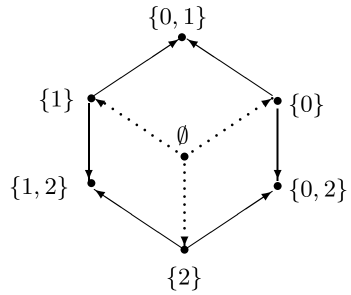
7.4. **cubical polyhedra.** If $A \subseteq B$ are two sets with $|B \setminus A| = r + 1$ (where $|S|$ denotes the cardinality of a set S), then the lattice of subsets $A \subseteq E \subseteq B$ is in one-to-one correspondence with the vertices of an $(r+1)$-dimensional cube, which we shall denote by $C^{r+1}(A, B)$, or simply $C(A, B)$. More generally, the k-dimensional faces of $C^{r+1}(A, B)$ are just the $C^k(D, E)$ where $A \subseteq D \subseteq E \subseteq B$ and $|E \setminus D| = k$. (We shall write v_E for $C^0(E, E)$, and $C(B)$ for $C(\emptyset, B)$).

Definition 7.5. If in the above description we require that the second inclusion – i.e., $E \subset B$ – be strict $(E \ne B)$, we obtain an r-dimensional sub-polyhedron $L^r(A, B) \subset C^{r+1}(A, B)$, which is the star of the vertex v_A in the boundary $\partial C^{r+1}(A, B)$ of the cube. We call this an *r-lattice polyhedron*.

The boundary of $L^r(A, B)$, denoted by $\partial L^r(A, B)$, corresponds to the subsets $A \subset D \subseteq E \subset B$ (both ends strict inclusions). It is homeomorphic to $\mathbf{S}^{r-1}$.

Example 7.6. The 2-dimensional lattice polyhedron $L^2(\emptyset, \{0, 1, 2\})$ is isomorphic to a hexagon; the solid lines indicate $\partial L^2(\emptyset, \{0, 1, 2\})$ (compare [42]):

The 3-dimensional lattice polyhedron $L^3(A, B)$ has 15 vertices, 26 edges, 17 square faces, and four 3-cube facets.



7.7. **higher homotopy operations.** We shall use the lattice polyhedra to define appropriate higher homotopy operations, analogous to those of section 5, as follows:

Let n be a (fixed) natural number; we associate to any set $A = \{i_1, i_2, \ldots, i_k\} \subseteq \{0, 1, \ldots, n\}$ (with $i_1 < i_2 < \ldots < i_k$) the composite face map $d_A \stackrel{Def}{=} d_{i_1} \circ d_{i_2} \circ \ldots \circ d_{i_k} : mathbfn \to mathbfn–mathbfk$, with $d_\emptyset = id$. If $A \subseteq B \subseteq \{0, 1, \ldots, n\}$, the simplicial identity 2.2(i) guarantees that we can factor d_B as $d_{(A,B)} \circ d_A$ for some composite face map $d_{(A,B)} = d_{j_1} \circ \ldots \circ d_{j_m}$ (where $j_1 < \ldots < j_m$). We set $T(A, B) \stackrel{Def}{=} \{j_1, \ldots, j_m\}$, so that $d_{(A,B)} = d_{T(A,B)}$. (In general $T(A, B) \neq B \setminus A$.) For example, if $A = \{0, 4\}$ and $B = \{0, 1, 4, 7\}$, then $d_B = d_0 d_1 d_4 d_7 = (d_0 d_5) \circ (d_0 d_4)$, so $T(A, B) = \{0, 5\}$.

Note that for every $D \subseteq E$ with $T(D, E) = T$ there is a canonical (combinatorial) isomorphism $\phi^{D,E} : C^m(D, E) \to C^m(\emptyset, T)$.

Definition 7.8. Let there be given a (strict) Δ-simplicial space $\mathbf{V}_\bullet$, a space $\mathbf{Y}$, a sequence of homotopy classes $[h_\bullet] : \mathbf{V}_\bullet \to \mathbf{Y}$ satisfying (7.2) up to homotopy. Then for each $n \in \mathbb{N}$, an ∂L^n-*compatible collection* for this data is a set of maps $g^{T,m} : C(T) \ltimes \mathbf{V_m} \to \mathbf{Y}$, defined for each subset $T \subset \{0, 1, \ldots n\}$ with $|T| \leq n - 1$ and $0 \leq m \leq n - |T|$. We require that $g^{\emptyset,m} : C^0(\emptyset) \times \mathbf{V_m} \to \mathbf{Y}$ be in the homotopy class of $[h_m] : \mathbf{V_m} \to \mathbf{Y}$ for each $0 \leq m \leq n$.

Now for every $\emptyset \neq D \subseteq E \subset \{0, \ldots, n\}$ with $T = T(D, E)$ we may define maps $\bar{g}^{(D,E)} : C(D, E) \ltimes \mathbf{V_m} \to \mathbf{Y}$ by setting

$$\bar{g}^{(D,E)} = g^{T,n-|D|} \circ (\phi^{D,E} \ltimes id_{\mathbf{V_m}}). \tag{7.9}$$

These $\bar{g}$'s are required to satisfy a compatibility condition analogous to (5.2): namely, if $\emptyset \neq D \subseteq D' \subseteq E' \subseteq E \subset \{0, \ldots, n\}$ and $i : C(D', E') \hookrightarrow C(D, E)$ is the inclusion, we ask that

$$\bar{g}^{(D,E)}|_{C(D',E') \ltimes \mathbf{V_{n-|D|}}} = \bar{g}^{(D',E')} \circ (i \ltimes d_{T(D,D')}). \tag{7.10}$$

As in §5.3, such a compatible collection induces a map $f : \partial C^{n+1}(\{0, \ldots, n\}) \ltimes \mathbf{V_n} \to \mathbf{Y}$, since (7.10) guarantees that after precomposing all the $\bar{g}^{D,E}$'s with $i \ltimes d_D$ they will agree on all common faces of $\partial C^{n+1}(\{0, \ldots, n\})$.

We similarly define an L^n-*compatible collection* by allowing $|T| = n$ (that is, $D = \emptyset$) in the definition above.

Definition 7.11. Fix a splitting $\partial L^n(\{0, \ldots, n\}) \ltimes \mathbf{V_n} \simeq \Sigma^{\mathbf{r-1}} \mathbf{V_n} \vee \mathbf{V_n}$ We define the n-th order *higher homotopy operation* associated with $[h_\bullet] : \mathbf{V}_\bullet \to \mathbf{Y})$, which we denote by $\langle\langle 0, \ldots, n \rangle\rangle \subseteq [\Sigma^{n-1} \mathbf{V_n}, \mathbf{Y}]$, to be the image under the projection associated to this splitting of the set of classes in $[\partial L^n(\{0, \ldots, n\}) \ltimes \mathbf{V_n}, \mathbf{Y}]$ represented by those maps $f : \partial L^n(\{0, \ldots, n\}) \ltimes \mathbf{V_n} \to \mathbf{Y}$ which are induced by such ∂L^n-compatible collections $\{g^{T,k}\}$ (compare §5.4). Again we say that $\langle\langle 0, \ldots, n \rangle\rangle$ *vanishes* if it contains the null class.

Note that in our case $\mathbf{V_n}$ is homotopy equivalent to a wedge of spheres, so that the operations defined are subsets of the homotopy groups of $\mathbf{Y}$ (or $\mathbf{X}$). Moreover, $\partial L^2 \cong P_2(\delta)$, so the second order homotopy operations defined here are again Toda brackets (§5.10).

Remark 7.12. Observe also that any nullhomotopy for such a map $f : \partial L^n(\{0, \ldots, n\}) \ltimes \mathbf{V_n} \to \mathbf{Y}$ – i.e., an extension of f to $F : L^n(\{0, \ldots, n\}) \ltimes \mathbf{V_n} \to \mathbf{Y}$ – yields a choice of maps $\bar{g}(E) : C(E) \ltimes \mathbf{V_n} \to \mathbf{Y}$ for all $E \subset B$ (including $\bar{g}^\emptyset : v_\emptyset \times \mathbf{V_n} \to \mathbf{Y}$, in the class of $[h_n]$) which are compatible (in the sense of (7.10)) with the given $\bar{g}^{D,E}$'s.

Moreover, this in fact gives us an L^n-compatible collection $\{g^{T,m}\}$, since (7.9) can be used to define the missing $g^{T,m}$'s – see §7.14 below.

7.13. **obstructions to rectification.** The obstructions to the rectification of $[h_\bullet] : \mathbf{V}_\bullet \to \mathbf{Y}$ – and thus to the realization of ϕ – are the higher homotopy operations $\langle\langle 0, \dots, n\rangle\rangle \subseteq [\Sigma^{n-1}\mathbf{V_n}, \mathbf{Y}]$: if these all vanish, then $[h_\bullet]$ is rectifiable, and thus ϕ can be realized.

Unlike the situation in §5.6, however, in this case the vanishing of $\langle\langle 0, \dots, n\rangle\rangle$ is also a *sufficient* condition for the non-emptiness of $\langle\langle 0, \dots, n+1\rangle\rangle$ – i.e., for the existence of a ∂L^{n+1}-compatible collection extending the given collection $\{g^{T,m}\}$ for L^n (which was itself extended from a ∂L^n-collection by means of a nullhomotopy, as explained in §7.12).

The reason for this is that $\partial L^{n+1}(\{0, \dots, n+1\})$ is the union of the $(n+2)$ lattice polyhedra $\{L^n(\{k\}, \{0, \dots, n+1\})\}_{k=0}^{n+1}$, with $L^n(\{k\}, \{0, \dots, n+1\}) \cap L^n(\{j\}, \{0, \dots, n+1\}) = L^{n-1}(\{k, j\}, \{0, \dots, n+1\})$, and so on.

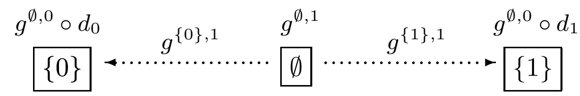
Now it follows from (2.2)(i) that $T(\{k\}, \{0, \dots, n+1\}) = \{0, 1, \dots, n\}$, so that

$$\phi^{\{k\},\{0,\dots,n+1\}} : L^n(\{k\}, \{0, \dots, n+1\}) \xrightarrow{\cong} L^n(\{0, \dots, n\}).$$

This, together with (7.9) and §7.12, implies that the L^n-compatible collection $\{g^{T,m}\}$ for $L^n(\{0, \dots, n\})$ is *already* a ∂L^{n+1}-compatible collection.

Note the lattice polyhedra described above provide a smaller combinatorial model than the mapping spaces needed in §7.3; however, the discussion of §4.10 carries over to the present situation.

Example 7.14. For $n = 1$ we have the following picture (where the maps at each vertex or face of $L^1(\{0, 1\})$ actually map $L^1(0, 1) \ltimes \mathbf{V_1} \to \mathbf{Y}$):



There is of course no first order obstruction to rectification, since we are assuming (7.2) holds up to homotopy – i.e., that the homotopy filling in $g^{\emptyset,0} \circ d_0 \coprod g^{\emptyset,0} \circ d_1 : \mathbf{S}^0 \ltimes \mathbf{V_1} \to \mathbf{Y}$ exists.

For $n = 2$ we then have the situation depicted in Figure 8, in which the maps on $\partial L^2(\{0, 1, 2\})$ are determined by precompositions of face maps with those of $L^1(\{0, 1, \})$, as noted. If the corresponding 2-order homotopy operation vanishes – i.e., if the prescribed map on $\partial L^2 \ltimes \mathbf{Y}$ extends to a map G on all of $L^2 \ltimes \mathbf{Y}$ – we obtain a choice of maps $g^{T,2}$ as indicated, by setting $g^{\emptyset,2} : \mathbf{V_2} \to \mathbf{Y}$ to be the restriction of G to the "barycenter" of L^2, and so on (cf. §7.12).

Thus Corollary 4.5 of [11] (the analogue of Theorem 3.5) yields the following

Theorem 7.15. *Given an morphism $\phi : \pi_*\mathbf{X} \to \pi_*\mathbf{Y}$ of Π-algebras, there is a map $f : \mathbf{X} \to \mathbf{Y}$ such that $\phi = \pi_* f$ if and only if the sequence of higher homotopy operations $\langle\langle 0, 1, \dots n\rangle\rangle$, defined in §7.13 and taking value in the homotopy groups of $\mathbf{Y}$, vanish for all $n \geq 2$.*

Remark 7.16. One should think of the above construction as comparing certain higher operations in $\pi_*\mathbf{X}$ – which necessarily vanish, since the augmented simplicial space $\mathbf{V}_\bullet \to \mathbf{X}$ is strict – with the corresponding operations (under $[f_\bullet]$) in $\pi_*\mathbf{Y}$, whose vanishing is thus a sufficient condition for ϕ to be realizable.

In the particular case where $G_* = H_*$ and ϕ is the identity, we are addressing the question of how many homotopy types can realize a given Π-algebra. In this case if $f : \mathbf{X} \to \mathbf{Y}$ is a weak equivalence (so a homotopy equivalence), corresponding higher homotopy operations must be mapped isomorphically under f_*.

Summary 7.17. Putting together the results of the last two sections, we obtain, for each space $\mathbf{X}$, a collection of non-vanishing higher homotopy operations, as follows:

1. Choose some fixed free simplicial resolution of Π-algebras $A_\bullet \to \pi_*\mathbf{X}$, together with a fixed realization by an augmented simplicial space $\mathbf{V}_\bullet \to \mathbf{X}$.
2. Consider all possible choices of coherently vanishing sequences of higher homotopy operations, thus obtaining all possible realizations $\mathbf{Y}$ of $\pi_*\mathbf{X}$ (by Proposition 6.11) – each with a chosen simplicial space $\mathbf{W}_\bullet \to \mathbf{Y}$ realizing $A_\bullet \to \pi_*\mathbf{X} (\cong \pi_*\mathbf{Y})$.

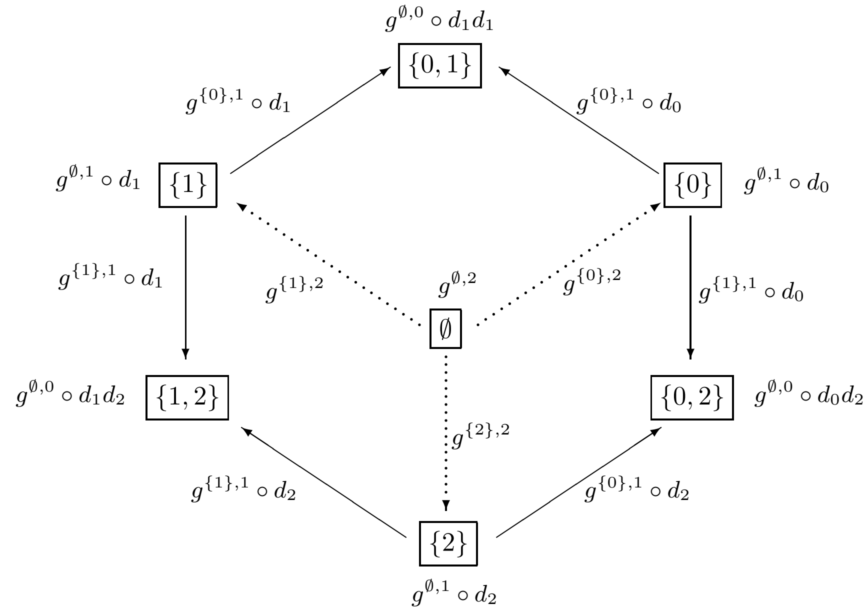


FIGURE 8. The second order operation $\langle\langle 0,1,2 \rangle\rangle$

3. For each such $\mathbf{W}_\bullet \to \mathbf{Y}$, try to construct a homotopy equivalence $f : \mathbf{Y} \to \mathbf{X}$ realizing the given isomorphism $\pi_*\mathbf{Y} \cong \pi_*\mathbf{X}$. If this is fails, we add the higher homotopy operation in $\pi_n\mathbf{X}$ which serves as an obstruction to the existence of f to our collection.

Thus we may interpret Theorem 7.15 as saying that the Π-algebra $\pi_*\mathbf{X}$, together with this collection of higher operations, constitute a complete set of "algebraic" invariants for the (weak) homotopy type of $\mathbf{X}$. This set of invariants is dual in some sense to the k-invariants of its Postnikov system.

Example 7.18. Consider the Π-algebra G_* with $G_r = \mathbb{Z}/\nvDash\langle\alpha\rangle$, $G_{r+1} = \mathbb{Z}/\nvDash\langle\alpha \circ \eta_\diagdown\rangle$, and $G_{r+2} = \mathbb{Z}/\nvDash\langle\beta\rangle$, where $\alpha \circ \eta_r \circ \eta_{r+1} = 0$, and all other groups are 0. (Here η_k is the suspended Hopf map, and we assume $r \geq 4$.)

The only higher order operation we need consider in counting realizations of G_* is $\langle\langle \eta_r, 2\iota_r, \alpha \rangle\rangle \subseteq G_{r+2} = \mathbb{Z}/\nvDash$, which by Lemma 5.12 coincides with the Toda bracket $\langle \eta_r, 2\iota_r, \alpha \rangle$, consisting of a single element (see (5.11)). Thus G_* has exactly two realizations in this case, depending on whether $\langle\langle \eta_r, 2\iota_r, \alpha \rangle\rangle = \{\beta\}$ or $\langle\langle \eta_r, 2\iota_r, \alpha \rangle\rangle = \{0\}$.

In the first case the realization $\mathbf{X}$ is just the $(r{+}2)$-nd stage in the Postnikov tower for $\Sigma^{r-1}\mathbf{RP}^2 \cup_{\eta^2\alpha} \mathbf{e}^{r+3}$ (cf. [2, §6.3]). In the second case the realization is the product of $\mathbf{K}(\mathbb{Z}/\nvDash, \diagdown + \nvDash)$ and the $(r+1)$-st stage in the Postnikov tower for $\Sigma^{r-1}\mathbf{RP}^2$.

8. An application

We now apply the above theory to a specific example (in which only second-order operations will figure) to prove:

Theorem 8.1. *For any prime p, the Π-algebra $\pi_*\mathbf{S}^r \otimes \mathbb{Z}/\imath$ cannot be realized for $r \geq 4(p-1)$ (for $r \geq 6$, if $p = 2$)*

Proof. Since the case $p = 2$ is contained in [4, Prop 3.3], we may fix a prime $p > 2$ and let $G_* = \pi_*\mathbf{S}^r \otimes \mathbb{Z}/\imath \in \Pi\text{-}Alg$. We shall show that G_* cannot even be realized in the stable range (see §2.11) $i \leq r + 2p - 3$ – so (as in the proof of Lemma 6.10) it suffices to produce an ordinary chain-complex resolution of modules over $\pi^s_*\mathbf{S}^0$. Such a resolution is given by:

$$\ldots \to \pi_*\mathbf{S}^{r+4p-6} \vee \mathbf{S}^{r+2p-3} \xrightarrow{(\alpha_1, p\iota)} \pi_*\mathbf{S}^{r+2p-3} \xrightarrow{\alpha_1} \pi_*\mathbf{S}^r \xrightarrow{p\iota} \pi_*\mathbf{S}^r \twoheadrightarrow G_*. \tag{8.2}$$

But $\langle \alpha_1, \alpha_1, p\iota \rangle = \{\alpha_2\} \subset \pi_{r+4p-5}\mathbf{S}^r$ by [43, p. 179 & Prop 3.4], since p annihilates $\pi_{r+4p-5}\mathbf{S}^r$, and thus by Lemma 5.12 also $\langle\langle \alpha_1, \alpha_1, p\iota \rangle\rangle = \{\alpha_2\} \neq 0$, so this serves as an obstruction to realizing any extension of resolution (8.2) to a full simplicial resolution of G_*. Thus by Proposition 6.11 and Theorem 3.5, G_* is not realizable. □

Remark 8.3. Since we only need to use it in the stable range, Proposition 6.11 can be proved very simply in our case:

Namely, assume that $G_* \cong \pi_*\mathbf{X}$, and (as in the proof of Lemma 6.10) it suffices to consider a Δ-simplicial space up to homotopy ${}^h\mathbf{W}_\bullet \to \mathbf{X}$ corresponding to (8.2) – in our example $\hat{\mathbf{W}}_0 = \mathbf{S}^r$, $\hat{\mathbf{W}}_1 = \mathbf{S}^r$, and so on. (All face maps but d_0 are 0). If we change the augmentation $\varepsilon : \mathbf{W_0} \to \mathbf{X}$ into a fibration, then $d_0 : \hat{\mathbf{W}}_1 \to \mathbf{W_0}$ (which in our example is the degree p map) factors through the fiber of ε, so $\varepsilon \circ d_0 = \varepsilon \circ d_1$ holds exactly. Proceeding in this way we obtain a strict Δ-simplicial space.

Remark 8.4. It should also be remarked that we are also faced here with the *algebraic* question posed at the beginning of section 1 – that is, what Π-algebra structure (if any) we can impose on $\pi_*\mathbf{S}^r \otimes \mathbb{Z}/p$. The problem is that for arbitrary $\alpha \in \pi_n\mathbf{S}^k$, the composition operation $\alpha^\# : \pi_k\mathbf{S}^r \to \pi_n\mathbf{S}^r$ need not be a homomorphism, so applying $-\otimes \mathbb{Z}/p$ to the graded abelian group $\pi_*\mathbf{S}^r$ will not always yield a well-defined operation $\alpha^\# \otimes \mathbb{Z}/p$.

To illustrate this problem, consider $\alpha \in \pi_n\mathbf{S}^k$ and $\beta \in \pi_k\mathbf{S}^r$. By [46, XI, (8.10)] we know that

$$\alpha^\#(3\beta) \equiv 3\alpha^\#(\beta) + 3(h_0(\alpha))^\#[\beta,\beta] \qquad \text{modulo } 2$$

where $h_0(\alpha) \in \pi_n\mathbf{S}^{2k-1}$ denotes the first non-trivial Hopf-Hilton invariant (cf. [46, XI, §8]). Thus if we denote by $\bar{\beta} \in \pi_k\mathbf{S}^r \otimes \mathbb{Z}/2$ the reduction modulo 2 of β, we see that the composition $\alpha^\#(\bar{\beta})$ is not well defined unless

$$(h_0(\alpha))^\#[\beta,\beta] \equiv 0 \qquad \text{modulo } 2 \tag{8.5}$$

Of course, this relation will not hold in general – for example, if ι_4 is a generator of $\pi_4\mathbf{S}^4$, then $[\iota_4,\iota_4] = 2\nu_4 - \alpha_4$ where ν_4, the Hopf map, generates an infinite cyclic summand in $\pi_7\mathbf{S}^4$ and α_4 generates a $\mathbb{Z}/12$ summand (cf. [44, IV]). Since ν_4 is an element of Hopf invariant 1, $h_0(\nu_4) = \iota_7$ (cf. [46, XI, (4.4) & (8.8)]), so (8.5) fails for $\alpha = \nu_4$, $\beta = \iota_4$.

This difficulty does not arise if we restrict attention to the stable range (see remark 2.11) – that is, if we try to find a space $\mathbf{X}$ with an isomorphism of truncated Π-algebras $\{\varphi_i : \pi_i\mathbf{X} \to \pi_i\mathbf{S}^r \otimes \mathbb{Z}/p\}_{i=1}^{2r-2}$. The proof of Theorem 8.1 shows this is impossible, regardless of how we may try to define $\pi_i\mathbf{S}^r \otimes \mathbb{Z}/p$ for $i \geq 2r-1$. This allows us to avoid the algebraic question as to the precise meaning (if any) of the expression $\pi_*\mathbf{S}^r \otimes \mathbb{Z}/p$, while still making sense of Theorem 8.1.

More generally, there is no problem in reducing modulo p any *abelian* Π-algebra – i.e., one in which all Whitehead products vanish (cf. [3, §3.2]) – since in this case all composition operations *are* homomorphisms. In particular, this will hold for $\pi_*\mathbf{S}^{2m+1}$ if p is odd by [46, X, (7.5)].

References

[1] H.J. Baues, "Geometry of loop spaces and the cobar construction", *Mem. AMS* **230**, AMS, Providence, RI, 1980.

[2] D. Blanc, "A Hurewicz spectral sequence for homology", *Trans. AMS* **318** (1990) No. 1, pp. 335-354.

[3] D. Blanc, "Abelian Π-algebras and their projective dimension", in M.C. Tangora, ed., *Algebraic Topology: Oaxtepec 1991* Contemp. Math. **146**, AMS, Providence, RI 1993.

[4] D. Blanc, "Operations on resolutions and the reverse Adams spectral sequence", *Trans. AMS* **342** (1994) No. 1, pp. 197-213.

[5] D. Blanc & C.R. Stover, "A generalized Grothendieck spectral sequence", in N. Ray & G. Walker, eds., *Adams Memorial Symposium on Algebraic Topology, Vol. 1*, Lond. Math. Soc. Lec. Notes Ser. **175**, Cambridge U. Press, Cambridge, 1992, pp. 145-161.

[6] J.M. Boardman & R.M. Vogt, *Homotopy Invariant Algebraic Structures on Topological Spaces*, Lec. Notes Math. **347**, Springer-Verlag, Berlin-New York, 1973.

[7] A.K. Bousfield & E.M. Friedlander, "Homotopy theory of Γ-spaces, spectra, and bisimplicial sets", in M.G. Barratt & M.E. Mahowald, eds., *Geometric Applications of Homotopy Theory, II*, Lec. Notes Math. **658**, Springer-Verlag, Berlin-New York, 1978, pp. 80-130.

[8] A.K. Bousfield & D.M. Kan, *Homotopy limits, Completions, and Localizations*, Lec. Notes Math. **304**, Springer-Verlag, Berlin-New York, 1972.

[9] J.-M. Cordier and T. Porter, "Vogt's theorem on categories of homotopy coherent diagrams", *Math. Proc. Camb. Phil. Soc.* **100** (1986) pp. 65-90.

[10] A. Dold, "Homology of symmetric products and other functors of complexes", *Ann. Math., Ser. 2* **68** (1958), pp. 54-80.

[11] W.G. Dwyer, D.M. Kan, & J.H. Smith, "Homotopy commutative diagrams and their realizations", *J. Pure Appl. Alg.* **57** (1989) No. 1, pp. 5-24.

[12] W.G. Dwyer, D.M. Kan, & C.R. Stover, "Obstructions to realizing Π-algebras", (in preparation).

[13] P. Gaiha & S.K. Gupta, "Adjacent vertices on a permutohedron", *SIAM J. Appl. Math.* **32** (1977) No. 2, pp. 323-327.

[14] H.H. Gershenson, "Higher composition products", *J. Math. Kyoto U.* **5** (1965), pp. 1-37.

[15] G.T. Guilbaud and P. Rosenstiehl, "Analyse algebrique d'un scrutin", *Math. et Sci. Humaines* **4** (1960), pp. 9-33.

[16] K.A. Hardie, "Derived homotopy constructions", *J. Lond. Math. Soc.* **35** (1960), pp. 465-480.

[17] K.A. Hardie, "On a construction of E.C. Zeeman", *J. Lond. Math. Soc.* **35** (1960), pp. 452-464.

[18] S.-T. Hu, "On the realizability of homotopy groups", *Pac. J. Math.* **1** (1951), pp. 583-602.

[19] K. Iwata, "On the realizability of Whitehead products", *Tôhoku Math. J., Ser. II* **16** (1964), pp. 203-208.

[20] K. Iwata & H. Miyazaki, "Remarks on the realizability of Whitehead products", *Tôhoku Math. J., Ser. II* **12** (1960), pp. 130-138.

[21] D.M. Kan, "Is an ss complex a css complex?", *Adv. in Math.* **4** (1970), pp. 170-171.

[22] M.M. Kapranov, "The permutoassociahedron, Mac Lane's coherence theorem, and assymptotic zones for the KZ equation", *J. Pure Appl. Alg.* **85** (1993) No. 2, pp. 119-142.

[23] M.M. Kapranov & V.A. Voevodsky, "Combinatorial-geometric aspects of polycategory theory: pasting schemes and higher Bruhat orders (list of results)", *Cah. Top. Géom. Diff. Cat.* **32** (1991) No. 1, pp. 11-27.

[24] M.M. Kapranov & V.A. Voevodsky, "∞-groupoids and homotopy types", *Cah. Top. Géom. Diff. Cat.* **32** (1991) No. 1, pp. 29-46.

[25] S. Mac Lane, "Natural associativity and commutativity", *Rice U. Stud.* **49** (1963) No. 4, pp. 28-46.

[26] J.P. May, *Simplicial Objects in Algebraic Topology*, U. Chicago Press, Chicago-London, 1967.

[27] R.J. Milgram, "Iterated loop spaces", *Ann. Math. (2)* **84** (1966) pp. 386-403.

[28] H. Miyazaki, "On realizations of some Whitehead products", *Tôhoku Math. J., Ser. II* **12** (1960), pp. 1-30.

[29] Y. Nomura, "An application of the path-space technique to the theory of triads", *Nagoya Math. J.* **22** (1963), pp. 169-188.

[30] G.J. Porter, "Spaces with vanishing Whitehead products", *Quart. J. Math., Ox. Ser. 2* **16** (1965), pp. 77-84.

[31] D.G. Quillen, *Homotopical Algebra.* Lec. Notes Math. No. 20, Springer-Verlag, Berlin-New York, 1963.

[32] D.G. Quillen, "Spectral sequences of a double semi-simplicial group", *Topology* **5** (1966), pp. 155-156.

[33] C.P. Rourke & B.J. Sanderson, "Δ-sets I: Homotopy theory", *Quart. J. Math., Ox. Ser. 2* **22** (1972), pp. 321-338.

[34] P.-H. Schoute, "Analytic treatment of the polytopes regularly derived from the regular polytopes", *Verh. Kon. Akad. Wet. Amst.*, **11** 1911.

[35] G.B. Segal, "Classifying spaces and spectral sequences", *Pub. Math. Inst. Hautes Et. Sci.* **34** (1968), pp. 105-112.

[36] G.B. Segal, "Categories and cohomology theories", *Topology* **13** (1974), pp. 293-312.

[37] E.H. Spanier, "Higher order operations", *Trans. AMS* **109** (1963), pp. 509-539.

[38] E.H. Spanier, "Secondary operations on mappings and cohomology", *Ann. Math.* **75** (1962) No. 2, pp. 260-282.

[39] J.D. Stasheff, *H-spaces from a Homotopy Point of View*, Lec. Notes Math. No. 161, Springer-Verlag, Berlin-New York, 1970.

[40] N. Stein, "Note on the realizability of Whitehead products", *Quart. J. Math., Ox. Ser. 2* **17** (1966), pp. 160-164.

[41] R.E. Street, "The algebra of oriented simplices", *J. Pure Appl. Alg.* **49** (1987) No. 3, pp. 283-335.

[42] R.E. Street, "Parity complexes", *Cah. Top. Géom. Diff. Cat.* **32** (1991) No. 4, p. 315-345.

[43] H. Toda, *Composition methods in the homotopy groups of spheres*, Adv. in Math. Study **49**, Princeton U. Press, Princeton, 1962.

[44] H. Toda, "Generalized Whitehead products and homotopy groups of spheres", *J. Inst. Polytech. Osaka City U., Ser. A, Math.* **3** (1952), pp. 43-82.

[45] G. Walker, "Long Toda brackets", in *Proc. Adv. Studies Inst. on Algebraic Topology, vol. III*, Aarhus U. Mat. Inst. Various Publ. Ser. **13**, Aarhus 1970, pp. 612-631.

[46] G.W. Whitehead, *Elements of Homotopy Theory*, Grad. Texts Math. **61**, SpringerVerlag, Berlin-New York, 1971.

[47] J.H.C. Whitehead, "On the realizability of homotopy groups", *Ann. Math.* **50** (1949), pp. 261-263.

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