HOMOTOPY OPERATIONS AND RATIONAL HOMOTOPY TYPE

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Abstract. We describe a collection of higher homotopy operations which determine the rational homotopy type of a simply-connected space $\mathbf{X}$. These are described in terms of simplicial resolutions of successive approximations $L^{(k,\alpha)}$ to the Quillen DGL model for $\mathbf{X}$. The operations lie in suitable cohomology groups $H^*(L^{(k,\alpha)};\pi_*\mathbf{X}_{\mathbb{Q}})$ of these DGLs. To facilitate the recovery of an integral version of the operations from the rational description, we also define a differential graded non-associative algebra model for rational spaces.

1. Introduction

The homotopy type of a space $\mathbf{X}$ is determined by its homotopy groups $\pi_*\mathbf{X}$, together with the action of all primary homotopy operations on it, and of certain higher homotopy operations (see [Bl2, §7.17]).

If we are interested only in the *rational* homotopy type of a simply-connected space $\mathbf{X}$, Whitehead products are the only non-trivial primary homotopy operations on the rational homotopy groups $\pi_*\mathbf{X}_{\mathbb{Q}} = \pi_*\mathbf{X}\otimes\mathbb{Q}$, which, after re-indexing, constitute a graded Lie algebra over $\mathbb{Q}$. The relevant higher order operations are also simpler than in the integral case. It is the purpose of this note to explain just how these determine the rational homotopy type, and make sense of

Theorem A. *For any simply-connected space $\mathbf{X}$, there is a sequence of higher homotopy operations taking value in $\pi_*\mathbf{X}$, which, together with the rational homotopy Lie algebra $\pi_{*-1}\mathbf{X}\otimes_{\mathbb{Z}}\mathbb{Q}$ itself, determine the rational homotopy type of $\mathbf{X}$. (See Theorem 7.15 below).*

These higher operations are certain subsets of $\pi_*\mathbf{X}$ which are indexed by elements in the homology of a certain inductively defined collection of differential graded Lie algebras (DGLs) defined below. Thus they take values in the corresponding cohomology groups, with coefficients in $\pi_*\mathbf{X}$.

Now it is clear intuitively that if L is any DGL, those cycles in the homology of L which are not generators, or products of other cycles, represent "higher homotopy operations" in L, in some sense. One of the objectives of this paper is to formalize this intuition within a more general framework. Moreover, if L represents the rational homotopy type of a topological space $\mathbf{X}$, it is not always evident how to represent these rational operations as *integral* higher order operations in $\pi_*\mathbf{X}$ (see §5.24 below). In order to address this problem, we must consider a somewhat "flabbier" model of rational

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homotopy than that provided by differential graded Lie algebras, namely a certain class of differential graded non-associative algebras.

Thus we also provide a (somewhat incomplete) answer to the following question: what additional structure on the ordinary homotopy groups $\pi_*\mathbf{X}$ of a simply-connected space $\mathbf{X}$, beyond the Whitehead products, is needed to determine its homotopy type up to rational equivalence?

1.1. **notation and conventions.** The ground field for all vector spaces, algebras, and tensor products will be $\mathbb{Q}$ (the rationals), unless otherwise stated.

$\mathcal{T}_*$ denotes the category of pointed CW complexes with base-point preserving maps, and by a *space* we shall always mean an object in $\mathcal{T}_*$, which will be denoted by a boldface letter: $\mathbf{X}, \mathbf{S}^n, \dots$. The subcategory of 1-connected spaces is denoted by $\mathcal{T}_1$, and the rationalization of a space $\mathbf{X} \in \mathcal{T}_1$ is $\mathbf{X}_{\mathbb{Q}}$. The category of rational 1-connected topological spaces is denoted by $\mathcal{T}_{\mathbb{Q}}$.

Let Δ denote the category of ordered sequences $\mathbf{n} = \langle 0, 1, \dots, n\rangle$ $(n \in \mathbb{N})$, with order-preserving maps. For any category $\mathcal{C}$, we let $s\mathcal{C}$ denote the category of simplicial objects over $\mathcal{C}$ – i.e., functors $\Delta^{op} \to \mathcal{C}$ (cf. [Ma, §2]); objects therein will be written $A_\bullet, \dots$. If we omit the degeneracies, we have a Δ-*simplicial* object, which we denote by $A^\Delta_\bullet, \dots$.

The category of non-negatively graded objects over a category $\mathcal{C}$ will be denoted by $gr\mathcal{C}$, with objects written $T_*, \dots$; we will write $|x| = p$ if $x \in T_p$. An upward shift by one in the indexing will be denoted by $\Sigma : gr\mathcal{C} \to gr\mathcal{C}$, so that $(\Sigma X_*)_{k+1} = X_k$, and $(\Sigma X_*)_0 = 0$. The category of graded vector spaces is denoted by $\mathcal{V}$.

The category of chain complexes (over $\mathbb{Q}$) will be denoted by $d\mathcal{V}$, and that of double chain complexes by $dd\mathcal{V}$. The differential of any differential graded object is written ∂ (to distinguish it from the face maps d_i of a simplicial object).

If $\mathcal{C}$ is a closed model category (cf. [Q1, I] or [Q3, II, §1]), we denote by $ho\mathcal{C}$ the corresponding homotopy category. If $X \in \mathcal{C}$ is cofibrant and $Y \in \mathcal{C}$ is fibrant, we denote by $[X, Y]_{\mathcal{C}}$ the set of homotopy classes of maps between them.

Let $\mathcal{S}et$ denote the category of sets, $\mathcal{V}ec$ the category of vector spaces (over $\mathbb{Q}$), $\mathcal{L}ie$ the category of Lie algebras, and $\mathcal{A}lg$ the category of non-associative algebras. We write $\mathcal{S}$ rather than $s\mathcal{S}et$ for the category of simplicial sets, and $\mathcal{S}_*$ for the category of *pointed* simplicial sets.

1.2. **organization:** In section 2 we review some background material on the Quillen DGL model for rational homotopy theory, and describe a bigraded variant of it; and in section 3 we give some more background on simplicial resolutions.

These are applied to the rational context in section 4, where we also define higher order homotopy operations for DGLs. These appear as the obstructions to realizing certain algebraic equivalences, and serve to determine the rational homotopy type of a simply-connected space. We give a first approximation to Theorem A in §4.15.

In section 5 we explain how to translate the usual bigraded and filtered DGL models into simplicial DGLs, which allows us to construct appropriate minimal simplicial resolutions. In section 6 we define the homology and cohomology of a DGL (after Quillen), and show that the obstructions we define above actually take value in the appropriate cohomology groups. Finally, in section 7 we describe a non-associative differential graded algebra model for rational homotopy theory, which facilitates the translation of the higher homotopy operations described above into integral homotopy operations. We summarize our main results in Theorem 7.15.

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2. Lie models

In this section we briefly recall some well-known definitions and facts of rational homotopy theory, and describe variants thereof.

2.1. **differential graded Lie algebras.** Let $\mathcal{L}$ denote the category of graded Lie algebras, or *GL*'s. An object $L_* \in \mathcal{L}$ is thus a graded vector space: $L_* = \oplus_{n=0}^{\infty} L_n$ over $\mathbb{Q}$, equipped with a bilinear graded product $[\ ,\ ] : L_p \otimes L_q \to L_{p+q}$ for each $p, q, \geq 0$, such that $[x,y] = (-1)^{|x||y|+1}[y,x]$ and $(-1)^{|x||z|}[[x,y],z] + (-1)^{|y||x|}[[y,z],x] + (-1)^{|z||y|}[[z,x],y] = 0$.

The free graded Lie algebra generated by a graded set X_* is denoted by $\mathbb{L}\langle X_* \rangle$. The functor $\mathbb{L} : gr\mathcal{S}et \to \mathcal{L}$ is left adjoint to the forgetful "underlying graded set" functor $U : \mathcal{L} \to gr\mathcal{S}et$, and it factors through $\mathcal{V}$: that is, $\mathbb{L}\langle X_* \rangle = L(\mathbb{V}\langle X_* \rangle)$, where $\mathbb{V}\langle X_* \rangle \in \mathcal{V}$ is the graded vector space with basis X_*.

The category of *differential graded Lie algebras*, or *DGL*s, will be denoted by $d\mathcal{L}$, with $d\mathcal{L}_0$ the subcategory of 0-*connected* Lie algebras (i.e., those with $L_0 = 0$). An object $L = (L_*, \partial_L) \in d\mathcal{L}$ is a graded Lie algebra $L_* \in \mathcal{L}$, together with a differential $\partial_L = \partial_L^n : L_n \to L_{n-1}$, for each $n > 0$, such that $\partial_L^{n-1} \circ \partial_L^n = \{0\}$ and $\partial_L[x,y] = [\partial_L x, y] + (-1)^{|x|}[x, \partial_L y]$.

The homology of the underlying chain complex of a DGL $L = (L_*, \partial)$ will be denoted $H'_* L$, to distinguish it from the DGL homology defined in §6.5 below. Because the differential ∂ is a derivation, $H'_* L$ inherits from L the structure of a graded Lie algebra.

A morphism of DGLs which induces an isomorphism in homology will be called a *quasi-isomorphism*, or *weak equivalence*.

In [Q3, II,§4-5], Quillen defined closed model category structures for the categories $d\mathcal{L}_0$ and $s\mathcal{L}ie$, as well as for topological spaces (and thus for $\mathcal{T}_{\mathbb{Q}}$), and proved:

2.2. Proposition. *There are pairs of adjoint functors* $\mathcal{T}_{\mathbb{Q}} \rightleftharpoons s\mathcal{L}ie$ *and* $s\mathcal{L}ie \rightleftharpoons d\mathcal{L}_0$, *which induce equivalences between the corresponding homotopy categories:* $ho\mathcal{T}_{\mathbb{Q}} \approx ho(s\mathcal{L}ie) \approx ho(d\mathcal{L}_0)$.

2.3. Notation. To every simply-connected space $\mathbf{X} \in \mathcal{T}_1$ one can thus associate a DGL $(L_*, \partial_L) \in d\mathcal{L}_0$, unique up to quasi-isomorphism, which determines its rational homotopy type. We denote any such DGL by L_X. In particular, $H'_*(L_X) \cong \pi_{*-1}\mathbf{X} \otimes_{\mathbb{Z}} \mathbb{Q}$, the rational homotopy algebra of $\mathbf{X}$, which we denote by $\Pi_*^X \in \mathcal{L}$.

2.4. Definition. The graded Lie algebra $H'_*(L_X)$ does not suffice to determine the rational homotopy type of $\mathbf{X} \in \mathcal{T}_1$: in fact, there may be infinitely many DGLs $\{L^{(n)}\}_{n=1}^{\infty}$ with $H'_*(L^{(n)}) \cong H'_*(L_X)$, no two of which are quasi-isomorphic as DGLs;

see e.g. [LS]. We shall denote by $d\mathcal{L}_0(\mathbf{X})$ the full subcategory of $d\mathcal{L}_0$ whose objects A satisfy $H'_*A \cong H'_*(L_X)$, with the isomorphism in $\mathcal{L}$ (see [SS], [LS], or [F1] for treatments of the cohomology analogue of $d\mathcal{L}_0(\mathbf{X})$ in terms of algebraic varieties).

The objects of $ho\, d\mathcal{L}_0(\mathbf{X})$ are thus all rational homotopy types which are indistinguishable from $\mathbf{X}_{\mathbb{Q}}$ on the primary homotopy operation level. Among these there is a distinguished simplest one: recall that a space $\mathbf{X}_{\mathbb{Q}} \in \mathcal{T}_{\mathbb{Q}}$ (or its corresponding DGL model $L_X \in d\mathcal{L}$) is called *coformal* (cf. [MN]) if L_X is weakly equivalent to the trivial DGL $(L_*, 0)$ (where of course $L_* = H'_*(L_X)$).

2.5. **minimal models.** Baues and Lemaire (in [BL, Cor. 2.4]; see also [N, Props. 5.6, 8.1 & 8.8]) showed that each connected DGL (L_*, ∂) has a *minimal model* $(\hat{L}_*, \hat{\partial})$, such that $\hat{L}_*$ is a free graded Lie algebra, $\hat{\partial} : \hat{L} \to \hat{L}$ factors through $[\hat{L}, \hat{L}]$, and there is a quasi-isomorphism of DGLs $\varphi : (\hat{L}_*, \hat{\partial}) \to (L_*, \partial)$ (unique up to chain homotopy). In particular, we can choose such a minimal model $\hat{L}_X$ for any space $\mathbf{X} \in \mathcal{T}_1$ (cf. §2.3).

As Neisendorfer observes in [N, §5], in general minimal models do not exist for non-connected DGLs (but see [Me] or [GHT] for ways around this).

2.6. **bigraded Lie algebras.** A differential bigraded Lie algebra, or *DBGL*, is a bigraded vector space $L_{*,*} = \oplus_{p=0}^{\infty} \oplus_{s=0}^{\infty} L_{p,s}$, equipped with a differential $\partial_L = \partial_L^{p,s} : L_{p,s} \to L_{p-1,s}$ and a bilinear graded product $[\ ,\ ] : L_{p,s} \otimes L_{q,t} \to L_{p+q,s+t}$ for each $p, q, s, t \geq 0$ satisfying:

(2.7i) $\quad [x, y] = (-1)^{(p+s)(q+t)+1}[y, x]$

(2.7ii) $\quad (-1)^{(p+s)(r+u)}[[x, y], z] + (-1)^{(p+s)(q+t)}[[y, z], x] + (-1)^{(q+t)(r+u)}[[z, x], y] = 0$

(2.7iii) $\quad \partial_L \circ \partial_L = 0$

(2.7iv) $\quad \partial_L[x, y] = [\partial_L x, y] + (-1)^{p+s}[x, \partial_L y]$

for $x \in L_{p,s}$, $y \in L_{q,t}$, and $z \in L_{r,u}$. The category of such DBGLs will be denoted by $db\mathcal{L}$, with $db\mathcal{L}_0$ the subcategory with $L_{p,0} = 0$ for all p.

2.8. Definition. Each DBGL $(L_{*,*}, \partial_L)$ has an *associated DGL* (L_*, ∂_L), defined $L_n = \bigoplus_{p+q=n} L_{p,q}$ (same ∂_L); some authors re-index $L_{*,*}$ so that $\hat{L}_{p,s} = L_{p,p+s}$, and then L_* is obtained from $\hat{L}_{*,*}$ by disregarding the first (homological) grading.

As for ordinary graded Lie algebras, one can define closed model category structures on $s\mathcal{L}_0$ and $db\mathcal{L}_0$ (see [BS, §2], and [Bl4, §4]), and we have the following analogue of [Q3, I, Props. 2.3 & 4.6, Thm. 4.4]:

2.9. Proposition. *There are adjoint functors* $s\mathcal{L}_0 \underset{N^*}{\overset{N}{\rightleftarrows}} db\mathcal{L}_0$, *which induce equivalences of the corresponding homotopy categories* $ho(s\mathcal{L}_0) \approx ho(db\mathcal{L}_0)$. N^* *takes free DBGLs to free simplicial graded Lie algebras.*

Proof. (We give the proof mainly to fix notation which will be needed later.) Given a simplicial graded Lie algebra $L_{\bullet,*} \in s\mathcal{L}_0$, let $(C_{*,*}, \partial)$ be its Moore chain complex (cf. [Ma, §22]), defined:

$$C_{p,s} = \bigcap_{i=1}^{p} [Ker(d_i^p)]_s \qquad \text{with} \quad \partial_p = (-1)^s \, d_0^p|_{C_{p,s}}. \tag{2.10}$$

The simplicial Lie bracket $[\![\ ,\ ]\!] : C_{p,s} \otimes C_{q,t} \to C_{p+q,s+t}$ is defined via the Eilenberg-Zilber map:

$$[\![x, y]\!] = \sum_{(\sigma,\tau)\in S_{p,q}} (-1)^{\varepsilon(\sigma)+pt} [s_{\tau_q} \dots s_{\tau_1} x, s_{\sigma_p} \dots s_{\sigma_1} y] \tag{2.11}$$

where $S_{p,q}$ denotes the set of all (p,q)-shuffles – that is, partitions of $\{0,1,\dots,p+q-1\}$ into disjoint sets $\sigma_1 < \sigma_2 < \dots < \sigma_p$, $\tau_1 < \tau_2 < \dots < \tau_q$ – and $\varepsilon(\sigma) = p + \sum_{i=1}^{p}(\sigma_i - i)$, so $(-1)^{\varepsilon(\sigma)}$ is the sign of the permutation corresponding to (σ,τ). (See [Mc1, VIII, §8]).

If we forget the Lie structure, the Moore chain complex functor N induces an equivalence between the categories of simplicial graded vector spaces and bigraded chain complexes (cf. [Do, Thm 1.9]), with the inverse functor ? defined for such a chain complex $(A_{*,*}, \partial)$ by

$$(?\, A_{*,*})_{n,s} := \bigoplus_{0\le\lambda\le n} \ \bigoplus_{I\in\mathfrak{I}_{n,\lambda}} A_{n-\lambda,s}$$

(where for each $n \ge 0$ and $0 \le \lambda \le n$, we let $\mathfrak{I}_{\lambda,n}$ denote the set of all sequences of λ non-negative integers $i_1 < \dots < i_\lambda (< n)$), with the obvious face maps (induced by ∂) and degeneracies (see [Ma, p. 95]).

The left adjoint $N^* : db\mathcal{L}_0 \to s\mathcal{L}_0$ to N is defined $N^*((L_{*,*}, \partial)) = L(?\,(L_{*,*}))/I(L_{*,*})$, where L is the free graded Lie algebra functor, and $I(L_{*,*})$ is the ideal generated by $[\![?\,(x), ?\,(y)]\!] - ?\,([x,y])$. The identities (2.7) follow from the corresponding ones in the singly-graded case and the simplicial identities. □

3. Simplicial resolutions

The proper algebraic setting for defining our higher homotopy operations is a suitable notion of a simplicial resolution of $\pi_* \mathbf{X}_{\mathbb{Q}}$:

3.1. Definition. Recall that a category of *universal graded algebras* (or variety of graded algebras, in the terminology of [Mc2, V,§6]) is a category $\mathcal{C}$ in which the objects are graded sets X_*, together with an action of a fixed set of n-ary graded operators $W = \{\omega : X_{k_1} \times X_{k_2} \times \dots \times X_{k_n} \to X_m\}$, satisfying a set of identities E, and the morphisms are functions on the sets which commute with the operators. Such categories always come equipped with a "free graded algebra" functor $F : gr\mathcal{S}et \to \mathcal{C}$, left adjoint to the "underlying graded set" functor $U : \mathcal{C} \to gr\mathcal{S}et$. In all the examples we shall be concerned with, the objects X_* will be "underlying-abelian" (see [BS, §2.1.1]), and in fact will have the underlying structure of a graded vector space over $\mathbb{Q}$.

Examples include $\mathcal{L}$, and the categories of associative (resp. non-associative) graded algebras. Note that any ordinary ungraded category of universal algebras may be thought of as a CUGA with all objects concentrated in degree 0.

3.2. Definition. A *free simplicial resolution* of an object B in a CUGA $\mathcal{C}$ is a weak equivalence from a cofibrant object $A_\bullet \in s\mathcal{C}$ to the constant simplical object associated to B (with respect to the closed model category structure on the category $s\mathcal{C}$ defined in [Q1, II, §4]). Such resolutions always exist, by [Q1, II, §4]; see section 5 below for a specific construction.

3.3. **bisimplicial objects.** We shall be interested in a particular type of simplicial resolution, which may be defined for an arbitrary CUGA $\mathcal{C}$ ((cf. [DKS] and [BS]), though we shall only need it for the case where $\mathcal{C}$ is a category of ungraded universal algebras, such as $\mathcal{L}ie$ or $\mathcal{A}lg$:

Consider the category $ss\mathcal{C}$ of bisimplicial objects over $\mathcal{C}$. We think of an object $A_{\bullet\bullet} \in ss\mathcal{C}$ as having *internal* and *external* simplicial structures, with corresponding homotopy group objects $\pi_t^i A_{\bullet\bullet}$ and $\pi_s^e A_{\bullet\bullet}$ (each taking value in $s\mathcal{C}$ – see [BS, App.]). Let $sF : gr\mathcal{S} \to s\mathcal{C}$ denote the free graded algebra functor, extended dimensionwise, and let $S^n(k)_\bullet$ be the graded simplicial set having the simplicial n-sphere $S^n_\bullet := \Delta[n]/\Delta[n]^{n-1}$ in degree k. We think of the simplicial graded algebras $F(S^n(k)_\bullet)$ as the $\mathcal{C}$-spheres, or *models*, for $s\mathcal{C}$ (cf. [BS, §3.1]). (In the ungraded case one can of course omit the extra degree k, and write simply $F(\mathbf{S}^n)$). The full subcategory of $s\mathcal{C}$ whose objects are weakly equivalent to coproducts of such models will be denoted by $\mathcal{M}_\mathcal{C}$, or simply $\mathcal{M}$.

One can use these models to define the so-called "E^2-model category structure" for $ss\mathcal{C}$, as in [DKS, §5], in which a map $f : X_{\bullet\bullet} \to Y_{\bullet\bullet}$ is a weak equivalence if

$$f_\star : \pi_s \pi_t^i X_{\bullet\bullet} \to \pi_s \pi_t^i Y_{\bullet\bullet} \text{ is an isomorphism for each } s, t \geq 0 \tag{3.4}$$

We shall not need an explicit description of the fibrations and cofibrations in $ss\mathcal{C}$, but only a particular type of cofibrant object, as follows:

3.5. Definition. A bisimplicial object $A_{\bullet\bullet} \in ss\mathcal{C}$ is called $\mathcal{M}$-*free* if for each $m \geq 0$ there are graded simplicial sets $X[m]_\bullet \simeq \bigvee_i \mathbf{S}^{n_i}(k_i)_\bullet$ such that $A_{\bullet,m} \cong F(X[m]_\bullet)$ (so that $A_{\bullet,m} \in \mathcal{M}$), and the external degeneracies of $A_{\bullet\bullet}$ are induced under F by maps $X[m]_\bullet \to X[m+1]_\bullet$ which are, up to homotopy, the inclusion of sub-coproduct summands. Any $X_\bullet \in s\mathcal{C}$ may be resolved by an $\mathcal{M}$-free bisimplicial algebra $A_{\bullet\bullet}$ (see [BS, §4.1]); this is called an $\mathcal{M}$-*free resolution* of $X_\bullet$.

3.6. Definition. The *diagonal* of a bisimplicial object $A_{\bullet\bullet} \in ss\mathcal{C}$ is a simplical object $diag(A_{\bullet\bullet}) \in s\mathcal{C}$ with $diag(A_{\bullet\bullet})_n := A_{n,n}$, face maps $d_k = d_k^i \circ d_k^e$, and degeneracies $s_j = s_j^i \circ s_j^e$.

3.7. Remark. There is a first quadrant spectral sequence with

$$E^2_{s,t} = \pi_s^e(\pi_t^i A_{\bullet\bullet}) \Rightarrow \pi_{s+t} diag(A_{\bullet\bullet})$$

(see [Q2], and compare [BF, Thm B.5]).

Thus in particular if $A_{\bullet\bullet} \to X_\bullet$ is a resolution (in the E^2-model category sense), we see that $\varepsilon : A_{0,\bullet} \to X_\bullet$ induces a weak equivalence $diag(A_{\bullet\bullet}) \simeq X_\bullet$.

Moreover, the same is true if we disregard the degeneracies and consider only the Δ-bisimplicial resolution $A^\Delta_{\bullet\bullet} \to X_\bullet$.

4. Resolutions for rational spaces

Given a simply-connected space $\mathbf{X} \in \mathcal{T}_1$, the first approximation to an algebraic description of its rational homotopy type is given by its rational homotopy Lie algebra $\Pi_*^X := \pi_{*-1}\mathbf{X}_{\mathbb{Q}} \in \mathcal{L}$.

If $\mathbf{X}_{\mathbb{Q}}$ were coformal (§2.4), then in particular all higher homotopy operations vanish in $\pi_*\mathbf{X}_{\mathbb{Q}}$, and no information beyond Π_*^X itself is needed to determine the rational homotopy type of $\mathbf{X}$. The higher homotopy operations we shall describe may thus be thought of as "obstructions to coformality", much in the spirit (though not the specific approach) of [HS].

4.1. **topological resolutions.** To proceed further, we need some kind of a "topological" simplicial object $C_\bullet$ which realizes a suitable "algebraic" simplicial resolution $V_{\bullet,*} \to \Pi_*^X$ in $s\mathcal{L}$, in the sense that $V_{\bullet,*} = \pi_{*-1}C_\bullet$. The higher homotopy operations we want then arise as the obstructions to realizing the "algebraic" augementation map $\pi_{*-1}C_\bullet \to \Pi_*^X$ topologically.

This can be done using actual topological spaces, as in the integral case (see [Bl2, §7], as simplified in [Bl3, §4.9]), but for rational spaces it is more convenient to use an algebraic model, in a category such as $d\mathcal{L}$. To allow us freedom in choosing this model, we give a general definition:

4.2. Assumptions. Let $g\mathcal{C}$ be a CUGA (which we may assume to have the underlying structure of a graded vector space), and $\mathcal{C}$ the category of (ungraded) universal algebras corresponding to objects of $g\mathcal{C}$ concentrated in degree 0. The cases we shall be interested in are $\mathcal{C} = \mathcal{L}ie$ (with $g\mathcal{C} = \mathcal{L}$) and $\mathcal{C} = \mathcal{A}lg$ (with $g\mathcal{C} = \mathcal{A}$).

As shown in [BS, App.], for each simplicial algebra $A_\bullet \in s\mathcal{C}$, the graded homotopy object $\pi_*A_\bullet$ actually takes value in $g\mathcal{C}$.

For a given $A_\bullet \in s\mathcal{C}$, let $C_{\bullet\bullet} \to A_\bullet$ be an $\mathcal{M}_{\mathcal{C}}$-free resolution (Definition 3.5). In particular, this implies that upon applying the functor π_* we obtain a free simplicial resolution $\pi_*^i C_{\bullet\bullet}$ (in the "external" direction!) of the graded algebra $\pi_*A_\bullet$. In fact, we only need a Δ-bisimplicial resolution (§3.7), but we shall nevertheless usually abuse notation by writing $C_{\bullet\bullet}$ for $C_{\bullet\bullet}^{\Delta}$.

Next, assume given another object $B_\bullet \in s\mathcal{C}$, together with an isomorphism $\varphi : \pi_*A_\bullet \cong \pi_*B_\bullet$ (in $g\mathcal{C}$). Define a sequence of morphisms $\psi_n : \pi_*C_{n,\bullet} \to \pi_*B_\bullet$ by $\psi_0 := \varphi \circ \varepsilon$ and $\psi_{n+1} := \psi_n \circ d_0$ (which implies that $\psi_{n+1} = \psi_n \circ d_i$ for all $0 \leq i \leq n$, by the simplicial identities).

We choose once and for all a fixed map $f_0 : C_{0,\bullet} \to B_\bullet$ realizing ψ_0 (this is possible because $C_{\bullet\bullet} \to A_\bullet$ is $\mathcal{M}$-free) and define $f_n : C_{n,\bullet} \to B_\bullet$ inductively by setting $f_{n+1} := f_n \circ d_n$, so that $\pi_*(f_n) = \psi_n$ for all $n \geq 0$. It is usually most convenient to set $f|_{\mathcal{D}_{(x)}^k} = 0$ for all $\mathcal{C}$-disks $\mathcal{D}_{(x)}^k \hookrightarrow C_{0,\bullet}$.

Note that, because $C_{\bullet\bullet}$ is $\mathcal{M}$-free, the maps $\{\psi_n\}_{n=0}^{\infty}$ define an augmented Δ-simplicial object ${}^hC_{\bullet\bullet}^{\Delta} \to B_\bullet$ *in the homotopy category* $ho(s\mathcal{C})$ – or equivalently, an augmented Δ-simplicial object up-to-homotopy.

4.3. Definition. Let $D[n] \in \mathcal{S}_*$ denote the standard simplicial n-simplex, together with an indexing of its non-degenerate k-dimensional faces $D[k]^{(\gamma)}$ by the composite face maps $\gamma = d_{i_{n-k}} \circ \ldots \circ d_{i_n} : \mathbf{n} \to \mathbf{k-1}$ in Δ^{op} (cf. [Bl3, §4]). Its $(n-1)$-skeleton, which is a simplicial $(n-1)$-sphere, is denoted by $\partial D[n]$. We shall take $* := D[0]^{(d_0 d_1 d_2 \ldots d_{n-1})}$ as the base point of $D[n] \in \mathcal{S}_*$, and we choose once and for all a fixed isomorphism $\varphi^{(\gamma)} : D[k]^{(\gamma)} \to D[k]$ for each face $D[k]^{(\gamma)}$ of $D[n]$ (see, e.g., [Bl3, (4.5)]).

4.4. Example. $\partial D[2]$ has three vertices: $D[0]^{(d_0 d_1)} = D[0]^{(d_0 d_0)}$ (the basepoint), $D[0]^{(d_0 d_2)} = D[0]^{(d_1 d_0)}$, and $D[0]^{(d_1 d_2)} = D[0]^{(d_1 d_1)}$. It has three non-degenerate edges $D[1]^{(d_0)}$, $D[1]^{(d_1)}$, and $D[1]^{(d_2)}$. $D[2]$ has in addition one non-degenerate 2-simplex, which we may denote by $D[2]^{(id)}$. See Figure 1 below for a depiction of $D[2]$, and Figure 2 for a depiction of $D[3]$.

4.5. Definition. Given $Y_\bullet \in s\mathcal{C}$ and a simplicial set $K_\bullet \in \mathcal{S}$, we define their *half-smash* (in $s\mathcal{C}$) by:

$$Y_\bullet \rtimes K_\bullet := Y_\bullet \otimes K_\bullet / (\{0\} \otimes K_\bullet)$$

(where $(Y_\bullet \otimes K_\bullet)_n := \coprod_{x \in K_n} (Y_n)_{(x)}$ – cf. [Q1, II, §1, Prop. 2]).

Similarly, the *smash product* (in $s\mathcal{C}$) of $Y_\bullet$ with a *pointed* simplicial set $K_\bullet \in \mathcal{S}_*$ is defined $Y_\bullet \wedge K_\bullet := Y_\bullet \rtimes K_\bullet / (Y_\bullet \rtimes \{*\})$, and if $K_\bullet = \mathbf{S}^r$ (the simplicial sphere), we write $\Sigma^r Y_\bullet$ for $Y_\bullet \wedge \mathbf{S}^r$.

4.6. Remark. If $Y_\bullet = F(\mathbf{S}^n)$ is a $\mathcal{C}$-sphere (see §3.3), then $\Sigma^r Y_\bullet \cong F(\mathbf{S}^{n+r})$ is also a $\mathcal{C}$-sphere. In fact, many of the usual properties of spheres in $ho\mathcal{T}$ also hold for $\mathcal{C}$-spheres – e.g., $\pi_r X_\bullet \cong [F(\mathbf{S}^n), X_\bullet]_{s\mathcal{C}}$ for any $X_\bullet \in s\mathcal{C}$ (cf. [Q1, I, §4]), and $Y_\bullet \simeq \coprod_i F(\mathbf{S}^{n_i}) \Rightarrow \Sigma^r Y_\bullet \simeq \coprod_i F(\mathbf{S}^{n_i+r})$ (cf. [Q1, I, §3]).

4.7. Definition. Under the assumptions of §4.2, for each $n \in \mathbb{N}$, we define a $\partial D[n]$-*compatible sequence* to be a sequence of maps $\{h_k : W_{k,\bullet} \rtimes D[k] \to B_\bullet\}_{k=0}^{n-1}$, such that $h_0 = f_0$ (under the natural identification $W_{0,\bullet} \rtimes D[0] = W_{0,\bullet}$), and for any iterated face maps $\delta = d_{i_{j+1}} \circ \cdots \circ d_{i_n}$ and $\gamma = d_{i_j} \circ \delta$ $(0 \leq j < n)$ we have

$$h_j \circ (d_{i_j} \rtimes id) = h_{j+1} \circ (id \rtimes \iota_\delta^\gamma) \quad \text{on } C_{j+1,\bullet} \rtimes D[j]], \tag{4.8}$$

where $\iota_\delta^\gamma : D[j] \to D[j+1]$ is the composite $\iota_\delta^\gamma := \varphi^\delta \circ \iota \circ (\varphi^\gamma)^{-1}$. Here φ^γ and φ^δ are the isomorphisms of Definition 4.3, and $\iota : D[j]^{(\gamma)} \to D[j+1]^{(\delta)}$ is the inclusion (compare [Bl3, Def. 4.10]).

A sequence of maps $\{h_k : W_{k,\bullet} \rtimes D[k] \to B_\bullet\}_{k=0}^{\infty}$ satisfying condition (4.8) for all γ, δ, and n is called a $\partial D[\infty]$-*compatible sequence.*

4.9. Definition. Given such a $\partial D[n]$-compatible sequence $\{h_k : C_{k,\bullet} \rtimes D[k] \to B_\bullet\}_{k=0}^{n-1}$ the *induced map* $\bar{h} : C_{n,\bullet} \rtimes \partial D[n] \to B_\bullet$ is defined on the "faces" $C_{n,\bullet} \rtimes D[n-1]^{(d_i)}$ of $C_{n,\bullet} \rtimes D[n]$ by: $\bar{h}|_{C_{n,\bullet} \rtimes D[n-1]^{(d_i)}} = h_{n-1} \circ (d_i \rtimes id)$. The compatibility condition (4.8) above guarantees that $\bar{h}$ is well-defined.

4.10. Definition. For each $n \geq 2$, the *n-th order homotopy operation* (associated to the choice of $C_{\bullet\bullet} \to A_\bullet$ in §4.2) is a subset $\langle\langle n \rangle\rangle$ of the track group $[\Sigma^{n-1} C_{n,\bullet}, B_\bullet]_{s\mathcal{C}}$ defined as follows:

Let $T_n \subseteq [C_{n,\bullet} \rtimes \partial D[n], B_\bullet]_{s\mathcal{C}}$ be the set of homotopy classes of maps $\bar{h} : C_{n,\bullet} \rtimes \partial D[n] \to B_\bullet$ induced as above by some $\partial D[n]$-compatible collection $\{h_k\}_{k=0}^{n-1}$. Since each $C_{n,\bullet}$ is a suspension, up to homotopy, by Remark 4.6, we have a splitting

$$C_{n,\bullet} \rtimes \partial D[n] \simeq (\mathbf{S}^{n-1} \wedge C_{n,\bullet}) \amalg C_{n,\bullet} \tag{4.11}$$

(as for topological spaces). We define $\langle\langle n\rangle\rangle \subseteq [\Sigma^{n-1}C_{n,\bullet}, B_\bullet]_{s\mathcal{C}}$ to be the image under the resulting projection of the subset $T_n \subseteq [C_{n,\bullet} \rtimes \partial D[n], B_\bullet]_{s\mathcal{C}}$.

Note that the projection of a class $[\bar{h}] \in T_n$ on the other summand $[C_{n,\bullet}, B_\bullet]_{s\mathcal{C}}$ coming from the splitting (4.11) is just the homotopy class of the map f_n of §4.2. On the other hand, since $C_{\bullet\bullet}$ was assumed to be $\mathcal{M}$-free, each $C_{n,\bullet} \simeq \coprod_{k=1}^{\infty}\coprod_{x\in T_{n,k}} F(\mathbf{S}^k_{(x)})$ is weakly equivalent to a wedge of spheres, so $\Sigma^{n-1}C_{n,\bullet} \simeq \coprod_{k=1}^{\infty}\coprod_{x\in T_{n,k}} F(\mathbf{S}^{k+n-1}_{(x)})$. Thus

$$[\Sigma^{n-1}C_{n,\bullet}, B_\bullet]_{s\mathcal{C}} \cong \prod_{k=1}^{\infty}\prod_{x\in T_{n,k}} [F(\mathbf{S}^{k+n-1}_{(x)}), B_\bullet]_{s\mathcal{C}}, \tag{4.12}$$

and we shall denote the components of $\langle\langle n\rangle\rangle$ under this product decomposition by $\langle\langle n, x\rangle\rangle \subseteq [F(\mathbf{S}^{k+n-1}_{(x)}), B_\bullet]_{s\mathcal{C}} = \pi_{k+n-1}B_\bullet$.

4.13. Definition. It is clearly a necessary condition for the subset $\langle\langle n\rangle\rangle$ to be non-empty that all the lower order operations $\langle\langle k\rangle\rangle$ $(2 \le k < n)$ *vanish* – i.e., contain the null class. A sufficient condition is that they *vanish coherently* (cf. [Bl2, Def. 5.7] – i.e., that the $\partial D[m]$-compatible collections $\{h^\gamma_k\}_{k=0}^{m-1}$ for the various faces γ of $\partial D[n]$ can be chosen to agree on their intersections, so that they in fact fit together to form a $\partial D[n+1]$-compatible collection $\{h_k\}_{k=0}^{n}$.

4.14. Remark. The coherent vanishing of all the operations $\{\langle\langle n\rangle\rangle\}_{n=2}^{\infty}$ is equivalent, by [BV, Cor. 4.21 & Thm. 4.49] and [Bl2, §4.11], to the *rectifiability* of the augmented Δ-simplicial object up-to-homotopy ${}^hC^\Delta_{\bullet\bullet} \to B_\bullet$: that is, its replacement by augmented Δ-simplicial object $\hat{C}^\Delta_{\bullet\bullet} \to B_\bullet$ over $s\mathcal{C}$ (with the simplicial identities now holding precisely, in $s\mathcal{C}$, rather than just in $ho(s\mathcal{C})$), such that $C_{n,\bullet} \simeq \hat{C}_{n,\bullet}$ for each n.

This in turn implies (by §3.7) that $diag(\hat{C}^\Delta_{\bullet\bullet}) \simeq B_\bullet$; but since $diag(\hat{C}^\Delta_{\bullet\bullet}) \simeq diag(C^\Delta_{\bullet\bullet})$, and $diag(C^\Delta_{\bullet\bullet}) \simeq A_\bullet$ by assumption, we conclude that $A_\bullet \simeq B_\bullet$ if and only if the higher homotopy operations $\{\langle\langle n\rangle\rangle\}_{n=2}^{\infty}$ vanish coherently.

4.15. Summary. This yields a first approximation to Theorem A, which may be described as follows:

We work in $\mathcal{C} = \mathcal{L}ie$ (and $g\mathcal{C} = \mathcal{L}$). Given a space $\mathbf{X} \in \mathcal{T}_1$ we consider the simplicial Lie algebra $B_\bullet$ corresponding to a DGL model $L_X \in d\mathcal{L}$ for $\mathbf{X}_\mathbb{Q}$ (under the functors of Proposition 2.2), and let $\Pi^X_* := \pi_{*-1}\mathbf{X}_\mathbb{Q} \in \mathcal{L}$ be its rational homotopy Lie algebra, with $A_\bullet \in s\mathcal{L}ie$ the simplicial Lie algebra corresponding to the trivial DGL $L^{(0)} := (\Pi^X_*, 0)$. Choose some $\mathcal{M}_{\mathcal{L}ie}$-free resolution $C_{\bullet\bullet} \in ss\mathcal{L}ie$ of $A_\bullet$.

$\mathbf{X}$ is coformal if and only if $A_\bullet \simeq B_\bullet$, and this happens if and only if all the higher homotopy operations $\{\langle\langle n\rangle\rangle\}_{n=2}^{\infty}$ associated to $C_{\bullet\bullet}$ vanish coherently, by Remark 4.14. If not, let n_0 denote the least $n \ge 2$ such that $0 \notin \langle\langle n\rangle\rangle$.

Note that we can apply the above procedure to any DGL in $d\mathcal{L}(\mathbf{X})$ (Def. 2.4), not only to L_X; and the existence and vanishing or non-vanishing of the higher homotopy operation $\langle\langle n_0\rangle\rangle \subset \pi_*\mathbf{X}_\mathbb{Q}$ is a homotopy invariant. Denote by $\mathcal{H}^{(1)}$ the set of all homotopy types in $ho\, d\mathcal{L}_0(\mathbf{X})$ for which $\langle\langle n_0\rangle\rangle$ is defined and has the same value as for $B_\bullet$ itself (i.e., those DGLs which are indistinguishable from L_X as far as the primary homotopy operations, and all the higher homotopy operations $\{\langle\langle n\rangle\rangle\}_{n=2}^{n_0}$ associated to $C_{\bullet\bullet}$, can see). For each $\alpha \in \mathcal{H}^{(1)}$, choose a representative DGL $L^{(1,\alpha)}$.

Next, choose a new $\mathcal{M}$-free resolution for the simplicial Lie algebra corresponding to $L^{(1,\alpha)}$, and repeat the above procedure, yielding a set of higher homotopy operations $\langle\langle n_{1,\alpha}\rangle\rangle \subset \pi_*\mathbf{X}_{\mathbb{Q}}$ which serve as obstructions to the existence of a homotopy equivalence $L^{(1,\alpha)} \xrightarrow{\simeq} L_X$. For each such higher operation $\langle\langle n_{1,\alpha}\rangle\rangle$, we denote by $\mathcal{H}^{(2,\alpha)}$ the set of all homotopy types in $\mathcal{H}^{(1)} \subseteq ho\, d\mathcal{L}_0(\mathbf{X})$ for which $\langle\langle n_{1,\alpha}\rangle\rangle$ has the same value as for L_X. Now choose representatives $L^{(2,\alpha,\alpha')}$ for each $\alpha' \in \mathcal{H}^{(2,\alpha)}$, and proceed as above.

In this way we obtain a tree T_X of rational homotopy types in $ho\, d\mathcal{L}_0(\mathbf{X})$, which also indexes a collection of higher homotopy operations of the form $\langle\langle n_{k,\alpha_1,\alpha_2,\ldots,\alpha_k}\rangle\rangle \subseteq \pi_*\mathbf{X}_{\mathbb{Q}}$, and $\lim_{k\to\infty} n_k = \infty$ along any branch of the tree T_X, so that in fact this collection of operations determines the rational homotopy type of $\mathbf{X}$.

In a future paper we hope to show how this tree of homotopy types in $ho\, d\mathcal{L}_0(\mathbf{X})$, and thus the corresponding collection of higher homotopy operations, may be described more effectively in terms of a "Postnikov tower" for an $\mathcal{M}_{\mathcal{L}ie}$-free resolution for $\mathbf{X}$.

5. Minimal resolutions

We now explain how the bisimplicial theory described in section 4 translates into a differential graded theory, when $\mathcal{C} = \mathcal{L}$. In particular, this allows an application of the Halperin-Stasheff perturbation theory to our context.

First, it is sometimes convenient to have *minimal* $\mathcal{M}$-free resolutions for a DGL, defined for any CUGA $\mathcal{C}$ as follows:

5.1. Definition. Any $B \in \mathcal{C}$ has a special kind of free simplicial resolution (see §refdsr) $A_\bullet \to B$, called a *CW-resolution*, defined as follows (cf. [Bl1, §5.3]):

If we let $\bar{A}_n$ denote the sub-algebra of A_n generated by the non-degenerate elements in T^n_*, we require that $d_i|_{\bar{A}_n} = 0$ for $1 \le i \le n$. The sequence $\bar{A}_0 = A_0, \bar{A}_1, \ldots, \bar{A}_n, \ldots$ is called a *CW-basis* for $A_\bullet$, and $\bar{d}_0 = d_0|_{\bar{A}_n}$ is the *attaching map* for $\bar{A}_n$.

Such a $A_\bullet \to B$ will be called *minimal* if each $\bar{A}_{n+1}$ is minimal among those free algebras in $\mathcal{C}$ which map onto the Moore n-cycles $Z_n A_\bullet = Ker(\partial_n)$ (see (2.10)).

5.2. Definition. When $\mathcal{C} = \mathcal{L}$, the category of graded Lie algebras, it will be more convenient at times to use of the adjoint functors of Proposition 2.9 to replace $A_{\bullet\bullet} \to X_\bullet$ by a simplicial DGL $L_{\bullet,*} \to X_*$. In this case the simplicial models are replaced by the corresponding DGLs, namely

1. A *$d\mathcal{L}$-n-sphere*, denoted by $\mathcal{S}^n_{(x)}$, is a DGL of the form $(\mathbb{L}\langle X_*\rangle, 0)$ where X_* is the graded set with $X_n = \{x\}$ and $X_i = \emptyset$ for $i \neq n$.
2. A *$d\mathcal{L}$-$(n+1)$-disk*, denoted $\mathcal{D}^{n+1}_{(x)}$, is the DGL $(\mathbb{L}\langle X_*\rangle, \partial_L)$ where $X_{n+1} = \{x\}$, $X_n = \{\partial_L x\}$, and $X_i = \emptyset$ for $i \neq n, n+1$. Its *boundary* is the $d\mathcal{L}$-n-sphere $\partial\mathcal{D}^{n+1}_{(x)} := \mathcal{S}^n_{(\partial_L x)}$.

3. A *two-stage DGL* is a DGL $(\mathbb{L}\langle X_*\rangle, \partial_L) \in d\mathcal{L}$, where for some $n \geq 0$ we have $X_i = \emptyset$ for $i \neq n, n+1$. Any coproduct (in $d\mathcal{L}$) of two-stage DGLs will be called a *free* DGL.

Evidently $d\mathcal{L}$-spheres and disks are free DGLs, and any free DGL may be described as the coproduct of $d\mathcal{L}$-spheres and disks – and more canonically, as a coproduct of $d\mathcal{L}$-spheres, disks, and collections of disks with their boundaries identified to a single sphere.

5.3. Definition. In fact, there is a comonad (cf. [EM, §2]) $F : d\mathcal{L} \to d\mathcal{L}$ defined for any $B = (B_*, \partial_B) \in d\mathcal{L}$ by

$$F(B) = (\coprod_{k=1}^{\infty} \coprod_{x \in B_k} \mathcal{D}^k_{(x)})/ \sim, \tag{5.4}$$

where we set $\mathcal{D}^k_{(x)} := \mathcal{S}^k_{(x)}$ if $\partial_B x = 0$, and let $\partial \mathcal{D}^{k+1}_{(x)} \sim \mathcal{S}^k_{(\partial_B x)}$ if $\partial_B x \neq 0$.

Clearly $F(B)$ is a free DGL, and by iterating F we obtain a free simplicial DGL $W_{\bullet,*}$ with $W_n = F^{n+1}(B)$ (see [Gd, App., §3]), which we call the *canonical free simplical DGL resolution* of $B = (B_*, \partial_B)$, which we denote by $W_{\bullet,*}(B)$. Observe that $W_{\bullet,*}$ (or equivalently, the corresponding bisimplicial Lie algebra $W_{\bullet\bullet}$) is an $\mathcal{M}$-free resolution of B.

5.5. Remark. Note that if $\partial_B \equiv 0$, by definition (5.4) $F(B_*, 0)$ has only spheres, and no disks, and thus the canonical resolution $W_{\bullet,*}(B)$ has $\partial_{W_n} = 0$ for all $n \geq 0$. Thus $W_{\bullet,*}$ may be identified with the usual canonical resolution of the graded Lie algebra B_* (coming from the "free graded Lie algebra on underlying graded set" comonad), which we shall denote by $V_{\bullet,*}(B_*)$.

Note further that by (3.4), if we apply the functor H'_* to $W_{\bullet,*} \to B_*$ – or equivalently, the functor π^i_* to $W_{\bullet\bullet} \to B_\bullet$ – we obtain a free simplicial resolution of the graded Lie algebra $L_* := H'_*(B_*, \partial_B)$.

5.6. Notation. If we write $\langle x\rangle \in F(B)$ for the generator corresponding to an element $x \in B_*$, then recursively a typical DGL generator for $W_n = W_{n,*}$ (in the canonical resolution $W_{\bullet,*}(B)$) is $\langle \alpha\rangle$, for $\alpha \in W_{n-1}$, so an element of W_n is a sum of iterated Lie products of elements of B_*, arranged within $n+1$ nested pairs of brackets $\langle\langle \cdots\rangle\rangle$. With this notation, the i-th face map of $W_{\bullet,*}$ is "omit i-th pair of brackets", and the j-th degeneracy map is "repeat j-th pair of brackets". We assume the bracket operation $\langle -\rangle$ is linear – i.e., that $\langle \alpha x + \beta y\rangle = \alpha\langle x\rangle + \beta\langle y\rangle$ for $\alpha, \beta \in \mathbb{Q}$ and $x, y \in B$.

In order to construct minimal $\mathcal{M}$-simplicial resolutions, first consider the coformal case:

5.7. **the bigraded model.** Any coformal DGL (§2.4), and in particular $L = (L_*, 0)$, has a *bigraded model* $A_{*,*} \to L_*$ – that is, a bigraded DGL $(A_{*,*}, \partial_A)$ (see §2.6) which is minimal in the sense of §2.5, along with a quasi-isomorphism $A_{*,*} \to L_*$. The bigraded model is unique up to isomorphism. See [O, I] for an explicit construction.

This is just the Lie algebra version of the bigraded model of [HS, §3] (see also [F2]), which is in turn essentially the Tate-Jozefiak resolution (see [J]) of a graded commutative algebra.

$A = (A_*, \partial_A)$ will denote the DGL associated to $A_{*,*}$ (Definition 2.8); by construction A is the minimal model (§2.5) for L (which is not minimal itself, unless L_* happens to be a *free* graded Lie algebra).

5.8. Example. Consider the graded Lie algebra $L_* = \mathbb{L}\langle a_1, b_1, c_2\rangle/I$, where I is the Lie ideal generated by $[a,a]$ and $[[c,a],[b,a]]$. The minimal model for the coformal DGL $L = (L_*, 0) \in d\mathcal{L}$ is (A_*, ∂_A), where A_* in dimensions ≤ 7 is $\mathbb{L}\langle a_1, b_1, c_2, x_3, y_5, w_6, z_7\rangle$, with $\partial_A(x) = [a,a]$, $\partial_A(y) = 3[x,a]$, $\partial_A(w) = [[c,a],[b,a]]$, and $\partial_A(z) = 4[y,a] + 3[x,x]$.

The bigraded model $A_{*,*}$ is obtained from A_* by introducing an additional (homological) grading: $a, b \in A_{0,1}$, $c \in A_{0,2}$, $x \in A_{1,3}$, $w \in A_{1,6}$, $y \in A_{2,5}$, $z \in A_{3,7}$, and so on.

5.9. Proposition. *Let $L = (L_*, 0) \in d\mathcal{L}$ be a coformal DGL, and $A_{*,*}$ its the bigraded model; then there is an $\mathcal{M}_{d\mathcal{L}}$-free simplicial resolution $C_{\bullet,*} \to L$, with a bijection $\theta : X_{**} \hookrightarrow C_{\bullet,*}$ between a bigraded set X_{**} of generators for $A_{*,*}$ and the set of non-degenerate $d\mathcal{L}$-spheres in $C_{\bullet,*}$. Moreover, $H'_*(C_{\bullet,*})$ is a minimal CW-resolution of $L_* = H'_*(A_{*,*})$, with CW basis generated by $im(\theta)$.*

Proof. By Proposition 2.9 there is a simplicial graded Lie algebra resolution $C_{\bullet,*} \to L_*$ corresponding to $A_{*,*}$, and thus a weak equivalence of simplicial graded Lie algebras $\psi : C_{\bullet,*} \to V_{\bullet,*} = V_{\bullet,*}(Ls)$ (see §5.5), which is one-to-one because $A_{*,*}$, and thus $C_{\bullet,*}$, are minimal (cf. [BL, §2]).

Now let $W_{\bullet,*}$ be the canonical free simplical DGL resolution of A_*; the fact that $\phi : A_{*,*} \to L_*$ is a quasi-isomorphism implies that there is a weak equivalence $\varphi : V_{\bullet,*} \to W_{\bullet,*}$ (as well as one in the other direction). The composite $\varphi \circ \psi : C_{\bullet,*} \to W_{\bullet,*}$ is again a one-to-one weak equivalence (by minimality); we may therefore think of $C_{\bullet,*}$ as a sub-simplicial object of $W_{\bullet,*}$.

Moreover, there is an embedding of bigraded vector spaces $\eta : A_{*,*} \to C_{\bullet,*}$ (see proof of Proposition 2.9), and thus another such embedding $\theta : A_{*,*} \to W_{\bullet,*}$, which may be defined explicitly as follows (using the notation of §5.6):

For $x \in X_{0,*}$, set $\theta(x) = \langle x\rangle \in C_{0,*} = F(A_*)$. Since ϕ maps $X_{0,*}$ onto a (minimal) set of Lie algebra generators for $L_* = H'_*(A_{*,*})$, each $\theta(x)$ is a ∂_W-cycle, so $C^{(0)}_{0,*} := \coprod_{k=1}^\infty \coprod_{x \in X_{0,k}} \mathcal{S}^k_{(\theta(x))}$ is a sub free DGL of $W_{0,*}$.

By minimality of $A_{*,*}$, any $x \in X_{n,*}$ $(n \geq 1)$ is uniquely determined by $\partial_A(x) \in A_{n-1,*}$. Thus if we require θ to be multiplicative (with respect to the ordinary bracket in $A_{*,*}$, and with respect to the simplicial Lie bracket $[\![\, , \,]\!]$ of (2.11) in $W_{\bullet\bullet}$), we may define $\theta : A_{*,*} \to W_{\bullet,*}$ inductively by

$$\theta(x) = \langle\theta(\partial_A(x))\rangle, \tag{5.10}$$

and we shall write $x^{(0)}$ for $\theta(x)$ if $x \in X_{**}$.

By definition (see Proposition 2.9), $d_0 \circ \theta = \theta \circ \partial_A$, so for $x \in X_{n,*}$ $(n \geq 2)$ we have $d_1(x^{(0)}) = d_1\langle\theta(\partial_A(x))\rangle = \langle d_0\theta(\partial_A(x))\rangle = \langle\theta(\partial_A^2(x))\rangle = 0$, while $\varepsilon(d_1(x^{(0)}))$ is a ∂_A-boundary for $x \in X_{1,*}$ (where $\varepsilon : W_{0,*} \to A_*$ is the augmentation). Thus Lemma 5.13 below implies that for $x \in X_{n,*}$ $(n \geq 1)$ we have $d_i(x^{(0)}) = 0$ for all $1 \leq i \leq n-1$, while $d_n(x^{(0)})$ is a ∂_W-boundary.

Therefore, if we set $C^{(0)}_{n,*} := \coprod_{k=1}^\infty \coprod_{x \in X_{n,k}} \mathcal{S}^k_{(x^{(0)})}$ for all $n \geq 0$, we see $\{H'_*(C^{(0)}_{n,*})\}_{n=1}^\infty$ is an $\mathcal{L}$-CW basis for $H'_*(C_{\bullet,*})$.

In order to give an explicit description of $C_{\bullet,*}$ in terms of $C^{(0)}_{\bullet,*}$, we need to know the Lie disks in which $d_n(x^{(0)})$ (and their faces) lie. By a double induction on $n \geq 1$ and $1 \leq r \leq n$, we shall now define, for all $x \in X_{n,k}$, elements $x^{(r)} \in W_{n-r,k+r}$ such that $\partial_W(x^{(r)}) = d_{n-r}(x^{(r-1)})$:

Note that for each $x \in A_{n,*}$ we have $\partial_A(x) = \sum_t a_t \omega_t[y_{i_1}, \dots, y_{i_{m_t}}]$, where $\omega_t[\dots]$ is some m_t-fold iterated Lie bracket, $y_{i_j} \in X_{n_j,*}$ with $\sum_{j=1}^{m_t} n_j = n$, and $a_t \in \mathbb{Q}$. Then

$$\theta(x) = \langle \theta(\partial_A(x)) \rangle = \langle \sum_t a_t \omega_t [\![y^{(0)}_{i_1}, \dots, y^{(0)}_{i_{m_t}}]\!] \rangle, \tag{5.11}$$

where $\omega_t[\![\dots]\!]$ is the same m_t-fold iterated Lie bracket as above, but now with respect to the simplicial Lie bracket $[\![\,,\,]\!]$, rather than $[\,,\,]$.

If we set $x^{(s)} = 0$ for $i > n$, we may define $x^{(s)}$ for $0 < s \leq n$ inductively by:

$$x^{(s)} = \langle \sum_t a_t \sum_{\substack{r_1 + \cdots + r_{m_t} = s \\ 0 \leq r_j}} \omega_t [\![y^{(r_1)}_{i_1}, \dots, y^{(r_{m_t})}_{i_{m_t}}]\!] \rangle \in C^{(s)}_{n-s,k-n+s}. \tag{5.12}$$

Thus if we assume by induction that we have chosen $y^{(r_j)}_{i_j}$ with $\partial_W(y^{(r_j)}_{i_j}) = d_{n_{j_0}}(y^{(r_j-1)}_{i_j})$, it follows from Lemma 5.13 below that indeed $\partial_W(x^{(s+1)}) = d_n(x^{(s)})$ and $d_i(x^{(s)}) = 0$ for $0 < i < n$.

For example, $y^{(0)} = \langle y \rangle$ and $\varepsilon(\langle y \rangle) = y \in A_k$ for any $y \in X_{0,k}$. Therefore, for $x \in X_{1,*}$ we have $\varepsilon d_1(x^{(0)}) = \varepsilon d_0(x^{(0)}) = \partial_A(x)$, so we may set $x^{(1)} = \langle x \rangle \in W_{1,k+1}$, with $\partial_W(x^{(1)}) = d_1(x^{(0)})$.

Now if we define by induction $C^{(r)}_{n,*} := C^{(r-1)}_{n,*} \amalg \coprod_{k=1}^{\infty} \coprod_{x \in X_{n+r,k}} \mathcal{D}^{k+r}_{(x^{(r)})}$, and let $C_{\bullet,*}$ be the sub-simplicial graded Lie algebra of $W_{\bullet,*}$ generated (under the degeneracies of $W_{\bullet,*}$) by $(C^{(r)}_{n,*})^n_{r=0}$ for all $n \in \mathbb{N}$, then $C_{\bullet,*}$ is closed under face maps and includes $im(\theta)$, and $\theta : A_{*,*} \to C_{\bullet,*}$ is a weak equivalence. The only non-degenerate Lie spheres in $C_{\bullet,*}$ are those of $C^{(0)}_{\bullet,*}$, as required. $\square$

5.13. Lemma. *If $A_\bullet \in s\mathcal{L}$ is a simplicial graded Lie algebra, $x \in A_p$ with $d_i x = 0$ for $1 \leq i \leq p-1$, and $y \in A_q$ with $d_j y = 0$ for $1 \leq j \leq q-1$, then $d_k([\![x,y]\!]) = 0$ for $1 \leq k \leq p+q-1$.*

Proof. By definition (2.11) we have

$$[\![x,y]\!] = \sum_{(\sigma,\tau) \in S_{p,q}} (-1)^{\varepsilon(\sigma)+p|y|} [s_{\tau_q} \dots s_{\tau_1} x, s_{\sigma_p} \dots s_{\sigma_1} y] \in A_{p+q}. \tag{5.14}$$

Now for each summand $w_{\sigma,\tau} := [s_\tau x, s_\sigma y]$ in (5.14), with (σ,τ) a (p,q)-shuffle, there are two cases to consider:

The first is that there exist ℓ, m such that $\tau_\ell = k$, $\sigma_m = k-1$ – in which case there is an associated (p,q)-shuffle (σ',τ'), differing from (σ,τ) only in that τ_ℓ and σ_m are switched, so that $d_k(w_{\sigma,\tau}) = d_k(w_{\sigma',\tau'})$ but $(-1)^{\varepsilon(\sigma)} = -(-1)^{\varepsilon(\sigma')}$, and these pairs thus cancel in the sum (5.14).

In the second case, $k, k-1 \in \{\sigma_1, \ldots, \sigma_p\}$, say, and then there is some $0 \leq \ell \leq q$ with $\tau_\ell < k-1$ and $\tau_{\ell+1} > k$. Since necessarily $k+1-p \leq \ell \leq k-1$, we find that $d_k s_\tau x = s_{\tau_q -1} \cdots s_{\tau_{\ell+1}-1} s_{\tau_\ell} \cdots s_{\tau_1} d_{k-\ell} x = 0$. $\square$

5.15. Example. Consider the graded Lie algebra $L_* = \mathbb{L}\langle a_1, b_1, c_2\rangle / \langle [a,a], [[c,a],[b,a]]\rangle$ of Example 5.8, with $L = (L_*, 0)$. The $\mathcal{M}$-free simplicial resolution $C_{\bullet,*} \to L$ may be described (in homological dimensions ≤ 3) as follows:

(1) $C^{(0)}_{0,*}$ is the coproduct (in $\mathcal{L}$) of $\mathcal{S}^1_{(a^{(0)})} = \mathcal{S}^1_{(\langle a\rangle)}$, $\mathcal{S}^1_{(b^{(0)})} = \mathcal{S}^1_{(\langle b\rangle)}$, and $\mathcal{S}^2_{(c^{(0)})} = \mathcal{S}^2_{(\langle c\rangle)}$.
(2) $C^{(0)}_{1,*} = \mathcal{S}^2_{(x^{(0)})} \amalg \mathcal{S}^6_{(w^{(0)})}$, where $x^{(0)} = \langle [\langle a\rangle, \langle a\rangle]\rangle$ and $w^{(0)} = \langle [[\langle c\rangle, \langle a\rangle], [\langle b\rangle, \langle a\rangle]]\rangle$.
(3) $C^{(0)}_{2,*}$ consists of $\mathcal{S}^5_{(y^{(0)})}$, where $y^{(0)} = \langle 3[\langle [\langle a\rangle, \langle a\rangle]\rangle, \langle\langle a\rangle\rangle]\rangle$.
(4) $C^{(0)}_{3,*}$ consists of $\mathcal{S}^6_{(z^{(0)})}$, where
$$z^{(0)} = \langle 4[\langle 3[\langle [\langle a\rangle, \langle a\rangle]\rangle, \langle\langle a\rangle\rangle]\rangle, \langle\langle\langle a\rangle\rangle\rangle] + 6[\langle [\langle\langle a\rangle\rangle, \langle\langle a\rangle\rangle]\rangle, \langle\langle [\langle a\rangle, \langle a\rangle]\rangle\rangle]\rangle.$$

For $C^{(1)}_{\bullet,*}$ we need in addition

(1) $\mathcal{D}^3_{(x^{(1)})} \hookrightarrow C^{(1)}_{0,*}$ with $\partial_W(x^{(1)}) = d_1(x^{(0)}) = \langle [a,a]\rangle$.
(2) $\mathcal{D}^6_{(y^{(1)})} \hookrightarrow C^{(1)}_{1,*}$ with $y^{(1)} = \langle 3[\langle x\rangle, \langle a\rangle]\rangle$ and $\partial_W(y^{(1)}) = d_2(y^{(0)}) = \langle 3[\langle [a,a]\rangle, \langle a\rangle]\rangle$.
(3) $\mathcal{D}^7_{(z^{(1)})} \hookrightarrow C^{(1)}_{2,*}$ with $z^{(1)} = \langle 4[\langle 3[\langle x\rangle, \langle a\rangle]\rangle, \langle\langle a\rangle\rangle] + 6[\langle [\langle a\rangle, \langle a\rangle]\rangle, \langle\langle x\rangle\rangle]\rangle$ and $\partial_W(z^{(1)}) = d_3(z^{(0)}) = \langle 4[\langle 3[\langle [a,a]\rangle, \langle a\rangle]\rangle, \langle\langle a\rangle\rangle] + 6[\langle [\langle a\rangle, \langle a\rangle]\rangle, \langle\langle [a,a]\rangle\rangle]\rangle$.

For $C^{(2)}_{\bullet,*}$ we need in addition

(1) $\mathcal{D}^7_{(y^{(2)})} \hookrightarrow C^{(2)}_{0,*}$ with $\partial_W(y^{(2)}) = d_1(y^{(1)}) = \langle 3[x,a]\rangle$.
(2) $\mathcal{D}^8_{(z^{(2)})} \hookrightarrow C^{(2)}_{1,*}$ with $z^{(2)} = \langle 4[\langle y\rangle, \langle a\rangle] + 3[\langle x\rangle, \langle x\rangle]\rangle$ and $\partial_W(z^{(2)}) = d_2(z^{(1)}) = \langle 4[\langle 3[x,a]\rangle, \langle a\rangle] + 6[\langle [a,a]\rangle, \langle x\rangle]\rangle$.

For $C^{(3)}_{\bullet,*}$ we must add $\mathcal{D}^9_{(z^{(3)})} \hookrightarrow C^{(3)}_{0,*}$ with $\partial_W(z^{(1)}) = d_1(z^{(2)}) = \langle 4[y,a] + 3[x,x]\rangle$.

5.16. **the filtered model.** If $B = (B_*, \partial_B) \in d\mathcal{L}$ is an arbitrary DGL, it no longer has a bigraded model, as in §5.7 above. However, if $L_* := H'_*(B)$ is the homotopy Lie algebra of B_*, the bigraded model $(A_{*,*}, \partial_A)$ for $(L_*, 0)$ may be *perturbed* into a *filtered model* for B: that is, one may define a increasing filtration $0 = \mathcal{F}^{-1}(A) \subset \mathcal{F}^0(A) \subset \cdots \mathcal{F}^r(A) \subset \mathcal{F}^{r+1}(A) \subset \cdots$ on $A_{*,*}$ by $\mathcal{F}^r(A) := \bigoplus_{i=0}^r A_{i,*}$, and a new differential $D_A = \partial_A + \delta_A$ on $A_{*,*}$ such that $\delta_A : A_{n,*} \to \mathcal{F}^{n-2}(A)$ (and of course D_A is a still a derivation). We may decompose $D_A : A_{n,*} \to A_{*,*}$ as $D_A = \partial_0 + \partial_1 + \cdots + \partial_{n-1}$, where $\partial_r : A_{n,*} \to A_{n-r-1,*}$ (and $\partial_0 = \partial_A$, the original differential of the bigraded model).

See [O, II] or [Har]; this is again the Lie algebra version of a construction of Halperin and Stasheff in [HS, §4] (see also [F1]).

Note that the filtered model is no longer unique, since its construction depends on choices; in particular, it is not necessarily minimal. One again has the associated DGL (A_*, D_A), which is quasi-isomorphic to the original DGL B, and $A_{*,*}$ is obtained by filtering A_*.

5.17. Proposition. *Let $B = (B_*, \partial_B)$ be a DGL, and $(A_{*,*}, D_A)$ a filtered model for B; then there is an $\mathcal{M}_{d\mathcal{L}}$-free simplicial resolution $E_{\bullet,*} \to B$, with a bijection*

$\theta : X_{**} \hookrightarrow E_{\bullet,*}$ between a bigraded set X_{**} of generators for $A_{*,*}$ and the set of non-degenerate $d\mathcal{L}$-spheres in $E_{\bullet,*}$.

Proof. We start with the minimal $\mathcal{M}$-free resolution $C_{\bullet,*} \to L_*$ for $L_* = H'_*(B_*)$, constructed as in the proof of Proposition 5.9, and deform it into an $\mathcal{M}$-free resolution for B, using the filtered model $(A_{*,*}, D_A)$ as a guideline. This time we shall embed the resulting $\mathcal{M}$-free resolution in the canonical free DGL resolution $W_{\bullet,*}$ of (A_*, D_A), the DGL associated to the filtered model:

For each $x \in X_{n,k}$ (where X_{**} is a bigraded set of generators for the bigraded Lie algebra $A_{*,*}$, as above), set $x^{(n)} = \langle x\rangle \in W_{0,k}$, and let $D_A(x) = \partial_0(x) + \partial_1(x) + \cdots + \partial_{n-1}(x)$ as above, with

$$\partial_r(x) = \sum_t a_t^{(r)} \omega_t^{(r)}[y_{i_1}, \ldots, y_{i_{m_t}}] \in A_{n-r-1,*},$$

where $\omega_t^{(r)}[\ldots]$ is some m_t-fold iterated Lie bracket, as above, and each $y_{i_j} \in X_{n_j,*}$ with $\sum_{j=1}^{m_t} n_j = n - r - 1$.

If we set $x^{(s)} = 0$ for $i > n$, we may define $x^{(s)}$ for $0 < s \le n$ inductively by:

$$x^{(s)} = \langle \sum_{r=0}^{s} \sum_t a_t^{(r)} \sum_{\substack{r_1+\cdots+r_{m_t}=s-r \\ 0\le r_j}} \omega_t^{(r)} [\![y_{i_1}^{(r_1)}, \ldots, y_{i_{m_t}}^{(r_{m_t})}]\!] \rangle \in C_{n-s,k-n+s}^{(s)} \tag{5.18}$$

Using Lemma 5.13 and the fact that for any $A_\bullet \in s\mathcal{L}$, $x \in A_p$ and $y \in A_q$ we have $d_{p+q}([\![x,y]\!]) = [\![d_p(x), y]\!] + (-1)^q [\![x, d_q(y)]\!]$, one may then verify inductively that $d_{n-s}(x^{(s)}) = \partial_W(x^{(s+1)})$ and $d_i(x^{(s)}) = 0$ for $0 < i < n-s$, for all $0 \le s < n$. The rest of the construction is as in the proof of Proposition 5.9. $\square$

5.19. Example. Consider the DGL $B = (B_*, \partial_B) \in d\mathcal{L}$ where B_* is the free Lie algebra $\mathbb{L}\langle a_1, b_1, c_2, x_3, y_5, z_7, \ldots\rangle$, with $\partial_B(x) = [a,a]$, $\partial_B(y) = 3[x,a] - [[b,a],c]$, $\partial_B(z) = 4[y,a] + 3[x,x]$, and so on.

Here $L_* := H'_*(B) = \mathbb{L}\langle a_1, b_1, c_2\rangle / \langle [a,a], [[c,a],[b,a]]\rangle$, so the bigraded model for $(L_*, 0)$ is $(A_{*,*}, \partial_A)$ of Example 5.8 above, and the filtered model is obtained from it by setting $D_A y = 3[x,a] - [[b,a],c]$ and $D_A(z) = 4[y,a] + 3[x,x] - 4w + 2[[x,b],c]$.

The corresponding $\mathcal{M}$-free resolution is obtained from $C_{\bullet,*}$ of Example 5.15 by making the following changes:

1. Set $y^{(1)} := \langle 3[\langle x\rangle, \langle a\rangle] - [[\langle b\rangle, \langle a\rangle], \langle c\rangle]\rangle$, with $\partial_W(y^{(1)}) = d_2(y^{(0)}) = \langle 3[\langle [a,a]\rangle, \langle a\rangle]\rangle$ as before, (but now $\partial_W(y^{(2)}) = \langle 3[x,a] - [[b,a],c]\rangle$, of course).
2. Set $z^{(1)} := \langle 4[\langle 3[\langle x\rangle, \langle a\rangle]\rangle, \langle\langle a\rangle\rangle] + 6[\langle[\langle a\rangle, \langle a\rangle]\rangle, \langle\langle x\rangle\rangle] - 4\langle[[\langle c\rangle, \langle a\rangle], [\langle b\rangle, \langle a\rangle]]\rangle + 2[[\langle[\langle a\rangle, \langle a\rangle]\rangle, \langle\langle b\rangle\rangle], \langle\langle c\rangle\rangle]\rangle$ (with $\partial_W(z^{(1)})$ unchanged).
3. Set $z^{(2)} := \langle 4[\langle y\rangle, \langle a\rangle] + 6[\langle x\rangle, \langle x\rangle] - 4\langle w\rangle + 2[[\langle x\rangle, \langle b\rangle], \langle c\rangle]\rangle$, with $\partial_W(z^{(2)}) = d_3(z^{(1)}) =$
$\langle 4[\langle 3[x,a] - [[b,a],c]\rangle, \langle a\rangle] + 6[\langle [a,a]\rangle, \langle x\rangle] - 4\langle [[c,a],[b,a]]\rangle + 2[[\langle [a,a]\rangle, \langle b\rangle], \langle c\rangle]\rangle$.
4. Finally, $\partial_W(z^{(3)}) = \langle 4[y,a] + 3[x,x] - 4w + 2[[x,b],c]\rangle$.

We have the following analogue of Definition 4.5:

5.20. Definition. Given a DGL $L = (\mathbb{L}\langle X_*\rangle, \partial_L) \in d\mathcal{L}$ and a simplicial set $A \in \mathcal{S}$, we define their *half-smash* $L \rtimes A = (\mathbb{L}\langle Y_*\rangle, \partial') \in d\mathcal{L}$ by setting $Y_n := \coprod_{k=0}^{n} X_k \times \hat{A}_{n-k}$, where $\hat{A}_i$ denotes the set of non-degenerate i-simplices of A. For $a \in \hat{A}_k$ and $x \in X_m$, we set

$$\partial'(x,a) = \sum_{i=0}^{k} (-1)^{i+m}(x, d_i a) + (\partial_L x, a)$$

(and extend ∂' by requiring that it be a derivation).

5.21. Example. $\mathcal{S}^2_{(x)} \rtimes D[1] = (\mathbb{L}\langle X_* \rangle, \partial')$, where $X_2 = \{(x,(d_0)), (x,(d_1))\}$, $X_3 = \{(x,(id))\}$, and $\partial'(x,(id)) = (x,(d_0)) - (x,(d_0))$.

Similarly, $\mathcal{S}^3_{(y)} \rtimes D[2] = (\mathbb{L}\langle Y_* \rangle, \partial')$, where $Y_3 = \{(y,(d_0 d_1)), (y,(d_0 d_2)), (y,(d_1 d_2))\}$, $Y_4 = \{(y,(d_0)), (y,(d_1)), (y,(d_2))\}$, and $Y_5 = \{(y,(id))\}$, with $\partial'(y,(id)) = -(y,(d_0)) + (y,(d_1)) - (y,(d_2))$, $\partial'(y,(d_0)) = -(y,(d_0 d_1)) + (y,(d_0 d_2))$, $\partial'(y,(d_1)) = -(y,(d_0 d_1)) + (y,(d_1 d_2))$, and $\partial'(y,(d_2)) = -(y,(d_0 d_2)) + (y,(d_1 d_2))$.

5.22. Remark. In order to apply the obstruction theory described in §4.15, note that all the definitions of section 4 pass over to the DGL setting in a straightforward manner. However, if we now start with the trivial DGL $A = L^{(0)}_* := (\Pi^X_*, 0)$, we may take $C_{\bullet,*} \to L^{(0)}_*$ to be the minimal $\mathcal{M}$-free resolution of Proposition 5.9, corresponding to the bigraded model for $(\Pi^X_*, 0)$, and let $B = (B_*, \partial_B)$ (corresponding to $B_\bullet$ in §4.15) be the filtered model for L_X. We assume that $A \not\simeq B$.

As explained in §4.15, there is a least $n_0 \geq 2$ such that $0 \notin \langle\langle n_0 \rangle\rangle \subseteq H'_*(B)$, and we write $\langle\langle n_0 \rangle\rangle = (\langle\langle n_0, x_i \rangle\rangle)_{i \in I}$, in the notation of 4.10, where $x_i \in X_{n_0,i}$ and $\mathcal{S}^{k_i}_{(x_i)}$ are corresponding DGL spheres in $C_{n_0,*}$ (we include in the index set I only those coordinates of (4.12) which do not vanish).

Again let $\mathcal{H}^{(1)}$ denote the set of all homotopy types in $ho\, d\mathcal{L}(\mathbf{X})$ for which $\langle\langle n_0 \rangle\rangle$ has the same value as for L_X, and choose a representative $L^{(1,\alpha)} \in d\mathcal{L}(\mathbf{X})$ for each $\alpha \in \mathcal{H}^{(1)}$. By [HS, §3], we may assume $L^{(1,\alpha)}$ is obtained from B by perturbation of ∂_B. Proceeding as in §4.15 we obtain a tree of DGLs $L^{(k,\alpha_1,\alpha_2,\ldots,\alpha_k)} \in d\mathcal{L}(\mathbf{X})$, and by [Bl1, Theorem 3.1], we know that $L^{(k,\alpha_1,\alpha_2,\ldots,\alpha_k)}$ may be chosen to agree with L_X through degree $n+1$ at least, so $\operatorname{colim}_n L^{(n)} \simeq L_X$ along any branch of the tree.

Note also that because $H'_*(C_{\bullet,*})$ is a (minimal) CW resolution of $H'_*(B)$, in each case, the maps $\psi_n : H'_*(C_{n,*}) \to H'_*(B)$ are null for all $n \geq 1$ (see §4.2). Thus any $\partial D[n]$-compatible collection $\{h_k\}_{k=0}^{n-1}$ in §4.10 induce a map $C_{n,*} \wedge \partial D[n] \to B$ directly, without need of the splitting (4.11).

5.23. Example. For the DGL $B = (B_*, \partial_B)$ of example 5.19, with $C_{\bullet,*}$ as in example 5.15, we define $h_0 = f_0 : C_{0,*} \to B$ by setting $f_0(\langle a \rangle) = a$, $f_0(\langle b \rangle) = b$, $f_0(\langle c \rangle) = c$, and $f_0 = 0$ for all other disks in $C_{0,*}$ (so for example $f_0(\langle [a,a] \rangle) = 0$).

Thus on $\mathcal{S}^2_{(x^{(0)})} \rtimes D[1]$ we have $h_1(x^{(0)},(d_0)) = [a,a]$ and $h_1(x^{(0)},(d_1)) = 0$, by definition (4.8), so we must choose $h_1(x^{(0)},(id)) = x \in B_3$.

Now on $\mathcal{S}^3_{(y^{(0)})} \rtimes D[2]$ we have $h_2(y^{(0)},(d_0 d_1)) = h_2(y^{(0)},(d_0 d_2)) = h_2(y^{(0)},(d_1 d_2)) = 0$, and $h_2(y^{(0)},(d_0)) = (h_1 \circ d_0)(y^{(0)},(d_0) = 3[x,a]$, while $h_2(y^{(0)},(d_1)) = h_2(y^{(0)},(d_2)) = 0$.

This defines a $\partial D[2]$ compatible sequence for $C_{\bullet,*}$, and the resulting secondary operation is $\langle\langle 2, y^{(0)} \rangle\rangle = \{\langle 3[x,a] \rangle]\} \subseteq H'_4(B)$; but since $3[x,a]$ does not bound in B, $\langle\langle 2 \rangle\rangle$ does not vanish, and we have found the (expected) obstruction to the coformality of B.

5.24. Remark. The second order operation described in the previous example is actually a secondary Whitehead product. Unlocalized higher order Whitehead products were defined by G. Porter in [P, 1.3], and the relation between this definition and the rational version has been studied by several authors – see [AA], [A1, A2], [R2, R1] and [T, V.1].

However, there are other higher order rational homotopy operations, too: for example, in the DGL $L_* = (\mathbb{L}\langle a_1, b_1, c_1, d_1, x_4, y_4, z_4, w_4\rangle, \partial)$, with $\partial(x) = [[b,a],c]$, $\partial(y) = [[b,a],d]$, $\partial(z) = [[d,c],a]$ and $\partial(w) = [[d,c],b]$, the cycle $[x,d] + [y,c] + [z,b] + [w,a]$ represents such an operation. There appears to be no general procedure for representing these as *integral* higher order operations in $\pi_* \mathbf{X}$; we shall offer a (partial) answer to this difficulty in section 7.

6. Homology of DGLs

Obstructions in algebraic topology traditionally take values in suitable cohomology groups. In order to show that this holds in our seting, too, we recall Quillen's definition of homology and cohomology in model categories:

6.1. Definition. An object X in a category $\mathcal{C}$ is said to be *abelian* if it is an abelian group object – that is, if $Hom_{\mathcal{C}}(Y, X)$ has a natural abelian group structure for any $Y \in \mathcal{C}$. When $\mathcal{C}$ is $\mathcal{L}ie$, $\mathcal{A}lg$, $s\mathcal{L}ie$, $s\mathcal{A}lg$, $\mathcal{L}$, or $d\mathcal{L}$, for example, this is equivalent to requiring that all products vanish in X (cf. [BS, §5.1.3]).

The full subcategory of abelian objects in $\mathcal{C}$ is denoted by $\mathcal{C}_{ab} \subset \mathcal{C}$. In the cases of interest to us, this will itself be an abelian category. It is equivalent to the category $\mathcal{V}ec$ of vector spaces if $\mathcal{C} = \mathcal{L}ie$ or $\mathcal{A}lg$, to $\mathcal{V}$ if $\mathcal{C} = \mathcal{L}$, to the category $s\mathcal{V}ec$ of simplicial vector spaces if $\mathcal{C} = s\mathcal{L}ie$ or $s\mathcal{A}lg$, and to the category $d\mathcal{V}$ if $\mathcal{C} = d\mathcal{L}$ (see §1.1).

In these cases, we have an *abelianization* functor $Ab : \mathcal{C} \to \mathcal{C}_{ab}$, along with a natural transformation $\theta : Id \to Ab$ having the appropriate universal property. In all the examples above, $Ab(X) = X/I(X)$, where $I(X)$ is the ideal in $X \in \mathcal{C}$ generated by all non-trivial products.

6.2. Definition. Let $\mathcal{C}$ be a category as above, which also has a closed model category structure: in [Q1, II, §5] (or [Q4, §2]), Quillen defines the *homology* of an object $X \in \mathcal{C}$ to be the total left derived functor $\mathbf{L}(Ab)$ of Ab, applied to X (cf. [Q1, I, §4]).

In more familiar terms, this means that we construct a resolution $A \to X$ (i.e., replace X by a weakly equivalent cofibrant object $A \in \mathcal{C}$), and then define the i-th homology group of X by $H_i X := \pi_i(Ab(A))$, for an appropriate concept of homotopy groups π_* in $\mathcal{C}_{ab}$ (see [Q1, II, §4]). One must verify, of course, that this definition is independent of the choice of the resolution $A \to X$.

Similarly, the *cohomology* of X with coefficients in $M \in \mathcal{C}_{ab}$ is defined:

$$H^i(X; M) := [\mathbf{L}(Ab)X, \Omega^{i+N}\Sigma^N M]_{\mathcal{C}_{ab}} \qquad \text{for } N \qquad \text{large enough}$$

(where the loop and suspension functors Ω and Σ are defined in [Q1, I, §2]).

Again, in the cases that interest us, Ω is essentially the shift operator Σ^{-1} of §1.1, and so the i-th cohomology group of X with coefficients in M is then $H^i(X; M) := [\Sigma^i Ab(A), M]_{\mathcal{C}_{ab}}$.

6.3. Definition. If $\mathcal{C}$ itself does not have a closed model category structure, one often defines the homology of $\mathbf{X} \in \mathcal{C}$ by embedding $\mathcal{C}$ in some category which does have such a structure, which in many cases may be taken to be $s\mathcal{C}$, the category of simplicial objects over $\mathcal{C}$ (see [Q1, II, §4]). Thus, if $\iota : \mathcal{C} \hookrightarrow s\mathcal{C}$ is the embedding of categories defined by taking $\iota(C)$ to be the constant simplicial object equal to C in all dimensions, then $H_i(C) := \pi_i(\mathbf{L}(Ab \circ \iota)C)$.

This is the approach usually taken for $\mathcal{C} = \mathcal{L}ie$, $\mathcal{A}lg$, or $\mathcal{L}$: to define the homology of a graded Lie algebra $L_* \in \mathcal{L}$, say, one chooses a free simplicial resolution $A_{\bullet,*} \to L_*$ (such as the canonical resolution – cf. §5.3), and then calculate the homotopy groups of the simplicial graded vector space $Ab(A_{\bullet,*}) \in s\mathcal{V}$ (or the homology groups of the bigraded chain complex in $db\mathcal{V}$ corresponding to $Ab(A_{\bullet,*})$ – see proof of Proposition 2.9).

6.4. Remark. Note that if we apply Definition 6.2 as is to a DGL $L = (L_*, \partial_L) \in d\mathcal{L}$, we may take the resolution A to be the minimal model $\hat{L} = (\hat{L}_*, \hat{\partial})$ for L (cf. §2.5), and since its abelianization is just the graded vector space $Q(\hat{L})$ of indecomposables, and $Q(\hat{\partial}) = 0$ by definition, $H_i(L)$ would be isomorphic to the vector space spanned by a set of generators for $\hat{L}$ in dimension i.

If we want cohomology with coefficients in an object $M_* \in d\mathcal{L}_{ab} \approx d\mathcal{V}$ with *trivial* differential – i.e., M_* is just a graded vector space – we find

$$H^i(X; M_*) \cong \prod_{j=1}^{\infty} Hom_{\mathcal{V}ec}(H_j(X),\ M_{i+j}),$$

by the universal coefficients theorem.

However, since L is itself graded, we would like H_*L to be bigraded (with a "homological" degree, as well as a "topological" one). This requires a combination of Definitions 6.2 and 6.3, as follows:

6.5. Definition. The *homology* $H_{*,*}(L_\bullet)$ of a simplicial Lie algebra $L_\bullet \in s\mathcal{L}ie$ is defined to be the left derived functors of the abelianization, with respect to the E^2-closed model category structures (§3.3) on $ss\mathcal{L}ie$ and $ss\mathcal{L}ie_{ab} \approx dd\mathcal{V}$ respectively. More precisely,

$$H_{s,t}(L_\bullet) := \pi_s(\mathbf{L}(Ab \circ \iota)L_\bullet)_t = \pi_s\pi^i_t(AbA_{\bullet\bullet}), \tag{6.6}$$

where $A_{\bullet\bullet} \to L_\bullet$ is some $\mathcal{M}$-free bisimplicial resolution of $L_\bullet$.

Similarly, for any DGL $L \in d\mathcal{L}$ we may define $H_{s,t}(L) := \pi_s H'_t(Ab(A_{\bullet,*}))$, for a $\mathcal{M}_{d\mathcal{L}}$-free simplicial resolution $A_{\bullet,*} \to L$; and these two definitions agree under the equivalence of homotopy categories $ho(s\mathcal{L}ie) \approx ho(d\mathcal{L})$ of Proposition 2.2.

The bigraded cohomology of a DGL L with coefficients in the abelian DGL (i.e., chain complex) M is defined analogously as $H^s_t(L) := \pi^s(Hom_{d\mathcal{L}_{ab}}(Ab(A_{\bullet,*}), M)_t)$

We note that the homology and cohomology of differential graded (commutative) algebras have been defined by Goodwillie (in [Go]) and Burghelea & Vigué-Poirrier (in [BV]), in a manner analogous to the traditional definitions of Hochschild homology. See [Lo, §5.3].

6.7. Proposition. *For any DGL* $L \in d\mathcal{L}$, *there is a monomorphism of graded vector spaces* $H_{n,t}(L) \hookrightarrow H_{n,t}(L')$, *where* $L' \simeq (H'_*(L), 0)$ *is the coformal model for L; the same holds for cohomology with trivial coefficients.*

Proof. If $A_{*,*}$ is the bigraded model for L', and $C_{\bullet,*} \to L'$ the simplicial resolution of Proposition 5.9, then the non-degenerate spheres $\mathcal{S}^k_{(x^{(0)})} \subset C_{n,t}$, which correspond to a vector space basis for $H_{n,t}(L')$, are in bijective correspondence with the generators $x \in X_{n,t}$ for $A_{*,*}$.

Now let $B_{*,*}$ be a filtered model for L obtained by perturbing $(A_{*,*}, \partial_A)$, and $E_{\bullet,*} \to L$ the associated simplicial resolution of Proposition 5.17: since $B_{*,*}$ need no longer be minimal (§5.16), a vector space basis for $H_{n,t}(L)$ now corresponds to a *subset* of the collection of non-degenerate spheres $\mathcal{S}^k_{(x^{(0)})} \subset E_{n,t}$, (which are still in bijective correspondence with the generators $x \in X_{n,t}$ for $A_{*,*}$ or $B_{*,*}$). $\square$

Note that this description of the homology implies also that $H_{*,*}(L)$ is indeed just a bigraded version of the DGL homology defined in §6.3.

6.8. Proposition. *The collection of higher homotopy operations* $\langle\langle n_{k,\alpha_1,\alpha_2,\dots,\alpha_k}, x\rangle\rangle$ *which determine the rational homotopy type of* $\mathbf{X} \in \mathcal{T}_1$ *(described in §4.15 above) are indexed by elements* $x \in H_{n_{k,\alpha_1,\alpha_2,\dots,\alpha_k},t}(L^{(k,\alpha_1,\alpha_2,\dots,\alpha_k)})$ *in the homology of the DGLs of §5.22, and take value in the cohomology of these DGLs, with* $\langle\langle n_{k,\alpha_1,\alpha_2,\dots,\alpha_k}, x\rangle\rangle \subseteq H^{n_{k,\alpha_1,\alpha_2,\dots,\alpha_k}}_{t+n_k-1}(L^{(k,\alpha_1,\alpha_2,\dots,\alpha_k)}; \pi_*\mathbf{X}_{\mathbb{Q}})$.

Proof. We may construct a simplicial resolution $E_{\bullet,*}$ for each successive DGL $L^{(k)} = L^{(k,\alpha_1,\alpha_2,\dots,\alpha_k)}$, corresponding to the filtered models obtained as perturbations $(A_{*,*}, D_A)$ of the bigraded model $(A_{*,*}, \partial_A)$ for $L^{(0)}$, as above. The non-degenerate spheres $\mathcal{S}^m_{(x)} = \mathcal{S}^m_{(x^{(0)})} \subset E_{m,t}$ which index the higher homotopy operations $\langle\langle m, x\rangle\rangle$ are thus in bijective correspondence with the generators $x \in X_{m,t}$ for $A_{*,*}$. However, if x is not minimal – in the sense that $D_A(x) \notin [A, A]$, or $x + \alpha = D_a(y)$ for some $\alpha \in A_{*,*}$ and $y \in X_{m+1,t}$ – then we can construct a new simplicial resolution $E'_{\bullet,*}$ of $L^{(k)}$ in which $\mathcal{S}^m_{(x)}$ has been eliminated (though of course new spheres may appear in higher simplicial dimensions). By the universal property of resolutions (i.e., of cofibrant objects in the E^2 model category for $sd\mathcal{L}$ – see §3.3) there is a map of resolutions $E_{\bullet,*} \to E'_{\bullet,*}$, and there can be no non-vanishing higher operation $\langle\langle n_{k,\alpha_1,\dots,\alpha_k}, x\rangle\rangle$ which serves as an obstruction to rectifying the augmentation up-to-homotopy $\varphi : E_{\bullet,*} \to L^X$, since $\varphi_n|_{\mathcal{S}^m_{(x)}}$ can be factored through $0 \in E'_{\bullet,*} \to L^X$. Thus the only homotopy operations which can appear are those corresponding to *non-trivial* homology classes in $H_*(L^{(k,\alpha_1,\alpha_2,\dots,\alpha_k)})$. $\square$

Proposition 6.7 thus implies that we may if we like think of all the higher homotopy operations described in §4.15 (associated to the various deformations of $L^{(0)}$) as lying in one fixed bigraded group $H^*_*(L^{(0)}; \pi_*\mathbf{X}_{\mathbb{Q}})$, which is of course just the usual cohomology of a graded Lie algebra, and is easier to compute than the cohomology of a non-trivial DGL.

7. Non-associative algebra models

The DGL higher homotopy operations, of which an example was given in §5.23, are unsatisfactory from a topological point of view because there is no obvious way to translate them, in general, into *unlocalized* topological homotopy operations. We now describe an algebraic model for rational homotopy theory which may serve to answer this objection.

7.1. **non-associative graded algebras.** Let $\mathcal{A}$ denote the category of *non-associative graded algebras*: an object $A_* \in \mathcal{A}$, is just a graded vector space equipped with a bilinear graded product $A_p \otimes A_q \to A_{p+q}$. Let $\mathbb{A}\langle X\rangle$ denote the free non-associative graded algebra generated by a graded set X_*. As in §2.1, the functor $\mathbb{A} : gr\mathcal{S}et \to \mathcal{A}$ factors through $A : \mathcal{V} \to \mathcal{A}$.

$d\mathcal{A}$ will denote the category of differential graded non-associative algebras (A_*, ∂_A), called *DGNAs*; the differential ∂_A must satisfy $\partial_A \circ \partial_A = 0$ and $\partial_A(x \cdot y) = \partial_A x \cdot y + (-1)^{|x|} x \cdot \partial_A y$, as for DGLs.

For simplicity we assume each $A_* \in \mathcal{A}, d\mathcal{A}$ is connected – that is, $A_0 = \{0\}$. Again, we have a category $db\mathcal{A}$ of differential bigraded non-associative algebras (*DBGNA*s), as in §2.6.

As for any CUGA (§3.1), one can define a closed model category structure on $s\mathcal{A}$ (see [Q1, II, §4]) and thus by [Bl4, §4] on $db\mathcal{A}$, and one has the expected analogues of Propositions 2.2 and 2.9:

7.2. Proposition. *There are adjoint functors* $s\mathcal{A}lg \underset{N^*}{\overset{N}{\rightleftarrows}} d\mathcal{A}$, *which induce equivalences of the corresponding homotopy categories* $ho(s\mathcal{A}lg) \approx ho(d\mathcal{A})$.

7.3. Proposition. *There are adjoint functors* $s\mathcal{A} \underset{N^*}{\overset{N}{\rightleftarrows}} db\mathcal{A}$, *which induce equivalences of the corresponding homotopy categories* $ho(s\mathcal{A}) \approx ho(db\mathcal{A})$; *and* N^* *takes free DBGNAs to free simplicial non-associative algebras.*

7.4. Notation. For any $(X_*, \cdot) \in \mathcal{A}$, let $[x, y]$ denote $\frac{1}{2}(x \cdot y + (-1)^{|x||y|+1} y \cdot x)$. We then have $[y, x] = (-1)^{|x||y|+1}[x, y]$, so $(X_*, [\,,\,])$ is now a non-associative graded algebra with a graded-commutative multiplication. Moreover, any graded derivation ∂ on $(X_*, \cdot)$ is also a derivation with respect to $[\,,\,]$, and any morphism of algebras from $(X_*, \cdot)$ to a graded-commutative algebra will also respect $[\,,\,]$. Therefore we can (and will) assume that our non-associative algebras are all graded-commutative, and denote the product by $[\,,\,]$.

Moreover, if $A_\bullet \in s\mathcal{A}$ is a simplicial graded algebra, we shall also use the notation $[\![x, y]\!] = \sum_{(\sigma,\tau)\in S_{p,q}} (-1)^{\varepsilon(\sigma)+p|y|}[s_{\tau_q} \dots s_{\tau_1} x, s_{\sigma_p} \dots s_{\sigma_1} y]$ for the corresponding simplicial bracket (compare (2.11)).

7.5. Definition. Any simplicial Lie algebra $L_\bullet \in s\mathcal{L}ie$ is in particular an object in $s\mathcal{A}lg$; let $\iota : d\mathcal{L} \hookrightarrow d\mathcal{A}$ be the inclusion functor. Note that even if each L_n is free as a Lie algebra, it is not free as a non-associative algebra: a free simplicial resolution $J_\bullet \to \iota(L_\bullet)$ in the category $s\mathcal{A}lg$ (see §3.2) will be called a *$s\mathcal{A}lg$-model* for $L_\bullet$. Such models can be constructed functorially, for example by a variant of §5.5.

There is also the analogous concept of a *$d\mathcal{A}$-model* $J_* \in d\mathcal{A}$ of a DGL L; we can of course translate back and forth between these two types of models using Proposition 7.2.

Since the DGL $L = (L_*, \partial_L)$ has an internal grading, and its $d\mathcal{A}$-model $J = (J_*, \partial_J)$ is constructed as a resolution of L, it is natural to define a second "homological" degree on J_*, so as to have a *filtered* $d\mathcal{A}$-model (cf. §5.16). If the DGL is trivial (i.e., $\partial_L = 0$), a $d\mathcal{A}$-model for $(L_*, 0)$ will be a differential bigraded non-associative algebra (DBGNA) $J = (J_{*,*}, \partial_J)$ (cf. §5.7).

7.6. Remark. Define a *Jacobi algebra* to be a DGNA $J = (J_*, \partial_J) \in d\mathcal{A}$ such that $H'_* J \in \mathcal{L}$. In particular, any $d\mathcal{A}$-model J of a DGL L is a Jacobi algebra, since $H'_* J \cong H'_* L$. We denote by $\mathcal{J} \subset d\mathcal{A}$ the full subcategory of Jacobi algebras. These algebras are clearly related to the strongly homotopy Lie algebras of [SS] (see also [LM]), though in general a Jacobi algebra is just a "Lie algebra up to homotopy".

Note that even though $d\mathcal{A}$ itself is a CUGA, $\mathcal{J}$ apparently is not, and it does not inherit many desirable properties from $d\mathcal{A}$: for example, $\mathcal{J}$ is not closed under the coproduct in $d\mathcal{A}$. However, one still has *free* Jacobi algebras, in the following sense:

7.7. Lemma. *There is a functor $J : d\mathcal{V} \to d\mathcal{A}$, and a natural transformation $\theta : J \to L$ such that:*

(a) *JV_* is free as an algebra, for any chain complex V_*;*
(b) *θ_{V_*} is a surjective quasi-isomorphism;*
(c) *any chain map $\varphi : V_* \to K_*$ (where $V_* \in d\mathcal{V}$ and $K_* \in \mathcal{J}$) extends to a $d\mathcal{A}$ map $\hat{\varphi} : JV_* \to K_*$.*

Proof. As noted above, we may define $J_{*,*} = J(V_*)$ by induction on the homological filtration:

Start with $J_{0,*} = A(V_*)$ (the free non-associative algebra on the differential graded vector space (V_*, ∂_V), with ∂_J extending ∂_V as a derivation), and let $\theta_0 : J_{0,*} \to L(V_*)$ be the obvious surjection, with $K_{0,*} = Ker(\theta_0)$ a two-sided $d\mathcal{A}$-ideal of $J_{0,*}$.

Choose once and for all some collection of generators $M_0 = \{\mu_i\}_{i \in I}$ for $K_{0,*}$ as a $J_{0,*}$-bimodule: for each choice of a $d\mathcal{V}$-basis $\{x_\gamma\}_{\gamma \in \Gamma}$ for V_* – that is, of a graded vector space basis of the form $\{x_\alpha, \partial_V x_\alpha, x_\beta\}$, with $\partial_V x_\beta = 0$ – we may write each μ_i as some expression $\mu_i(x_{\gamma_{i_1}}, \ldots, x_{\gamma_{i_n}})$. If we then choose some other $d\mathcal{V}$-basis $\{x'_{\gamma'}\}_{\gamma' \in \Gamma'}$ for V_*, again we will have $\mu'_i := \mu_i(x'_{\gamma'_{i_1}}, \ldots, x'_{\gamma'_{i_n}}) \in K_{0,*}$; define $J_{1,*}$ to be the free non-associative algebra on the DG vector subspace of $K_{0,*}$ spanned by all such "canonical operations" μ'_i $(i \in I)$, for all possible choices of $d\mathcal{V}$-bases $\{x'_{\gamma'}\}_{\gamma' \in \Gamma'}$ for V_*. Again one has the obvious augmentation $J_{1,*} \to J_{0,*}$ to serve as ∂_J (with $\partial_J \circ \partial_J = 0$ by construction), and one takes the kernel $K_{1,*}$ of this augmentation for the next step.

Proceeding in this way we may define the functor J by induction on the homological filtration; if the collections of operations M_n are chosen canonically, the functor itself will be canonical. Properties (a)-(c) are readily verified. □

7.8. Example. Let $L = (\mathbb{L}\langle X_* \rangle, 0)$ be the trivial free DGL on a graded set X_*; in this case the canonical DBGNA model $(J_{*,*}, \partial_J) = J(X_*, 0)$ for L may be described in part as follows:

Let $J_{0,*} = \mathbb{A}\langle X_*\rangle$ (the free non-associative algebra on X_*). Since the Jacobi identity holds in $\mathbb{L}\langle X_*\rangle$, but not in $\mathbb{A}\langle X_*\rangle$, we have $\mu(x,y,z) = [x,[y,z]] - [[x,y],z] + (-1)^{qr}[[x,z],y] \in K_{0,*}$ for all $x,y,z \in X_*$. Thus $J_{1,*}$ will be generated as a $J_{0,*}$-bimodule by the image of $(J_{0,*})^{\otimes 3}$ under the Σ_3-equivariant multilinear map $\lambda_3 : J_{i,p}\otimes J_{j,q}\otimes J_{k,r} \to J_{i+j+k+1,p+q+r}$. Here the symmetric group Σ_n acts on $J_{*,p_1}\otimes\cdots\otimes J_{*,p_n}$ by permutations, and on $J_{*,p_1+\cdots+p_n}$ via the *Koszul sign* homomorphism $\varepsilon_I : \Sigma_n \to \{1,-1\}$ (defined by letting $\varepsilon_I((k,k+1)) = (-1)^{i_k i_{k+1}+1}$ for any adjacent transposition $(k,k+1)\in\Sigma_n$). We set

$$\partial_J(\lambda_3(x\otimes y\otimes z)) = [x,[y,z]] - [[x,y],z] + (-1)^{qr}\ [[x,z],y].$$

However, there are relations among these elements $\lambda_3(x\otimes y\otimes z)$, so we define a Σ_4-equivariant multilinear map $\lambda_4 : J_{i,p}\otimes J_{j,q}\otimes J_{k,r}\otimes J_{\ell,s} \to J_{i+j+k+\ell+2,p+q+r+s}$,

$$\begin{aligned}\partial_J(\lambda_4(x\otimes y\otimes z\otimes w)) = {} & [x,\lambda_3(y\otimes z\otimes w)] \ + \ [\lambda_3(x\otimes y\otimes z),w] \\ & - (-1)^{|z||w|}\ [\lambda_3(x\otimes y\otimes w),z] \ + \ (-1)^{|y|(|z|+|w|)}[\lambda_3(x\otimes z\otimes w),y] \\ & - (\lambda_3([x,y]\otimes z\otimes w) \ + \ \lambda_3(x\otimes y\otimes [z,w])) \\ & + epsyz\ (\lambda_3([x,z]\otimes y\otimes z) \ + \ \lambda_3(x\otimes z\otimes [y,w])) \\ & - (-1)^{|w|(|y|+|z|)}\ (\lambda_3([x,w]\otimes y\otimes z) \ + \ \lambda_3(x\otimes w\otimes [y,z])).\end{aligned}$$

In fact, one can define a sequence of "higher Jacobi relations" $\lambda_n(x_1\otimes\cdots\otimes x_n)$, for all $n\geq 3$, which yield an explicit construction of $J(X_*)$ for a the free (graded) Lie algebra LX_*. Compare [LM, 2.1].

7.9. **$\mathcal{A}$-homotopy operations.** One can now apply the theory of section 4 verbatim to any space $\mathbf{X}\in\mathcal{T}_1$ with $\mathcal{C} = \mathcal{A}lg$ rather than $\mathcal{L}ie$, to obtain a sequence of higher homotopy operations as in §4.15 which determining the rational homotopy type of $\mathbf{X}$ – the only difference being that the simplicial resolutions $C_{\bullet\bullet}$ of the successive simplicial Lie algebras $L^{(k)}_\bullet$ are now $\mathcal{M}_{\mathcal{A}lg}$ free resolutions of $L^{(k)}_\bullet$ in $ss\mathcal{A}lg$.

This is the reason that the theory of section 3 was stated for an arbitrary CUGA, rather than specifically for $\mathcal{C} = \mathcal{L}ie$. The reason that our general theory was stated for *simplicial* rather than *differential graded* universal algebras is that there seems to be no reasonable version of Proposition 7.2 for an arbitrary CUGA.

7.10. **minimal resolutions.** To make the construction more accessible, it is again useful to have *minimal* $\mathcal{M}_{\mathcal{A}lg}$ resolutions, as in section 5. For this purpose, we consider a variant of the above approach:

Even though $\mathcal{J}$ does not inherit a closed model category structure from $d\mathcal{A}$, one may define *models* for $\mathcal{J}$, in the sense of §3.3, by letting a *Jacobi sphere* be any $d\mathcal{A}$-model of a $\mathcal{L}$-sphere (§5.2), and more generally let $\mathcal{M}_{\mathcal{J}}$ denote the full subcategory of $\mathcal{J}$ consisting of DGNAs weakly equivalent to objects in $\mathcal{M}_{d\mathcal{L}}$ – i.e., Jacobi models of DGLs which are (up to homotopy) coproducts of $d\mathcal{L}$-spheres.

An $\mathcal{M}_{\mathcal{J}}$ resolution of a DGL L, which we shall call simply a *Jacobi resolution*, is then defined to be a free simplicial resolution of DGNAs $A_{\bullet,*}\to\iota(L)$ (Def. 3.2), with each $A_{n,*}\in\mathcal{M}_{\mathcal{J}}$. Note that such an $A_{\bullet,*}\to\iota(L)$ is at the same time also an $\mathcal{M}_{\mathcal{J}}$-Jacobi resolution of the $d\mathcal{A}$-model J_* of L, and it is usually more convenient to think of it as such.

There is a comonad $F : d\mathcal{A} \to d\mathcal{A}$ as in (5.4), which yields the *canonical Jacobi resolution* $U_{\bullet,*}$ for any $C \in \mathcal{J}$, as in §5.3. Again we may use the notation of §5.6.

One also has an analogue of Propositions 5.9 and 5.17, as follows:

7.11. Proposition. *Let $B = (B_*, \partial_B) \in d\mathcal{L}$ be any DGL, and $(A_{*,*}, D_A)$ a filtered model for B: then there is a Jacobi resolution $J_{\bullet,*} \to \iota(B)$, with a bijection $\theta : X_{**} \hookrightarrow J_{\bullet,*}$ between a bigraded set X_{**} of generators for $A_{*,*}$ and the set of non-degenerate $d\mathcal{A}$-spheres in $J_{\bullet,*}$.*

Proof. Let $G = G_B$ be a $d\mathcal{A}$-model for the DGL B, and $U_{\bullet,*} \to G$ the canonical Jacobi resolution. As in the proof of Proposition 5.9, we may define a map $\theta : A_{*,*} \to U_{\bullet,*}$ inductively by the equation $\theta(x) = \langle \theta(\partial_A(x)) \rangle$ (compare (5.10)), and we shall again write $x^{(0)}$ for $\theta(x)$ if $x \in X_{**}$ (a set of generators for $A_{*,*}$), and let $V_{*,*}$ be the bigraded vector space spanned by $\theta(X_{**})$. For simplicity of notation we consider first the case where B has trivial differential and $A_{*,*}$ is bigraded (with $D_A = \partial_A$).

For each $n \in \mathbb{N}$, define the sub-DGNA $J^{(0)}_{n,*}$ of $U_{n,*}$ to be $J(V_{n,*})$, in the notation of Lemma 7.7 – that is, $J^{(0)}_{n,*}$ is the coproduct, in $\mathcal{J}$, of a set of Jacobi spheres $\mathcal{S}^k_{(x^{(0)})}$, one for each generator $x \in X_{n,k}$ of $A_{*,*}$. By Lemma 7.7(c), it is enough to define the face and degeneracy maps of $J_{\bullet,*}$ on each x – where we may use the description of §5.6.

Once again, we want $d_i(x^{(0)})$ to be a ∂_U-boundary for each $1 \leq i \leq n$; but the analogues of the elements $x^{(s)}$ of Propositions 5.9 and 5.17 are more complicated, so we need some definitions:

For each $0 \leq s \leq n$, let $\mathcal{K}_{n,s}$ denote the set of all sequences $I = (i_1, \ldots, i_s)$ of integers $1 \leq i_1 < \cdots < i_s \leq n$, corresponding to the s-fold face map $d_I = d_{i_1} \circ \cdots \circ d_{i_s} : \mathbf{n} \to \mathbf{n-s}$ in Δ^{op} (compare Definition 4.3 and the proof of Proposition 5.9). Given $I = (i_1, \ldots, i_s) \in \mathcal{K}_{n,s}$, for each $1 \leq j \leq s$ let $I(\hat{\jmath}) := (i_1, \ldots, \widehat{i_j}, \ldots, i_s) \in \mathcal{K}_{n,s-1}$ be obtained from I by omitting the j-th entry. By repeatedly using the identity $d_k d_m = d_{m-1} d_k$ $(k < m)$, we can find a unique $\kappa(j) \in \{1, 2, \ldots, n\}$ such that $d_{\kappa(j)} \circ d_{I(\hat{\jmath})} = d_I$.

For each $x \in X_{n,k}$, $0 \leq s \leq n$, and $I \in \mathcal{K}_{n,s}$, we want to choose choose an element $x^{(s;I)} \in J^{(s)}_{n-s,k-n+s} \subset U_{n-s,k-n+s}$ by induction on $n-s$, starting with $x^{(0,\emptyset)} := x^{(0)} = \theta(x)$, so that:

$$\partial_U(x^{(s;I)}) = \sum_{j=1}^{s} (-1)^j d_{\kappa(j)}(x^{(s-1;I(\hat{\jmath}))}) \tag{7.12}$$

for $s \geq 1$. (The index s is not really needed, since $s = |I|$, but it is useful for keeping the analogy with the notation of (5.12) in mind.)

Note that since $d_0 \circ \theta = \theta \circ \partial_A$ no longer holds here (because d_0 is a morphism in $d\mathcal{A}$, not in $d\mathcal{L}$), it is not generally true that $d_1(x^{(0)}) = 0$ for all $x \in X_{**}$. However, since applying H'_* still yields a CW resolution $H'_* J^{(0)}_{\bullet,*} \to H'_* C = H'_* B$, (where C is the $d\mathcal{A}$-model for the DGL B), we know that $d_i(x^{(0)})$ must be a ∂_U-boundary for each $1 \leq i \leq n$. Thus we can choose an element $x^{(1;1)} \in U_{n-1,k-n+1}$ with

$\partial_U(x^{(1;1)}) = d_1(x^{(0)})$ (a special case of (7.12)) – and in fact $x^{(1;1)}$ may be expressed in terms of the "canonical operations" μ_i of Lemma 7.7.

Now let $\partial_A(x) = \sum_t a_t\omega_t[y_{i_1},\dots,y_{i_{m_t}}]$ for $y_{i_j}\in A_{n_j,*}$, so

(7.13)
$$x^{(0)} = \langle\sum_t a_t\omega_t[\![y^{(0)}_{i_1},\dots,y^{(0)}_{i_{m_t}}]\!]\rangle = \langle\sum_t a_t\sum_{(J_1,\dots,J_{m_t})}(-1)^{\varepsilon_t}\omega_t[s_{J_1}(y^{(0)}_{i_1}),\dots,s_{J_{m_t}}(y^{(0)}_{i_{m_t}})]\rangle$$

as in (5.11), where the $(n-n_j)$-multi-index $J_j\subseteq\{0,1,\dots,n-1\}$ is obtained by repeated shuffles, which also determine the sign $(-1)^{\varepsilon_t}$ (see (2.11) ff.). Therefore,

$$d_k(x^{(0)}) = \langle\sum_t a_t\sum_{(I_1,\dots,I_{m_t})}(-1)^{\varepsilon_t}\omega_t[d_{k-1}s_{J_1}(y^{(0)}_{i_1}),\dots,d_{k-1}s_{J_{m_t}}(y^{(0)}_{i_{m_t}})]\rangle.$$

The proof of Lemma 5.13 (which is valid in $d\mathcal{A}$, too) implies by induction on $t\geq 2$ that for each summand $v_t=\omega_t[s_{J_1}(y^{(0)}_{i_1}),\dots,s_{J_{m_t}}(y^{(0)}_{i_{m_t}})]$, there is exactly one $1\leq j\leq t$ and $0\leq \ell\leq k-1$ such that

$$d_{k-1}(v_t) = \omega_t[s_{J'_1}(y^{(0)}_{i_1}),\dots,s_{J'_{j-1}}(y^{(0)}_{i_{j-1}}),s_{J''_j}d_\ell(y^{(0)}_{i_j}),s_{J'_{j+1}}(y^{(0)}_{i_{j+1}}),\dots,s_{J'_{m_t}}(y^{(0)}_{i_{m_t}})],$$

for suitable multi-indices $J'_1,\dots,J'_{j-1},J'_{j+1},\dots,J'_{m_t}$ and J''_j.

Since $\ell<k$ and $n_j<n$, we may assume by induction that we have defined $y^{(1;\ell)}_{i_j}\in U_{n_j-1,*}$ such that $\partial_U(y^{(1;\ell)}_{i_j}) = d_\ell(y^{(0)}_{i_j})$, and then let $x^{(1;k)}$ be

$$\langle\sum_t\sum_{(J_1,\dots,J_{m_t})}(-1)^{\varepsilon t}\omega_t[s_{J'_1}(y^{(0)}_{i_1}),\dots,s_{J'_{j-1}}(y^{(0)}_{i_{j-1}}),s_{J''_j}(y^{(1;\ell)}_{i_j}),s_{J'_{j+1}}(y^{(0)}_{i_{j+1}}),\dots,s_{J'_{m_t}}(y^{(0)}_{i_{m_t}})]\rangle.$$

The rest of the construction of the elements $x^{(s;I)}$, as well as the generalization to the filtered case, is similar to that in the proofs of Propositions 5.9 and 5.17. □

7.14. Example. Consider the coformal DGL $L=(\mathbb{L}\langle a_1,b_1,c_2\rangle/\langle[a,a],[[c,a],[b,a]]\rangle,0)$ of Example 5.8, with $G\in\mathcal{J}$ its $d\mathcal{A}$-model. We construct the minimal Jacobi resolution $J_{\bullet,*}\to G$ corresponding to the bigraded model for L (embedded in the canonical Jacobi resolution $U_{\bullet,*}\to G$) by modifying the $\mathcal{M}_{d\mathcal{L}}$-free resolution $C_{\bullet,*}\to L$ of Example 5.15, as follows:

For each $n\in\mathbb{N}$, define the Jacobi algebras $J^{(0)}_{n,*}$ $(n=0,1,\dots)$ to be the coproducts, in $\mathcal{J}$: $J^{(0)}_{0,*}=\mathcal{S}^1_{(\langle a\rangle)}\amalg\mathcal{S}^1_{(\langle b\rangle)}\amalg\mathcal{S}^2_{(\langle c\rangle)}$, $J^{(0)}_{1,*}=\mathcal{S}^2_{(x^{(0)})}\amalg\mathcal{S}^6_{(w^{(0)})}$, $J^{(0)}_{2,*}=\mathcal{S}^5_{(y^{(0)})}$, $J^{(0)}_{3,*}=\mathcal{S}^6_{(z^{(0)})}$, and so on.

The face maps d_i $(i=0,1,\dots,n)$ are defined by §5.6, where $[\,,\,]$ is now as in §7.4:

(1) For $y^{(0)}=\langle 3[\langle[\langle a\rangle,\langle a\rangle]\rangle,\langle\langle a\rangle\rangle]\rangle\in J^{(0)}_{2,3}$, we have $d_0(y^{(0)})=3[\langle[\langle a\rangle,\langle a\rangle]\rangle,\langle\langle a\rangle\rangle]=3[\![x^{(0)},a^{(0)}]\!]$ and $d_2(y^{(0)})=\langle 3[\langle[a,a]\rangle,\langle a\rangle]\rangle$, while $d_1(y^{(0)})=\langle 3[[\langle a\rangle,\langle a\rangle],\langle a\rangle]\rangle$. This no longer vanishes as in §5.15, since the Jacobi identity deos not hold in $\mathcal{A}$, but we have an element $y^{(1;1)}:=\langle\lambda_3(\langle a\rangle\otimes\langle a\rangle\otimes\langle a\rangle)\rangle\in J^{(1)}_{1,4}$ (in the notation of Example 7.8), with $\partial(y^{(1;1)})=d_1(y^{(0)})$.

On the other hand, we also have an element $y^{(1;2)}:=\langle 3[\langle x\rangle,\langle a\rangle]\rangle\in J^{(1)}_{1,4}$ (which we denoted simply by $y^{(1)}$ in §5.15), with $\partial(y^{(1;2)})=d_2(y^{(0)})$. The simplicial identity $d_1 d_2 = d_1 d_1$ implies that $d_1(y^{(1;2)}) - d_1(y^{(1;1)}) = \langle 3[x,a] - \lambda_3(a \otimes a \otimes a)\rangle$ is a ∂_J-cycle, so we have $y^{(2;1,2)} \in J^{(2)}_{0,5}$ with $\partial_J(y^{(2;1,2)}) = \langle 3[x,a] - \lambda_3(a \otimes a \otimes a)\rangle$.

(2) For $z^{(0)} = \langle 4[\langle 3[\langle[\langle a\rangle,\langle a\rangle]\rangle,\langle\langle a\rangle\rangle]\rangle,\langle\langle\langle a\rangle\rangle\rangle] + 6[\langle[\langle\langle a\rangle\rangle,\langle\langle a\rangle\rangle]\rangle,\langle\langle[\langle a\rangle,\langle a\rangle]\rangle\rangle]\rangle$ in $J^{(0)}_{3,4}$:

(a) $z^{(1;1)} := \langle \lambda_3(x^{(0)} \otimes \langle\langle a\rangle\rangle \otimes \langle\langle a\rangle\rangle)\rangle$, with

$$d_1(z^{(0)}) = \langle 6(2[[x^{(0)},\langle\langle a\rangle\rangle],\langle\langle a\rangle\rangle], +[[\langle\langle a\rangle\rangle,\langle\langle a\rangle\rangle],x^{(0)}])\rangle = \partial_J(z^{(1;1)}),$$

(b) $z^{(1;2)} := \langle 4[\lambda_3(\langle a\rangle \otimes \langle a\rangle \otimes \langle a\rangle),\langle\langle a\rangle\rangle]\rangle$, with

$$d_2(z^{(0)}) = \langle 4[[[a^{(0)},a^{(0)}],a^{(0)}],\langle\langle a\rangle\rangle]\rangle = \partial_J(z^{(1;2)}) \quad \text{since} \quad [x^{(0)},x^{(0)}] = 0).$$

(c) $z^{(1;3)} = \langle 4[\langle 3[\langle[a,a]\rangle,\langle a\rangle]\rangle,\langle\langle a\rangle\rangle] + 6[\langle[\langle a\rangle,\langle a\rangle]\rangle,\langle\langle[a,a]\rangle\rangle]\rangle \in J^{(1)}_{2,5}$, with

$$d_3(z^{(0)}) = \partial_J(z^{(1;3)}).$$

Next, in $J^{(2)}_{\bullet,*}$:

(a) The simplicial identity $d_1 d_1 = d_1 d_2$ implies that

$$d_1(z^{(1;1)}) - d_1(z^{(1;2)}) = \langle 4[\lambda_3(\langle a\rangle \otimes \langle a\rangle \otimes \langle a\rangle),\langle a\rangle] - 6\lambda_3([\langle a\rangle,\langle a\rangle] \otimes \langle a\rangle \otimes \langle a\rangle)\rangle$$

is a ∂_J-cycle – and indeed we have $z^{(2;1,2)} := \langle \lambda_4(\langle a\rangle \otimes \langle a\rangle \otimes \langle a\rangle \otimes \langle a\rangle)\rangle \in J^{(2)}_{2,6}$ with $\partial(z^{(2;1,2)}) = d_1(z^{(1;1)}) - d_1(z^{(1;2)})$.

(b) Similarly,

$$d_1(z^{(1;3)}) - d_2(z^{(1;1)}) = \langle 12[[\langle x\rangle,\langle a\rangle],\langle a\rangle] + 6[[\langle a\rangle,\langle a\rangle],\langle x\rangle] - 6\lambda_3(\langle[a,a]\rangle \otimes \langle a\rangle \otimes \langle a\rangle)\rangle$$

is a ∂_J-cycle, so we have $z^{(2;1,3)} := \langle 6\lambda_3(\langle x\rangle \otimes \langle a\rangle \otimes \langle a\rangle)\rangle \in J^{(2)}_{2,6}$ with $\partial(z^{(2;1,3)}) = d_1(z^{(1;3)}) - d_2(z^{(1;1)})$.

(c) $d_2(z^{(1;3)}) - d_2(z^{(1;2)}) = \langle 4[\langle 3[x,a]\rangle,\langle a\rangle] + 6[\langle[a,a]\rangle,\langle x\rangle] - 4[\langle\lambda_3(a \otimes a \otimes a)\rangle,\langle a\rangle]\rangle$ is a ∂_J-cycle, hit by $z^{(2;2,3)} := \langle 4[\langle y\rangle,\langle a\rangle] + 3[\langle x\rangle,\langle x\rangle]\rangle \in J^{(2)}_{2,6}$.

Finally, we have

(a) For $d_1 z^{(2;1,2)} = \langle \lambda_4(a \otimes a \otimes a \otimes a)\rangle$ we have

$$\partial_J(d_1 z^{(2;1,2)}) = d_1 d_2(z^{(1;2)}) - d_1 d_2(z^{(1;1)}) = \langle 4[\lambda_3(a \otimes a \otimes a),a] - 6\lambda_3([a,a] \otimes a \otimes a)\rangle$$

(b) For $d_1 z^{(2;1,3)} = \langle 6\lambda_3(x \otimes a \otimes a)\rangle$ we have

$$\partial_J(d_1 z^{(2;1,3)}) = d_1 d_2(z^{(1;1)}) - d_1 d_1(z^{(1;3)}) = \langle 6\lambda_3([a,a] \otimes a \otimes a) - 12[[x,a],a] - 6[[a,a],x]\rangle$$

(c) For $d_1 z^{(2;2,3)} = \langle 4[y,a] + 3[x,a]\rangle$ we have

$$\partial_J(d_1 z^{(2;2,3)}) = d_1 d_2(z^{(1;2)}) - d_1 d_1(z^{(1;3)}) = \langle 4[\lambda_3(a \otimes a \otimes a),a] - 12[[x,a],a] - 6[[a,a],x]\rangle,$$

So there is an element $z^{(3;1,2,3)} \in J^{(3)}_{0,7}$ with

$$\partial_J(z^{(3;1,2,3)}) = \langle \lambda_4(a \otimes a \otimes a \otimes a) - 6\lambda_3(x \otimes a \otimes a) + 4[y,a] + 3[x,a]\rangle.$$

We can now summarize the main result of this paper in the following

7.15. Theorem. *Let* $\mathbf{X}$ *be a simply connected space, and* $\Pi^X_* := \pi_{*-1}\mathbf{X}_{\mathbb{Q}} \in \mathcal{L}$ *its rational homotopy Lie algebra. There is a tree* T_X *of DGLs* $L^{(k,\alpha_1,\dots,\alpha_k)}$*, starting with* $L^{(0)} \simeq (\Pi^X_*, 0)$*, and for each branch* $\alpha_1,\dots,\alpha_k,\dots$ *of* T_X*, an increasing sequence of positive integers (or* ∞*)* $(n_k = n_{k,\alpha_1,\dots,\alpha_k})_{k=1}^{\infty}$ *such that*

(a) $H'_*(L^{(k,\alpha_1,\dots,\alpha_k)}) \cong \Pi^X_*$*;*

(b) *The higher homotopy operations* $\langle\langle m\rangle\rangle \subset H^m_*(L^{(k,\alpha_1,\dots,\alpha_k)};\Pi^X_*)$ *associated to a minimal Jacobi resolution of* $L^{(k,\alpha_1,\dots,\alpha_k)}$ *as in §4.10, vanish for* $m < n_k$.

(c) *The operation* $\langle\langle n_k\rangle\rangle = \langle\langle n_{k,\alpha_1,\dots,\alpha_k}\rangle\rangle \subset H^{n_k}_*(L^{(k)};\Pi^X_*)$ *does not vanish (unless* $n_k = \infty$*).*

(d) *For any* α_{k+1} *along the branch of* $(\alpha_1,\dots,\alpha_k)$*, the DGL* $L^{(k+1,\alpha_1,\dots,\alpha_k,\alpha_{k+1})}$ *may be chosen so that it agree with* $L^{(k,\alpha_1,\dots,\alpha_k)}$ *in degrees* $\leq n_k+1$*, so the sequential colimit* $L^{(\infty)} = \mathrm{colim}_k\, L^{(k)}$*, along any branch, is well defined, and is a DGL model for* $\mathbf{X}$.

The main difference between the construction described here (in $s\mathcal{J}$) and that of Proposition 5.17 is that the "higher order information" in $J_{\bullet,*}$ is no longer concentrated in the last face map $d_n : J_{n,*} \to J_{n-1,*}$. As a result, the higher homotopy operations associated to the (minimal) Jacobi resolution (as in section 4) are true simplicial operations, which can be translated more directly into topological ones. This is perhaps best illustrated by an example.

7.16. Example. Consider a space $\mathbf{X} \in \mathcal{T}_1$ whose minimal model is the DGL $B = (B_*, \partial_B)$ of example 5.19: $B_* = \mathbb{L}\langle a_1, b_1, c_2, x_3, y_5, z_7, \dots\rangle$, and $\partial_B(x) = [a,a]$, $\partial_B(y) = 3[x,a] - [[b,a],c]$, $\partial_B(z) = 4[y,a] + 3[x,x]$, and so on, and let $G \in \mathcal{J}$ be a $d\mathcal{A}$-model for B. The corresponding coformal DGL $L^{(0)} \simeq \mathbb{L}\langle a_1,b_1,c_2\rangle/\langle [a,a],[[c,a],[b,a]]\rangle$, with $G^{(0)} \in \mathcal{J}$ as its $d\mathcal{A}$-model, is that considered in Example 7.14.

The first obstruction to the coformality of B is again $\langle\langle 2; y^{(0)}\rangle\rangle \in \Pi^X_4$, as in Example 5.23 – but now it is represented by the map $\bar{h} : \mathcal{S}^3_{(y^{(0)})} \rtimes \partial D(2) \to G$ depicted in Figure 1:

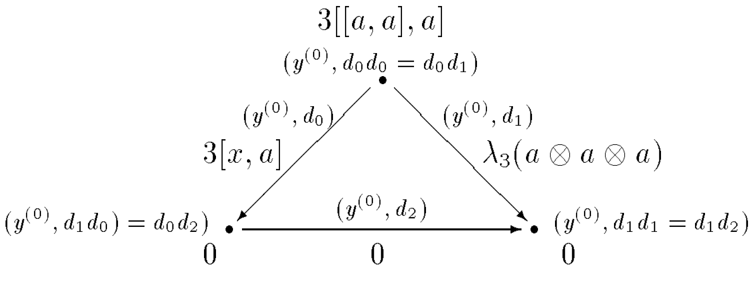

FIGURE 1. The operation $\langle\langle 2; y^{(0)}\rangle\rangle$ on $\mathcal{S}^2_{(y^{(0)})} \rtimes \partial D(2)$

Thus $\bar{h}$ maps the 3-dimensional generator of $\mathcal{S}^3 \simeq \mathcal{S}^2_{(y^{(0)})} \rtimes \partial D(2)$ to the cycle $\langle 3[x,a] - \lambda_3(a\otimes a\otimes a)\rangle \in G_4$ (which does not bound, because $\partial_G(y) = 3[x,a] - \lambda_3(a\otimes a\otimes a) - [[b,a],c]$ in G). This may be interpreted as a proper "Toda bracket", in the sense that $3[[a,a],a] = 0$ in Π^X_* "for two different reasons". (Compare this with the analogous description of the corresponding integral operation – namely, the second-order Whitehead product – in [Bl2, Ex. 4.13].)

We may next construct the DGL $L^{(1)}$ (in which $\partial_L(y) = 3[x,a] - [[b,a],c]$), perturb the Jacobi resolution $J_{\bullet,*} \to G^{(0)}$ of Example 7.14 to obtain a Jacobi resolution of $L^{(1)}$ (or rather, of the corresponding $d\mathcal{A}$-model $G^{(1)}$), and identify the next obstruction as $\langle\langle 3; z^{(0)}\rangle\rangle \in \Pi^X_6$.

However, to avoid cluttering the picture we describe instead the third order homotopy operation $\langle\langle 3; z^{(0)}\rangle\rangle$ for $G^{(0)}$ itself (where of course it vanishes) in Figure 2. Again we simply depict $\partial D(3)$, representing $\mathcal{S}^4_{(z^{(0)})} \rtimes \partial D(3)$, where the maps h_i are marked in the corresponding face of $D(3)$ (except where they are 0):

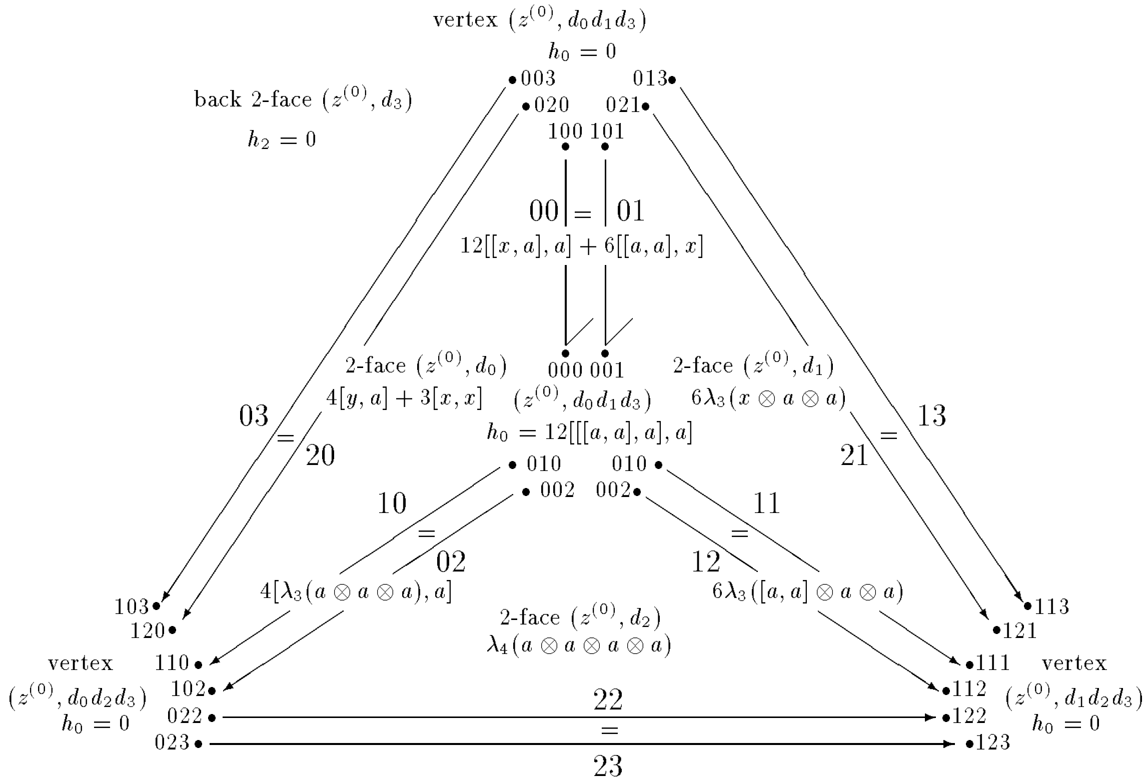

FIGURE 2. Depiction of $\mathcal{S}^4_{(z^{(0)})} \rtimes \partial D(3)$

7.17. Remark. While it is clear that the simplicial higher homotopy operations we obtain by means of Jacobi resolutions are closer to topological (unlocalized) ones, it is still by no means a trivial task to translate the algebraic description so obtained into a precise topological one. Moreover, care must be taken when dealing with unlocalized operations, since the ones we describe are essentially in the $\mathcal{F}$ category (in which, essentially, we disregard maps of finite order in the ordinary homotopy category). See [Ba, ch. III] for a sample computation, illustrating the pitfalls involved.

Nevertheless, we claim that the Jacobi resolutions point the way towards a precise integral description of the homotopy operations involved to a much greater extent than the corresponding DGL operations. Of course, we could get an even fuller description of these operations if we replace $\mathcal{A}lg$ by the category of algebras with a non-cummutative and non-bilinear product, which is (skew) commutative and bilinear up to homotopy. But working in such a setting is clearly impracticable.

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