# NEW MODEL CATEGORIES FROM OLD

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ABSTRACT. We review Quillen's concept of a model category as the proper setting for defining derived functors in non-abelian settings, explain how one can transport a model structure from one category to another by mean of adjoint functors (under suitable assumptions), and define such structures for categories of cosimplicial coalgebras.

## 1. INTRODUCTION

Model categories, first introduced by Quillen in [Q1], have proved useful in a number of areas – most notably in his treatment of rational homotopy in [Q2], and in defining homology and other derived functors in non-abelian categories (see [Q3]; also [BoF, BlS, DwHK, DK, DwS, Goe, ScV]). From a homotopy theorist's point of view, one interesting example of such non-abelian derived functors is the $E^2$-term of the mod $p$ unstable Adams spectral sequence of Bousfield and Kan. They identify this $E^2$-term as a sort of $Ext$ in the category $\mathcal{CA}$ of unstable coalgebras over the mod $p$ Steenrod algebra (see §7.4).

The original purpose of this note was to provide an element in this identification which appears to be missing from the literature: namely, an explicit model category structure for the category $c\mathcal{CA}$ of cosimplicial coalgebras as above. What one would really like is a model category for arbitrary categories of cosimplicial *universal coalgebras*, analogous to Quillen's treatment of simplicial universal algebras in [Q1, II, §4]. This treatment is based on Quillen's "small object argument" (see Proposition 4.8 below), and implicitly uses a procedure for transfering model category structures by means of adjoint functors (in the direction of the left adjoint). The procedure is made explicit in Theorem 4.15 below.

Unfortunately, Quillen's procedure cannot be dualized, in the categorical sense. The reason is essentially set-theoretic: more can be said about maps *into* a sequential colimit of sets than about maps *out of* a sequential limit (and thus, for example, colim is exact, for $R$-modules, while lim is not).

Therefore, for our purposes we describe, in Theorem 4.14, alternative (and less elegant) conditions for using adjoint functors to create new model category structures. The dual version, Theorem 7.6, then allows us to define model category structures for certain categories of cosimplicial universal coalgebras – including $c\mathcal{CA}$ (see Proposition 7.7).

1.1. **notation and conventions.** For any category $\mathcal{C}$, we denote by $gr\mathcal{C}$ the category of *non-negatively graded objects* over $\mathcal{C}$, by $gr_+\mathcal{C}$ the category of positively graded

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objects, by $s\mathcal{C}$ the category of *simplicial* objects over $\mathcal{C}$ (cf. [May, §2]), and by $c\mathcal{C}$ the category of *cosimplicial* objects over $\mathcal{C}$. For an abelian category $\mathcal{M}$, we let $c_*\mathcal{M}$ denote the category of *chain complexes* over $\mathcal{M}$ (in non-negative degrees); similarly $c^*\mathcal{M}$ is the category of cochain complexes.

The category of sets will be denoted by $\mathcal{S}et$, that of topological spaces by $\mathcal{T}op$, that of groups by $\mathcal{G}p$, and that of simplicial sets by $\mathcal{S}$ (rather than $s\mathcal{S}et$). For any ring $R$, the categories of left (respectively, right) $R$-modules are denoted by $R$-$Mod$ (resp. $Mod$-$R$). $\mathbb{F}_p$ denotes the field with $p$ elements. We have tried to be consistent in using $\mathcal{A}$ for a category of universal algebras (§3.6 below), $\mathcal{B}$ for a category of universal coalgebras (§7.3), and $\mathcal{M}$ for an abelian category.

Throughout we shall use "dual" to refer to the categorical dual (cf. [Mac1, II,§1]); other duals (such as the vector space dual) will be called by other names (e.g., "conjugate").

For any functor $F : I \to \mathcal{C}$ we denote the (inverse) limit of $F$ simply by $\lim F$ or $\lim_I F$, (rather than $\lim_{\leftarrow}$), and the colimit (i.e., direct limit) by $\operatorname{colim} F$. In particular, *sequential* limits *of type* $\kappa$ are limits indexed by an (infinite) ordinal $\kappa$: $\lim_{\nu<\kappa} X_\nu$, and similarly for colimits. An initial object (in any category $\mathcal{C}$) will be denoted by $*_I$, and a terminal object by $*_T$.

1.2. **organization.** In section 2 we review the definition of model categories and some related concepts, as well as their relevance to derived functors. In section 3 we make explicit the relation between adjoint functors and limits, and in section 4 we explain their relation to defining new model category structures. In sections 5 & 6 respectively we discuss simplicial and cosimplicial objects over abelian categories. Finally, in section 7 we describe the "universal coalgebras" we are interested in, and apply our results to define a model category structure on such categories of cosimplicial coalgebras.

*Acknowledgements* 1.3. I would like to thank the referee for many useful comments, and in particular for suggesting Theorem 4.15 in its present generality.

I understand that in [CaG], Cabello and Garzón have also given conditions for defining model category structures by means of adjoint functors.

## 2. Model categories

We begin with an exposition of Quillen's theory of model categories, in a form suited to our (algebraic) purposes:

**Definition 2.1.** A class $\mathfrak{W}$ of morphisms in a category $\mathcal{C}$ will be called a *class of quasi-isomorphisms* if there is a functor $\gamma : \mathcal{C} \to \mathcal{D}$ such that $f \in \mathfrak{W} \iff \gamma(f)$ is an isomorphism in $\mathcal{D}$.

**Definition 2.2.** A map $f : X \to Y$ is called a *retract* of a map $g : K \to L$ if there are maps $k, \ell, r, s$ making the following diagram commute:

Note that any class of quasi-isomorphisms is closed under retracts (i.e., $g \in \mathfrak{W} \Rightarrow f \in \mathfrak{W}$ in Figure 1).

2.3. **axioms for model categories.** Let $\mathcal{C}$ be a category with three distinguished classes of morphisms: $\mathfrak{W}$, $\mathfrak{C}$, and $\mathfrak{F}$. Consider the following two axioms:
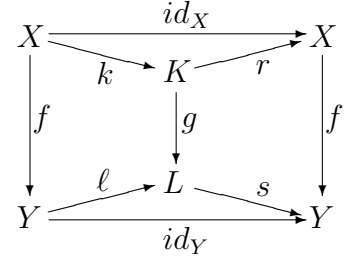
FIGURE 1

**Axiom 1.** For any morphism $f : A \to B$ in $\mathcal{C}$

  (i) there is a factorization $A \xrightarrow{i} C \xrightarrow{p} B$ $(f = p \circ i)$ with $i \in \mathfrak{C} \cap \mathfrak{W}$ and $p \in \mathfrak{F}$.

  (ii) Moreover, if also $f = p' \circ i'$ with $i' \in \mathfrak{C} \cap \mathfrak{W}$ and $p' \in \mathfrak{F}$ for $A \xrightarrow{i'} C' \xrightarrow{p'} B$, then there is a map $h : C \to C'$ making the following diagram commute:
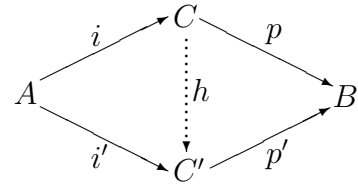


FIGURE 2

Note that if $\mathfrak{W}$ is a class of quasi-isomorphisms, necessarily $h \in \mathfrak{W}$ – so that (ii) says the factorization in (i) is unique up to quasi-isomorphism.

**Axiom 2.** For any morphism $f : A \to B$ in $\mathcal{C}$

  (i) there is a factorization $A \xrightarrow{i} C \xrightarrow{p} B$ with $i \in \mathfrak{C}$ and $p \in \mathfrak{F} \cap \mathfrak{W}$.

  (ii) If $f = p' \circ i'$ is another such factorization, there is an $h$ making the diagram in Figure 2 commute.

**Definition 2.4.** Let $\mathcal{C}$ be a category and $\mathfrak{W}$, $\mathfrak{C}$ and $\mathfrak{F}$ classes of morphisms in $\mathcal{C}$. Assume that $\mathfrak{W}$ is a class of quasi-isomorphisms and $\mathfrak{C}$ and $\mathfrak{F}$ are each closed under compositions. Then

  (i) If $\mathcal{C}$ has all finite limits, and $\langle \mathcal{C}; \mathfrak{W}, \mathfrak{C}, \mathfrak{F} \rangle$ satisfy Axiom 1, we call this a *right model category* (RMC) structure on $\mathcal{C}$, or say that $\mathcal{C}$ is an RMC.
  (ii) If $\mathcal{C}$ has all finite colimits, and $\langle \mathcal{C}; \mathfrak{W}, \mathfrak{C}, \mathfrak{F} \rangle$ satisfy Axiom 2, we call this a *left model category* (LMC) structure on $\mathcal{C}$.
  (iii) If both hold, $\langle \mathcal{C}; \mathfrak{W}, \mathfrak{C}, \mathfrak{F} \rangle$ is called a *model category*.

*Remark* 2.5. In order to "do homotopy theory" in $\mathcal{C}$ one requires the full force of a model category; in fact, it is often convenient to have additional structure, such as simplicial <u>Hom</u>-objects (cf. [Q1, II, §1]), *properness* (cf. [BoF, Def. 1.2]), and so on (see [Bau, I] and [He, II] for more general treatments). However, for the purposes of "homotopical algebra" – i.e., homological algebra in non-abelian categories – it is enough to have an RMC or an LMC (see §§2.14-2.16 below).

**Example 2.6.** The original motivating example of a model category is the category $\mathcal{T}op$ of topological spaces, with $\mathfrak{W}$ the class of homotopy equivalences, $\mathfrak{C}$ the class of cofibrations, and $\mathfrak{F}$ the class of (Hurewicz) fibrations (cf. [St]). An alternative model category structure on $\mathcal{T}op$ is given in [Q1, II,§3].

However, for our purposes the basic example of a model category will be the category $\mathcal{S}$ of simplicial sets, with $\mathfrak{W}$ the class of weak equivalences (maps inducing an isomorphism in $\pi_*(-)$), $\mathfrak{C}$ the class of one-to-one maps, and $\mathfrak{F}$ the class of Kan fibrations (cf. [May, §7]). See sections 5 & 6 below for further examples.

*Remark* 2.7. Given $\langle \mathcal{C}; \mathfrak{W}, \mathfrak{C}, \mathfrak{F} \rangle$ satisfying axioms 1 and 2, in order for $\mathfrak{W}$ to be a class of quasi-isomorphisms it suffices that:

(a)    $\mathfrak{W}$ include all isomorphisms,
(b)    $\mathfrak{W}$ be closed under retracts, and
(c)    Given $A \xrightarrow{f} B \xrightarrow{g} C$ with two out of $\{f, g, g \circ f\}$ in $\mathfrak{W}$, the third is, too.

In this situation Quillen constructs in [Q1, I,§1] a *localization* of $\mathcal{C}$ with respect to $\mathfrak{W}$, which comes with a functor $\gamma : \mathcal{C} \to Ho\mathcal{C}$ such that $f \in \mathfrak{W} \Leftrightarrow \gamma(f)$ is an isomorphism in $Ho\mathcal{C}$. However, in almost all known examples of model categories $\mathfrak{W}$ is given to begin with as a class of quasi-isomorphisms.

**Definition 2.8.** We call the closure of $\mathfrak{F}$ under retracts (§2.2) the class of *fibrations* in $\mathcal{C}$; similarly, the closure of $\mathfrak{C}$ under retracts is called the class of *cofibrations*.

A fibration which is in $\mathfrak{W}$ will be called a *trivial fibration*, and a cofibration in $\mathfrak{W}$ will be called a *trivial cofibration*.

*Remark* 2.9. In Quillen's definition no distinguished subclasses $\mathfrak{F}$, $\mathfrak{C}$ of the classes of (co)fibrations appear (nor are right or left model categories mentioned). But such classes occur naturally in many examples (just as *free $R$-modules* form a distinguished subclass of projective $R$-modules), and allow a convenient simplification of the axioms.

There is in fact no loss of generality in our definition, in light of the following facts:

**Definition 2.10.** Given any commutative square

$$
\begin{array}{ccc}
A & \xrightarrow{\ \alpha\ } & X \\
{\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\
B & \xrightarrow[\ \beta\ ]{} & Y
\end{array}
$$

we say that $f$ has the *right lifting property* (RLP) with respect to $i$ – or equivalently, that $i$ has the *left lifting property* (LLP) with respect to $f$ – if a dotted arrow exists making the diagram commute.

**Lemma 2.11.** *If $\langle \mathcal{C}; \mathfrak{W}, \mathfrak{C}, \mathfrak{F} \rangle$ is an RMC, any fibration in $\mathcal{C}$ has the RLP with respect to any trivial cofibration; dually, if $\langle \mathcal{C}; \mathfrak{W}, \mathfrak{C}, \mathfrak{F} \rangle$ is an LMC, any trivial fibration in $\mathcal{C}$ has the RLP with respect to any cofibration.*

*Proof.* Let $\mathcal{C}$ be a right model category, and assume given a diagram as in Figure 1. First, if $f \in \mathfrak{F}$ and $i \in \mathfrak{W} \cap \mathfrak{C}$, one can factor $\alpha$ and $\beta$ using Axiom 1(i):

Thus we have $h' : B' \to X'$ by Axiom 1(ii) (since $\mathfrak{C}$ and $\mathfrak{F}$ are closed under compositions, and $(f \circ p) \circ k$ and $q \circ (j \circ i)$ are two factorizations of $f \circ \alpha = \beta \circ i$), and $h = p \circ h' \circ j$ is the required lifting.

Next, assume that $f$ is a fibration – i.e., a retract of a map $g \in \mathfrak{F}$ – so we have a commutative diagram:



and by the first case (for $g \in \mathfrak{F}$) there is a lifting



so $r \circ h : B \to X$ is the required lifting for the $i$ with respect to the fibration $f$. The case where $i$ is any trivial cofibration is dealt with similarly; and the case of a left model catgeory is of course dual. $\square$

**Fact 2.12.** *([Q2, II, Prop. 1.1]). The fibrations of a right model category are precisely those morphisms having the RLP with respect to all $i \in \mathfrak{C} \cap \mathfrak{W}$, and conversely, the trivial cofibrations are those morphisms which have the LLP with respect to all $f \in \mathfrak{F}$. The cofibrations of a left model category are characterized by having the LLP with respect to all $f \in \mathfrak{F} \cap \mathfrak{W}$, and the trivial fibrations are those morphisms which have the RLP with respect to all $i \in \mathfrak{C}$.*

*Proof.* If $f : A \to B$ has the RLP with respect to all $i \in \mathfrak{C} \cap \mathfrak{W}$, use Axiom 1 to factor $f$ as $A \xrightarrow{i} C \xrightarrow{p} B$ with $i \in \mathfrak{C} \cap \mathfrak{W}$; the lifting $r : X \to A$ which exists by hypothesis shows that $f$ is a retract of $p \in \mathfrak{F}$, so a fibration. Similarly for the other cases. $\square$

**Corollary 2.13.** *The (trivial) fibrations of an RMC (resp. LMC) are preserved under base change, products, and sequential limits – that is,*

*(a) If $p : X \to Y$ is a (trivial) fibration, $f : X \to Z$ is any map, and W is the pushout of $(q, f)$, with structure maps $q : Z \to W$, $g : Y \to W$, then q is a (trivial) fibration.*

*(b) If* $\{p_\alpha : X_\alpha \to Y_\alpha\}_{\alpha \in A}$ *are all (trivial) fibrations, so is* $\prod_\alpha p_\alpha : \prod_\alpha X_\alpha \to \prod_\alpha Y_\alpha$.

*(c) If* $(p_\nu : X_{\nu+1} \to X_\nu)_{\nu < \kappa}$ *is a sequence of (trivial) fibrations, the map* $q_\nu : X = \lim_{\nu < \kappa} X_\nu \to X_\nu$ *is a (trivial) fibration for each* $\nu < \kappa$.

   *Similarly, the (trivial) cofibrations of a LMC (resp., RMC) are preserved under cobase change, coproducts, and sequential colimits.*

*Proof.* The constructions (a)-(c) all preserve the lifting property with respect to any (fixed) map.                                                                                        □

   We now recall how model categories are used to define derived functors in non-abelian categories. Let $\langle \mathcal{C}; \mathfrak{W}, \mathfrak{C}, \mathfrak{F} \rangle$ be a model category.

**Definition 2.14.** The *homotopy category* $Ho\mathcal{X}$ of any model category $\mathcal{X}$ is obtained from it by localizing with respect to the weak equivalences, with $\gamma : \mathcal{X} \to Ho\mathcal{X}$ the localization functor. Quillen shows that $Ho\mathcal{X}$ is equivalent to the category $\pi(\mathcal{X}_{cf})$, whose objects are those objects $X \in \mathcal{X}$ for both fibrant (i.e., $X \to *_T$ is a fibration) and cofibrant (i.e., $*_I \to X$ is a cofibration), and whose morphisms are homotopy classes of maps (cf. [Q1, I, §1]).

   Under this equivalence of $Ho\mathcal{X}$ and $\pi(\mathcal{X}_{cf})$, the localization functor is determined by the choice, for each object $X \in \mathcal{X}$, of a cofibrant and fibrant object $A$ with a weak equivalence $A \to X$. This is called a *resolution* of $X$, and all such are homotopy equivalent. However, we can sometimes make do with less:

**Definition 2.15.** If $H : \mathcal{X} \to \mathcal{Y}$ is a functor between model categories which preserves weak equivalences between cofibrant objects, the *total left derived functor* of $H$ is the functor $\boldsymbol{L}H = \tilde{H} \circ \gamma : \mathcal{X} \to Ho\mathcal{Y}$, where $\tilde{H} : Ho\mathcal{X} \to Ho\mathcal{Y}$ is induced by $H$ on $\mathcal{X}_c$ (the subcategory of cofibrant objects in $\mathcal{X}$).

*Remark* 2.16. In fact we need only a left model category structure on $\mathcal{X}$ in order for $\boldsymbol{L}$ to be defined. Of course right derived functors are defined analogously in any right model category.

   In the particular case where $\mathcal{Y} = s\mathcal{C}$ is a category of simplicial objects over some concrete category $\mathcal{C}$, the usual *n*-th *derived functor* of any $T : \mathcal{X} \to s\mathcal{C}$, denoted $L_n T$, assigns to an object $X \in \mathcal{C}$ the object $(L_n T)X = \pi_n(\boldsymbol{L}T)X = \pi_n TA$, where $A \to X$ is any resolution.

   If also $\mathcal{X} = s\mathcal{D}$ for some $\mathcal{D}$, and $T : \mathcal{C} \to \mathcal{D}$ is *prolonged* to a functor $sT : s\mathcal{C} \to s\mathcal{D}$ (by applying it dimensionwise to simplicial objects), then for $C \in \mathcal{C}$ we have $(L_n T)C = \pi_n(sT)A_\bullet$, where $A_\bullet$ is a resolution of the constant simplicial object which is equal to $C$ in each dimension. When $T$ is an additive functor between abelian categories with enough projectives, this reduces to the usual definition of derived functors (see also [Bo2, §7], [DoP], [EM2], & [Hu]).

   We have avoided the question of when a functor will in fact preserve weak equivalences between cofibrant objects. This depends on the specific model categories in question (see Remark 7.8 below).

## 3. ADJOINT FUNCTORS AND LIMITS

   We next recall some general facts about limits and adjoint functors: Let $\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be a pair of adjoint functors (i.e., $F$ is (left) adjoint to $U$), with the natural adjunction

isomorphism $\vartheta : Hom_{\mathcal{C}}(FD, C) \cong Hom_{\mathcal{D}}(D, UC)$. We denote $\vartheta^{-1}(id_{UC}) : FUD \to D$ by $\varepsilon_D$.

*Remark* 3.1. It is not hard to see that $U$ preserves all limits which exist in $\mathcal{C}$, and dually, $F$ preserves all colimits which exist in $\mathcal{D}$ (cf. [Mac1, V, §6]).

$F$ evidently preserves projectivity, so if $\mathcal{D}$ is a category in which all objects are projective (e.g., $\mathcal{D} = \mathcal{S}et$) and $\varepsilon_D$ is always an epimorphism, then $im(F)$ consists of projectives and $\varepsilon_D : FUD \to D$ is a functorial projective cover.

**Definition 3.2.** Given a diagram $S : I \to \mathcal{C}$, we say that a functor $T : \mathcal{C} \to \mathcal{D}$ *creates* the limit $\lim_I S$ in $\mathcal{C}$ (cf. [Mac1, V,1]) if $\lim_I T \circ S$ exists in $\mathcal{D}$, $\lim_I S$ exists in $\mathcal{C}$, and $T(\lim_I S) = \lim_I (TS)$. Similarly for creation of colimits.

**Definition 3.3.** Given adjoint functors $\mathcal{C} \overset{F}{\underset{U}{\rightleftarrows}} \mathcal{D}$ and a diagram $S : I \to \mathcal{C}$, we say the pair $(U, F)$ *produces* the colimit $\operatorname{colim} S \in \mathcal{C}$ if $\operatorname{colim}_I(US) \in \mathcal{D}$ exists, and $\operatorname{colim}_I S \in \mathcal{C}$ exists and is obtained as follows:

Let $L_0 = \operatorname{colim}_I(US)$, and $L_1 = \operatorname{colim}_I(UFUS)$. There are two natural morphisms $FL_1 \to FL_0$, namely: $d_0$, induced by the natural transformation $\varepsilon_{FU(j)}$ for every $j \in I$, and $d_1$, induced by $FU(\varepsilon_j)$. We require that $\operatorname{colim} S$ be the coequalizer (in $\mathcal{C}$) of $d_0$ and $d_1$ (see [L] or [Mac1, X, §1]; there seems to be no accepted name for this procedure).

In order for this construction to be of use, we need some information on coequalizers in $\mathcal{C}$ – at least for those which appear here. Such coequalizers, called *split* (or contractible), have a map $s : FL_0 \to FL_1$ such that $d_0 s = id$, $d_1 s d_0 = d_1 s d_1$. In the cases of interest to us split coequalizers are created by $U$, so the definition makes sense (cf. [BaW, 3.3, Prop. 3]).

*Remark* 3.4. Since we know that $F$ preserves all colimits – so that the colimit of a diagram in $\mathcal{C}$ which factors through $F$ is determined by the corresponding colimit in $\mathcal{D}$ – in our situation (see §3.6) the left model category structure we shall define on $s\mathcal{C}$ will allow us to identify any colimit in $\mathcal{C}$ as the 0-th left derived functor (§2.16) of the same colimit defined on the image of $F$.

This is analogous to viewing the usual tensor product of $R$-modules, say, as the 0-th derived functor of the more naturally defined functor of tensor product of *free* $R$-modules on specified sets of generators $X, Y$:

$$R\langle X \rangle \otimes R\langle Y \rangle \overset{Def}{=} R\langle X \times Y \rangle.$$

One may dually define "the pair $(F, U)$ produces the limit $\lim S$ in $\mathcal{D}$", (with equalizers replacing coequalizers, etc.).

**Example 3.5.** Let $\mathcal{G}p \overset{F}{\underset{U}{\rightleftarrows}} \mathcal{S}et$ denote the adjoint "underlying set" and "free group" functors between the categories of groups and sets respectively. Then $U$ creates all limits in $\mathcal{G}p$ – i.e., an inverse limit of a diagram of groups is just the corresponding limit for the underlying sets, endowed with a natural group structure. Likewise, the adjoint pair $(U, F)$ produces all colimits in $\mathcal{G}p$. For instance, the coproduct ("free product") $G \amalg H$ of two groups is obtained by choosing sets of generators $X, Y$ for $G$, $H$ respectively – say, $X = U(G)$, $Y = U(H)$ – and setting

$$G \amalg H = F(X \cup Y)/ \sim ,$$

where $[a][a'] \sim [aa']$, $[a]^{-1} \sim [a^{-1}]$ for $a, a'$ both in $X$ or both in $Y$ – which is precisely the coequalizer of the two obvious homomorphisms

$$F(UFU(G) \cup UFU(H)) \; \rightrightarrows \; F(U(G) \cup U(H)).$$

**Definition 3.6.** Recall that in a category of *universal algebras* (or variety of algebras) the objects are sets $X$, together with an action of a fixed set of $n$-ary operators $W = \cup_{n=0}^{\infty}\{\omega : X^n \to X\}$, satisfying a set of identities $E$; the morphisms are functions on the sets which commute with the operators.

(We can slightly modify the definition to cover the case where the set $X$ is *pointed*, non-negatively *graded*, and so on.)

**Example 3.7.** Categories of universal algebras include:

The category $\mathcal{G}p$ of groups; the category $R\text{-}Mod$ of left $R$-modules for any ring $R$, as well as that of (commutative or associative) algebras over $R$; the category of $\mathbb{F}_p$-algebras over the mod-$p$ Steenrod algebra; and the categories of Lie rings, or of restricted Lie algebras over $\mathbb{F}_p$.

All the above examples, and many others (though not the categories of monoids, semigroups, and semirings), have another convenient feature: their objects have the underlying structure of a (possibly graded) group.

*Remark* 3.8. For each category $\mathcal{C}$ of universal algebras there is a pair of adjoint functors $\mathcal{C} \underset{U}{\overset{F}{\rightleftharpoons}} \mathcal{S}et$, with $U(A)$ the *underlying* set of $A \in \mathcal{C}$, and $F(X)$ the *free* algebra on the set of *generators* $X$ (cf. [Co, III.5 & IV.2])

Thus in particular every category of universal algebras has enough projectives (§3.1). In fact, the functor $U : \mathcal{C} \to \mathcal{S}et$ is *monadic* in the sense that the category $\mathcal{C}$ can be reconstructed from the monad (=triple) $UF : \mathcal{S}et \to \mathcal{S}et$ (compare §7.1 below; cf. [BaW, 3.3, Prop. 4] or [Co, VIII.3] for the precise statement). Moreover:

**Proposition 3.9.** *For any category of universal algebras $\mathcal{C}$, the functor $F$ creates all limits and sequential colimits of monomorphisms in $\mathcal{C}$, and the pair $(U, F)$ produces all colimits in $\mathcal{C}$.*

*Proof.* Just as in example 3.5 above. The statement on colimits is due to Linton ([L]; see also [BaW, 9.3]). □

## 4. ADJOINT FUNCTORS AND MODEL CATEGORIES

We now explain how adjoint functors $\mathcal{C} \underset{U}{\overset{F}{\rightleftharpoons}} \mathcal{D}$ can be used to transfer an existing model category structure on $\mathcal{D}$ to the category $\mathcal{C}$.

*Convention* 4.1. To simplify the statements of our results, we assume that in all (left or right) model categories discussed in this section, all cofibrations are in particular monomorphisms. This need not hold in general (see Proposition 6.4 below), but it will hold in all situations we are interested in. It should be clear from the proofs how the statements must be modified without this assumption.

First, we require the following somewhat ad hoc

**Definition 4.2.** We say that a left model category $\langle \mathcal{C}; \mathfrak{W}_{\mathcal{C}}, \mathfrak{C}_{\mathcal{C}}, \mathfrak{F}_{\mathcal{C}} \rangle$ has *canonical factorizations of type $\kappa$* for Axiom 2 (for some ordinal $\kappa$) if the factorization $X \overset{j}{\longrightarrow} Z \overset{p}{\longrightarrow} Y$ of Axiom 2(i) for any $f : X \to Y$ in $\mathcal{C}$ is obtained as follows:

(a) There is a sequence of commuting diagrams

$$Z^{(-1)} = X \xrightarrow{j^{(0)}} Z^{(0)} \cdots \cdots Z^{(\nu)} \xrightarrow{j^{(\nu)}} Z^{(\nu+1)} \cdots \cdots$$

with maps $p^{(-1)} = f$, $p^{(0)}$, $p^{(\nu)}$, $p^{(\nu+1)}$ to $Y$.

for $\nu < \kappa$, such that $Z = \operatorname{colim}_{\nu < \kappa} Z^{(\nu)}$ and $p$ and $j$ are induced by the maps $(p^{(\nu)})_{\nu < \kappa}$ and $(j^{(\nu)})_{\nu < \kappa}$ respectively.

(b) For each $\nu \geq -1$, the object $Z^{(\nu+1)}$ is constructed as a pushout:

$$
\begin{array}{ccc}
V & \xrightarrow{g} & Z^{(\nu)} \\
{\scriptstyle i}\downarrow & \boxed{PO} & \downarrow{\scriptstyle j^{(\nu)}} \\
W & \longrightarrow & Z^{(\nu+1)}
\end{array}
\qquad\qquad PO^{(\nu)}
$$

(c) $i : V \to W$ is in turn constructed functorially as a coproduct:

$$V = \coprod_{\alpha \in \mathcal{K}_\nu} V_\alpha \xrightarrow{\coprod i_\alpha} \coprod_{\alpha \in \mathcal{K}_\nu} W_\alpha = W$$

where the set of maps $\{i_\alpha\}_{\alpha \in \mathcal{K}_\nu}$ depends functorially on $p^{(\nu)}$ (i.e., this is a functor on the comma category of maps in $\mathcal{C}$), and each $i_\alpha$ is in $\mathfrak{C}_\mathcal{C}$.

(d) There is a set of maps $h_\alpha : W_\alpha \to Y$ $(\alpha \in \mathcal{K}_\nu)$, also depending functorially on $p^{(\nu)}$, such that $p^{(\nu+1)} : Z^{(\nu+1)} \to Y$ is induced by $p^{(\nu)} : Z^{(\nu)} \to Y$ and $(\coprod_{\alpha \in \mathcal{K}_\nu} h_\alpha) : W \to Y$.

(e) For each limit ordinal $\nu$ we have $Z^{(\nu)} = \operatorname{colim}_{\mu < \nu} Z^{(\mu)}$.

*Remark* 4.3. Note that because each $i_\alpha$ is a cofibration, the maps $j^{(\nu)} : Z^{(\nu)} \to Z^{(\nu+1)}$, as well as the structure maps $Z^{(\nu)} \to Z$ (and thus $j : X \to Z$ itself) are cofibrations by Corollary 2.13.

Note also that canonical factorization implies in particular *functoriality* in Axiom 2(i) – that is, any $f : X \to Y$ may be factored $X \xrightarrow{j_f} Z_f \xrightarrow{p_f} Y$ (with $j_f \in \mathfrak{C}$ and $p_f \in \mathfrak{F} \cap \mathfrak{W}$), in such a way that, given maps $f' : X' \to Y'$, $x : X \to X'$ and $y : Y \to Y'$ such that $y \circ f = f' \circ x$, there is a map $z : Z_f \to Z_{f'}$ such that $z \circ i_f = i_{f'} \circ x$ and $y \circ p_f = p_{f'} \circ z$. While this functoriality is not part of Quillen's original definition, it is a useful property (which is in fact enjoyed by almost all model categories).

For examples of canonical factorizations, see Example 4.6 and Remark 5.3(ii) below. The most common situation is when $\kappa = \omega$.

The above is of course simply a partial axiomatization of Quillen's "small object argument" construction (see [Q1, II, 3.3-3.4]). Were we not interested in a dualizable version (see sections 6 and 7 below) – we could have started with a full axiomatization of Quillen's construction, as follows:

**Definition 4.4.** If $\langle \mathcal{C}; \mathfrak{W}_\mathcal{C}, \mathfrak{C}_\mathcal{C}, \mathfrak{F}_\mathcal{C} \rangle$ is a left model category, we say that a set of cofibrations $\{i_\gamma : V_\gamma \to W_\gamma\}_{\gamma \in \Gamma}$ is a collection of *$\kappa$-compact test cofibrations* for $\mathcal{C}$ if:

(a) any map $f : X \to Y$ in $\mathcal{C}$ which has the RLP with respect to each $i_\gamma$ $(\gamma \in \Gamma)$ is a trivial fibration, and

(b) the domain $V_\gamma$ of each test cofibration is *$\kappa$-compact* in $\mathcal{C}$ – that is, $Hom_\mathcal{C}(V_\gamma, -)$ commutes with sequential colimits of monomorphisms of type $\kappa$. (When $\kappa = \omega$, such objects are called (sequentially) *small* – cf. [Mit, II, §16]).

*Remark* 4.5. Note that if $\mathcal{C}$ is a concrete category, *any* object $C$ is $\kappa$-compact for any ordinal $\kappa$ of cofinality greater than the cardinality $|C|$ of $C$ (cf. [TZ, 10.51]), so (b) above is automatically satisfied for $\kappa = (\sup_{\gamma \in \Gamma}\{|V_\gamma|\})^+$, the successor cardinal of the supremum of the cardinalities of (the underlying sets of) all objects $V_\gamma$. (The idea of thus eliminating the requirement of "smallness" in Quillen's construction is due to Bousfield – cf. [Bo1, §11].)

It will also hold in other cases – for example, any finite simplicial set (i.e., one with finitely many non-degenerate simplices), or finitely generated $R$-module, is $\omega$-compact.

**Example 4.6.** The motivating example of test cofibrations is the model category of simplicial sets $\mathcal{S}$ (§2.6): by [Q1, II, §2, Prop. 1], a map $f : X \to Y$ is a trivial fibration if it has the RLP with respect to all the cofibrations $i_k : \overset{\bullet}{\boldsymbol{\Delta}}[k] \to \boldsymbol{\Delta}[k]$ $(k \geq 0)$, where $\boldsymbol{\Delta}[k]$ is the standard simplicial $k$-simplex, and $i_k : \overset{\bullet}{\boldsymbol{\Delta}}[k] \hookrightarrow \boldsymbol{\Delta}[k]$ is the inclusion of its $(k-1)$-skeleton.

**Definition 4.7.** If $\{i_\gamma : V_\gamma \to W_\gamma\}_{\gamma \in \Gamma}$ is a set of morphisms in some category $\mathcal{C}$, the associated *Quillen construction of type $\kappa$* is a functorial factorization $X \xrightarrow{j} Z \xrightarrow{p} Y$ of any morphism $f : X \to Y$ in $\mathcal{C}$, constructed as in §4.2, where for each $\nu < \kappa$, the set $\mathcal{K}_\nu$ (in 4.2(c)) is $\mathcal{K}_\nu = \coprod_{\alpha \in A} \mathcal{D}_{\nu,\alpha}$, with $\mathcal{D}_{\nu,\alpha} =$ the set all commutative diagrams $(d)$ of the form:

$$
\begin{array}{ccc}
V_\alpha & \xrightarrow{\quad g_d \quad} & Z^{(\nu)} \\
{\scriptstyle i_\alpha}\big\downarrow & & \big\downarrow {\scriptstyle p^{(\nu)}} \\
W_\alpha & \xrightarrow{\quad h_d \quad} & Y
\end{array}
\qquad (d) \quad,
$$

with $i_\alpha$ in the given collection $\{i_\gamma\}_{\gamma \in \Gamma}$. We set $h_\alpha = h_d$ in 4.2(d).

This is the ingredient needed to make Quillen's small object argument [Q1, II, 3.4] work:

**Proposition 4.8.** *If* $\langle \mathcal{C}; \mathfrak{W}_\mathcal{C}, \mathfrak{C}_\mathcal{C}, \mathfrak{F}_\mathcal{C} \rangle$ *is a left model category with a collection of $\kappa$-compact test cofibrations* $\{i_\gamma : V_\gamma \to W_\gamma\}_{\gamma \in \Gamma}$, *then the associated Quillen construction of type $\kappa$ yields canonical factorizations (which we shall call* canonical Quillen factorizations*) for Axiom 2.*

*Proof.* For any $f : X \to Y$ in $\mathcal{C}$, let $X \xrightarrow{j} Z \xrightarrow{p} Y$ be the Quillen construction associated to the set of test cofibrations; then $j$ is a cofibration by Corollary 2.13. To see that $p$ is a trivial fibration, we must show that it has the RLP with respect to all test cofibrations $i_\gamma : V_\gamma \to W_\gamma$ – i.e., we must produce liftings $\tilde{h}$ for any $h, g$ as below:

$$
\begin{array}{ccc}
V_\gamma & \xrightarrow{\ \ g\ \ } & Z \\
\downarrow{\scriptstyle i_\gamma} & \ \tilde{h} \nearrow\ \ & \downarrow{\scriptstyle p} \\
W_\gamma & \xrightarrow{\ \ h\ \ } & B
\end{array}
$$

But by the $\kappa$-compactness of $V_\gamma$, any map $g : V_\gamma \to Z = \operatorname{colim}_{\nu < \kappa} Z^{(\nu)}$ factors through $\hat{g} : V_\alpha \to Z^{(\nu)}$ for some $\nu < \kappa$, and the diagram

$$
\begin{array}{ccc}
V_\gamma & \xrightarrow{\ \ \hat{g}\ \ } & Z^{(\nu)} \\
\downarrow{\scriptstyle i_\gamma} & & \downarrow{\scriptstyle p^{(\nu)}} \\
W_\gamma & \xrightarrow{\ \ h\ \ } & B
\end{array}
$$

is one of the diagrams $(d)$ used to construct $Z^{(\nu+1)}$ in §4.7, so the structure map $W_\gamma \to Z^{(\nu+1)}$ defines the required lifting for the original $g$ and $h$. $\qquad\square$

*Remark* 4.9. The same definitions are possible for a right model category – although contrary to what one might expect, the construction is not dual to the above:

We say that a right model category $\langle \mathcal{C}; \mathfrak{W}_\mathcal{C}, \mathfrak{C}_\mathcal{C}, \mathfrak{F}_\mathcal{C} \rangle$ has *canonical factorizations* if the factorization $f = p \circ j$ ($j \in \mathfrak{C} \cap \mathfrak{W}$, $p \in \mathfrak{F}$) of Axiom 1(i) is obtained functorially precisely as in Definition 4.2, except that the maps $i_\alpha$ of 4.2(c) are required to be *trivial* cofibrations. Of course, Proposition 4.8 also holds for right model categories, with the Quillen construction using test trivial cofibrations.

**Example 4.10.** Let $\boldsymbol{V}(n, k) \subset \boldsymbol{\Delta}[n]$ $(0 \leq k \leq n)$ denote the simplicial set generated by all the $(n-1)$-dimensional faces of $\boldsymbol{\Delta}[n]$, except for the $k$-th one. The inclusions $i_{n,k} : \boldsymbol{V}(n, k) \hookrightarrow \boldsymbol{\Delta}[n]$ are the test trivial cofibrations for $\mathcal{S}$ (cf. [Q1, II, §2, Prop. 2]), so $\mathcal{S}$ has canonical Quillen factorizations (of type $\omega$) for Axiom 1, too.

**Definition 4.11.** Let $\mathcal{C} \overset{F}{\underset{U}{\rightleftarrows}} \mathcal{D}$ be adjoint functors, and $\mathcal{D}$ a left model category with canonical factorizations (of type $\kappa$) as in Definition 4.2. The *derived* factorization of any $f : A \to B$ in $\mathcal{C}$ is $A \xrightarrow{\ i\ } C \xrightarrow{\ q\ } B$, where $C = \operatorname{colim}_{\nu < \kappa} C^{(\nu)}$, and the maps $i : A \to C$ and $q : C \to B$, are obtained from a commutative diagram

$$
A \xrightarrow{\ i^{(0)}\ } C^{(0)} \cdots \cdots C^{(\nu)} \xrightarrow{\ \ i^{(\nu)}\ \ } C^{(\nu+1)} \cdots \cdots
$$
$$
{\scriptstyle f}\searrow \quad {\scriptstyle q^{(0)}}\searrow \quad {\scriptstyle q^{(\nu)}}\downarrow \quad {\scriptstyle q^{(\nu+1)}}\swarrow
$$
$$
B
$$

defined as follows: apply the construction in $\mathcal{D}$ to the map $U(f) : U(A) \to U(B)$ to get the pushout diagram $PO^{(-1)}$ of §4.2; then $C^{(0)}$ is the pushout of the *adjoint diagram* $\widehat{PO^{(-1)}}$:

$$
\begin{array}{ccc}
FV & \xrightarrow{\;\hat{g}\;} & A \\
{\scriptstyle Fi}\big\downarrow & \boxed{\text{PO}} & \big\downarrow{\scriptstyle i^{(0)}} \\
FW & \longrightarrow & C^{(0)}
\end{array}
\qquad\qquad \widehat{PO^{(-1)}}
$$

in which $\hat{g} : F(V) \to A$ is the adjoint of $g : V \to UA$, and $q^{(0)} : C^{(0)} \to B$ is induced by $f$ and the adjoint $\hat{h} : F(W) \to B$ of the coproduct of the maps $h_\alpha : W_\alpha \to UB$ (see §4.2).

More generally, for each $\nu < \kappa$, since the diagram $PO^{(\nu)}$ depends functorially only on the map $Up^{(\nu)} : UC^{(\nu)} \to UB$, we may define its adjoint $\widehat{PO^{(\nu)}}$ precisely as for $PO^{(-1)}$, and let $C^{(\nu+1)}$ be its pushout. Setting $C^{(\nu)} = \operatorname{colim}_{\mu<\nu} C^{(\mu)}$ for each limit ordinal $\nu < \kappa$ completes the construction.

Note that if the pair $(U, F)$ produces the colimits in $\mathcal{C}$ (§3.3), applying $U$ to the above factorization yields, for any map of the form $U\varphi : UA \to UB$, a construction of $UA \xrightarrow{Ui} UC \xrightarrow{Up} UB$ in $\mathcal{D}$ which can be described purely in terms of colimits in $\mathcal{D}$ and the monad (or triple) $UF : \mathcal{D} \to \mathcal{D}$ (which in fact determines the category $\mathcal{C}$ – see [Mac1, VI, §2]).

**Example 4.12.** If $\mathcal{A} \underset{U}{\overset{F}{\rightleftharpoons}} \mathcal{S}$ are adjoint functors, with $\mathcal{A}$ a category of universal algebras (§3.6), the derived factorization $A \to C \to B$ of any $\varphi : A \to B$ in $\mathcal{A}$ is constructed as in [Q1, II, §4, Prop. 3].

**Definition 4.13.** Let $\langle \mathcal{D}; \mathfrak{W}_\mathcal{D}, \mathfrak{C}_\mathcal{D}, \mathfrak{F}_\mathcal{D} \rangle$ be a left model category with canonical factorizations, and $\mathcal{C} \underset{U}{\overset{F}{\rightleftharpoons}} \mathcal{D}$ a pair of adjoint functors. We say that the pair $(U, F)$ *creates* a left model category structure $\langle \mathcal{C}; \mathfrak{W}_\mathcal{C}, \mathfrak{C}_\mathcal{C}, \mathfrak{F}_\mathcal{C} \rangle$ if

(i) $f \in \mathfrak{F}_\mathcal{C} \Leftrightarrow Uf \in \mathfrak{F}_\mathcal{D}$;
(ii) $f \in \mathfrak{W}_\mathcal{C} \Leftrightarrow Uf \in \mathfrak{W}_\mathcal{D}$;
(iii) $\mathfrak{C}_\mathcal{C} = \{ i = i_1 \circ \ldots \circ i_n \mid$ each $i_k$ is the first factor of some derived factorization$\}$.

**Theorem 4.14.** *Let $\langle \mathcal{D}; \mathfrak{W}_\mathcal{D}, \mathfrak{C}_\mathcal{D}, \mathfrak{F}_\mathcal{D} \rangle$ be a left model category with canonical factorizations of type $\kappa$, and $\mathcal{C} \underset{U}{\overset{F}{\rightleftharpoons}} \mathcal{D}$ a pair of adjoint functors. Assume that $U$ creates sequential colimits of monomorphisms of type $\kappa$, that $\mathcal{C}$ has all finite colimits, and that*

*(∗)   the derived factorization $A \xrightarrow{\;i\;} C \xrightarrow{\;p\;} B$ for any $f$ in $\mathcal{D}$ satisfies $Up \in \mathfrak{F}_\mathcal{D} \cap \mathfrak{W}_\mathcal{D}$.*

*Then $(U, F)$ creates a left model category structure with canonical factorizations.*

*Proof.* For Axiom 2(i) use the derived factorization of §4.11. For Axiom 2(ii), it suffices to show that any $i : A \to C$ constructed by the derived factorization of some $f : A \to B$ has the LLP with respect to any trivial fibration in $\mathcal{C}$; but if $j : V \to W$ is a cofibration in $\mathcal{D}$, then $Fj : FV \to FW$ has the LLP with respect to any trivial fibration in $\mathcal{C}$, by Def. 4.13(i)-(ii). We then see that the $i$'s constructed in §4.11 have the LLP by Corollary 2.13.                                                                     □

Hypothesis (∗) of Theorem 4.14 may seem hard to verify, but it seems unavoidable for our purpose (that is, for categorical dualizing). However, for other purposes the following version of the theorem, using the Quillen construction, may be more useful:

**Theorem 4.15.** *Let* $\langle \mathcal{D}; \mathfrak{W}_{\mathcal{D}}, \mathfrak{C}_{\mathcal{D}}, \mathfrak{F}_{\mathcal{D}} \rangle$ *be a left model category with a set* $\{i_{\gamma}\}_{\gamma \in \Gamma}$ *of* $\kappa$-*compact test cofibrations (and thus canonical Quillen factorizations), let* $\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *be a pair of adjoint functors, and assume that* $U$ *creates sequential colimits of monomorphisms of type* $\kappa$, *and that* $\mathcal{C}$ *has all finite colimits. Then* $(U, F)$ *create a left model category structure on* $\mathcal{C}$ *with a set* $\{Fi_{\gamma}\}_{\gamma \in \Gamma}$ *of* $\kappa$-*compact test cofibrations (and thus canonical Quillen factorizations).*

*Proof.* Since $U$ creates sequential colimits of monomorphisms of type $\kappa$, its left adjoint $F$ preserves $\kappa$-compactness, and by Definition 4.13, $F$ preserves the property of being test cofibrations. Thus $\{Fi_{\gamma}\}_{\gamma \in \Gamma}$ is a set of $\kappa$-compact test cofibrations for $\mathcal{C}$.

Alternatively, one could show directly (as in the proof of Proposition 4.8) that for any derived factorization $A \xrightarrow{i} C \xrightarrow{p} B$, the map $Up : UC \to UB$ has the RLP with respect to all test cofibrations – and then apply Theorem 4.14. Of course, the derived factorizations are just the canonical Quillen factorizations with respect to the new set of test cofibrations. $\square$

Again, the analogous theorem holds for right model categories, so that in fact we have a way to transport full model category structures using adjoint functors.

**Example 4.16.** When $\mathcal{C} = s\mathcal{A}$ for some category of universal algebras $\mathcal{A}$, the functors $s\mathcal{A} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{S}$ allow us to transfer the left model category structure of $\mathcal{S}$ to $s\mathcal{A}$, by Theorem 4.15. This yields the left model category structure on $\mathcal{C}$ described in [Q1, II, §4], in which $\mathfrak{C}_{\mathcal{C}}$ is the class of free maps of simplicial algebras (*ibid.*). This is actually a full model category structure on $s\mathcal{A}$, with the derived factorization obtained from the construction of §4.10 serving for Axiom 1(i).

## 5. SIMPLICIAL OBJECT OVER ABELIAN CATEGORIES

If $\mathcal{M}$ is any abelian category, there are adjoint functors $s\mathcal{M} \underset{N}{\overset{K}{\rightleftarrows}} c_{*}\mathcal{M}$ which are equivalences between the categories of (respectively) the simplicial objects and the chain complexes over $\mathcal{M}$ (cf. [Do, Thm 1.9]). In order to define a model category structure on $s\mathcal{M}$ it thus suffices to do so for the more familiar category of chain complexes, as Quillen does in [Q1, II, 4.11-4.12]:

**Definition 5.1.** Let $\mathcal{M}$ be an abelian category. Define $\mathfrak{W}_{c_{*}\mathcal{M}}$ to be the class of homology isomorphisms, $\mathfrak{F}_{c_{*}\mathcal{M}}$ the class of maps $f : A_{*} \to B_{*}$ which are surjective in positive degrees, and $\mathfrak{C}_{c_{*}\mathcal{M}}$ the class of one-to-one maps whose cokernel is projective in each dimension (if $\mathcal{M} = R\text{-}Mod$, we may require the cokernel to be dimensionwise *free*).

**Proposition 5.2.** *If* $\mathcal{M}$ *is an abelian category with enough projectives, then* $\langle c_{*}\mathcal{M}; \mathfrak{W}_{c_{*}\mathcal{M}}, \mathfrak{C}_{c_{*}\mathcal{M}}, \mathfrak{F}_{c_{*}\mathcal{M}} \rangle$ *is a model category.*

*Proof.* For notational simplicity we consider the case where every $A_{*} \in c_{*}\mathcal{M}$ has a *functorial* projective cover $\varepsilon_{A_{*}} : FA_{*} \to A_{*}$.

For Axiom 1(i), let $L : gr\mathcal{M} \to c_{*}\mathcal{M}$ denote the left adjoint of the forgetful functor $V : c_{*}\mathcal{M} \to gr\mathcal{M}$, with natural transformation $\vartheta_{B}^{L} : LVB_{*} \to B_{*}$, and use the factorization:

$$A_{*} \xrightarrow{\ i_{A_{*}}\ } A_{*} \amalg FLV(B_{*}) \xrightarrow{\ (f, \vartheta_{B}^{L} \circ \varepsilon_{LVB_{*}})\ } B_{*}$$

For Axiom 2(i), we wish to construct a sequence of commuting diagrams:

$$A_* \xrightarrow{j^{(0)}} C_*^{(0)} \cdots \cdots C_*^{(n-1)} \xrightarrow{j^{(n)}} C_*^{(n)} \cdots \cdots$$

with maps $f$, $p^{(0)}$, $p^{(n-1)}$, $p^{(n)}$ to $B_*$.

where (i) each $j^{(n)}$ is a cofibration;
(ii) each $p^{(n)}$ is a fibration;
(iii) $p^{(n)}$ induces an epimorphism $p_*^{(n)} : H_i C_*^{(n)} \twoheadrightarrow H_i B_*$ for all $i$;
(iv)(n) $p_*^{(n)} : H_i C_*^{(n)} \to H_i B_*$ is a monomorphism for $i < n$.

and then set $C_* = \mathrm{colim}\, C_*^{(n)}$.

To get a factorization $A_* \xrightarrow{j^{(0)}} C_*^{(0)} \xrightarrow{p^{(0)}} B_*$ of the given $f$ satisfying conditions (i), (ii), & (iii), let $T : gr\mathcal{M} \to c_*\mathcal{M}$ be the left adjoint of the functor $Z : c_*\mathcal{M} \to gr\mathcal{M}$ defined $Z(A_*)_n = Z_n^A \overset{Def}{=} Ker\{\partial_n : A_n \to A_{n-1}\}$, and set $C_*^{(0)} = A_* \amalg FTZ(B_*) \amalg FLV(B_*)$.

For the inductive step, assume given $f : A_* \to B_*$ satisfying conditions (ii), (iii), & (iv)(n) above; we wish to construct a factorization $A_* \xrightarrow{j} C_* \xrightarrow{p} B_*$ of $f$ satisfying conditions (i)-(iv)(n+1):

Let $K_n = Ker\{f_n : A_n \to B_n\} \cap Z_n^A \overset{i}{\hookrightarrow} A_n$ and $K_n \overset{q}{\twoheadrightarrow} Q_n = K_n/(Im\{\partial_{n+1} \cap K_n\})$, and let $\mathcal{L}$ denote the set of liftings $\lambda : FQ_n \to K_n$ in

$$
\begin{array}{ccc}
 & & FQ_n \\
 & \nearrow^{\lambda} & \downarrow^{\varepsilon_{Q_n}} \\
K_n & \xrightarrow{q} & Q_n \ .
\end{array}
$$

Let $E_*$ equal $A_*$ in degrees $\leq n$, $E_{n+1} = \coprod_{\lambda \in \mathcal{L}}(FQ_n)_\lambda$, and $E_i = 0$ for $i > n+1$, with $\partial_{n+1}^E : E_{n+1} \to A_n$ equal to $i \circ \lambda$ on $(FQ_n)_\lambda$. We have a map $g : E_* \to B_*$ equal to $f$ in dimensions $\leq n$ and $0$ elsewhere, and define $C_*$ to be the pushout of $A_* \hookleftarrow \tau_n A_* \hookrightarrow E_*$ (where $\tau_n A_*$ is $A_*$ truncated above degree $n$).

The lifting properties of Axioms 1(ii) & 2(ii) follow from those of projective objects in $\mathcal{M}$ in a straightforward manner. $\square$

*Remark* 5.3. We have given the proof because we have not seen it elsewhere; furthermore, it provides an illustration of the various types of model categories and factorizations which may occur:

(i) If $\mathcal{M}$ has enough projectives, we merely get a model category structure on $c_*\mathcal{M}$, as stated.

(ii) If $\mathcal{M}$ has *functorial* projective covers, the construction given in the proof shows the (left and right) model categories $c_*\mathcal{M}$ have canonical factorizations as in Def. 4.2. Thus if $\mathcal{C} \overset{F}{\underset{U}{\rightleftharpoons}} \mathcal{M}$ are adjoint functors satisfying suitable hypotheses, then the (left) model category structure for $c_*\mathcal{M}$, and thus on $s\mathcal{M}$, can be used to define a (left) model category structure on $s\mathcal{C}$ (in addition to the existence of suitable colimits in $\mathcal{C}$, we require that $UFX$ be projective in $\mathcal{M}$ for any $X \in \mathcal{M}$).

(iii) Of course, if $\mathcal{M} = R\text{-}Mod$ then $c_*\mathcal{M} \cong s\mathcal{M}$ has canonical Quillen factorizations – by 4.16, since then $\mathcal{M}$ is a category of universal algebras.

Note however that for categories of universal algebras over $R\text{-}Mod$ (in which the objects have the underlying structure of a $R$-module, and all operations are $R$-linear), the construction given in the proof above, combined with Theorem 4.14, yields a simpler description of the factorization of Axiom 2 – and thus of "projective resolutions" – than that provided for arbitrary universal algebras by Theorem 4.15 and §4.16.

(iv) The situation is of course greatly simplified when *all* objects in $\mathcal{M}$ are projective, (e.g., for $\mathcal{M} = \mathbb{F}\text{-}mod$ where $\mathbb{F}$ is a field), since then the fibrations are just epimorphims. In that case, if we let $\hat{B}_*$ denote the (shifted) *cone* on $B_*$ – i.e., $\hat{B}_n = B_{n+1} \oplus B_n$, with $\hat{\partial}_n(b, b') = (\partial^B_{n+1}b' - b, \partial^B_n b)$ – and $p$ the projection $\hat{B}_* \twoheadrightarrow B_*$, then a functorial factorization of $f$ for Axiom 2(i) is then given by:

$$A_* \xrightarrow{(id,0)} A_* \oplus \hat{B}_* \xrightarrow{(f,p)} B_*.$$

Nevertheless, this is not a canonical factorization in the sense of Def. 4.2, so it will not be suitable for the purposes of Theorem 4.14.

## 6. COSIMPLICIAL OBJECT OVER ABELIAN CATEGORIES

The definitions and results of section 5 are readily dualized to cosimplicial objects, as follows:

**Definition 6.1.** Recall that a *cosimplicial* object $X^\bullet$ over any category $\mathcal{C}$ is a sequence of objects $X^0, X^1, \ldots, X^n, \ldots$ in $\mathcal{C}$ equipped with *coface* and *codegeneracy* maps $d^i : X^n \to X^{n+1}$, $s^j : X^{n+1} \to X^n$ $(0 \le i, j \le n)$ satisfying the cosimplicial identities (cf. [BoK1, X, §2.1]).

We denote the category of cosimplicial objects over $\mathcal{C}$ by $c\mathcal{C}$. If $\mathcal{M}$ is an abelian category, we denote by $c^*\mathcal{M}$ the category of cochain complexes over $\mathcal{M}$.

Dual to [Do, Thm. 1.9] (noted in the beginning of section 5) we have:

**Proposition 6.2.** *For any abelian category $\mathcal{M}$ there is a natural isomorphism of categories $c\mathcal{M} \cong c^*\mathcal{M}$.*

*Proof.* Given $C^\bullet \in c\mathcal{M}$, the functor $N : c\mathcal{M} \to c^*\mathcal{M}$ is defined by $N^n = (NC^\bullet)^n = \bigcap_{j=0}^{n-1} Ker\{s^j : C^n \to C^{n-1}\}$, with $\delta^n : N^n \to N^{n+1}$ equal to $\sum_{i=0}^n (-1)^i (d^i|_{\cap_{j=0}^{n-1} Ker s^j})$.

Given $A^* \in c^*\mathcal{M}$, the inverse functor $L : c^*\mathcal{M} \to c\mathcal{M}$ is defined $LA^* = C^\bullet$, where $C^\bullet$ may be described explicitly in a manner dual to [May, p. 95] or [Bl, 5.2.1]:

For each $n \ge 0$ and $0 \le \lambda \le n$, let $\mathcal{I}_n^\lambda$ denote the set of all sequences $I = (i_1, \ldots, i_\lambda)$ of $|I| = \lambda$ integers such that $0 \le i_1 < i_2 < \ldots < i_\lambda \le n$; let $s^I = s^{i_1} \circ \cdots \circ s^{i_\lambda}$ be the corresponding $\lambda$-*fold codegeneracy map*. (We allow $\lambda = 0$, with the corresponding $s^\emptyset = 0$). Then

$$(6.3) \qquad\qquad C^n \stackrel{Def}{=} \prod_{0 \le \lambda \le n} \prod_{I \in \mathcal{I}_n^\lambda} A_{(I)}^{n-\lambda}.$$

We write $\pi_{(I)} : C^n \twoheadrightarrow A_{(I)}^{n-|I|}$ for the projection onto the copy of $A^{n-|I|}$ indexed by $I$.

For each $0 \le \lambda \le n$ and $0 \le k \le n-1$ there is a one-to-one function $\bar{s}_n^k : \mathcal{I}_{n-1}^{\lambda-1} \to \mathcal{I}_n^\lambda$, where $\bar{s}_n^k(I) = J$ is defined by the requirement that $s^I \circ s^k = s^J$ under the cosimplicial identities. The codegeneracy map $s^k : C^n \to C^{n-1}$ is then defined to be the composite:

$$C^n \longrightarrow \prod_{1 \le \lambda \le n} \prod_{I \in im(\bar{s}_n^k) \subset \mathcal{I}_n^\lambda} A^{n-\lambda}_{(\bar{s}_n^k I)} \xrightarrow{(\bar{s}_n^k)_*^{-1}} \prod_{0 \le \mu \le n-1} \prod_{I \in \mathcal{I}_{n-1}^\mu} A^{n-\mu}_{(I)} = C^{n-1} \ .$$

The coface map $d^j : C^n \to C^{n+1}$ is determined by the requirement that $\pi_{(\emptyset)} \circ d^0 = \delta^n : A^n \to A^{n+1}$ and $\pi_{(\emptyset)} \circ d^j = 0$ for $j > 0$, and by the cosimplicial identities – that is, given $J \in \mathcal{I}_{n+1}^\lambda$, use the cosimplicial identities to write $s^J \circ d^j = \phi \circ S^I$ (where $|I| = |J| + \epsilon - 1$, and either $\phi = id$, $\epsilon = 0$ or $\phi = d^i$, $\epsilon = 1$). Then $\pi_{(J)} \circ d^j : C^n \to A^{n+1-|J|}_{(J)}$ is the composite

$$C^n \xrightarrow{\pi_{(I)}} A^{n-|I|}_{(I)} \cong A^{n-|I|}_{(\emptyset)} \xrightarrow{\phi} A^{n-|I|+\epsilon}_{(\emptyset)} \cong A^{n+1-|J|}_{(J)}.$$

$\square$

One thus has a model category structure on $c\mathcal{M}$, induced by the following dual of Proposition 5.2:

**Proposition 6.4.** *If $\mathcal{M}$ is an abelian category with enough injectives, there is a model category structure on $c^*\mathcal{M}$ with $\mathfrak{W}_{c^*\mathcal{M}}$ the class of cohomology isomorphisms, $\mathfrak{C}_{c^*\mathcal{M}}$ the maps which are one-to-one in positive degrees, and $\mathfrak{F}_{c_*\mathcal{M}}$ the surjective maps with injective kernel.*

*Proof.* The proof is precisely dual to the case of chain complexes. For convenience of reference below we briefly recapitulate the factorization for Axiom 1(i):

Given $f : A^* \to B^*$ in $c^*\mathcal{M}$, we want $A^* \xrightarrow{i} C^* \xrightarrow{p} B^*$ with $i \in \mathfrak{C}_{c^*\mathcal{M}} \cap \mathfrak{W}_{c^*\mathcal{M}}$ and $p \in \mathfrak{F}_{c^*\mathcal{M}}$ ($f = p \circ i$), again under the assumption that every $A^* \in c^*\mathcal{M}$ has a functorial injective envelope $\varepsilon_{A^*} : A^* \hookrightarrow IA^*$. As in §4.2 we wish to construct a sequence of commuting diagrams:



where each $p_{(n)}$ is a fibration, and each $j_{(n)}$ is a cofibration which is monic in cohomology (in all dimensions), and epic in cohomology through dimension $n - 1$. We then set $C^* = \lim C^*_{(n)}$.

In this case the forgetful functor $V : c^*\mathcal{M} \to gr\mathcal{M}$ has a right adjoint $R : gr\mathcal{M} \to c^*\mathcal{M}$, and the functor $C : c^*\mathcal{M} \to gr\mathcal{M}$, defined by: $CA_n^* = Coker(\delta_A^{n-1})$, has a right adjoint $T : gr\mathcal{M} \to c^*\mathcal{M}$. Thus if we set $C^*_{(0)} \stackrel{Def}{=} B^* \times ITC(A^*) \times IRV(A^*)$, we find that the map $j_{(0)} : A^* \hookrightarrow C^*_{(0)}$ is a cofibration which is monic in cohomology, and the projection $\pi_{B^*} : C^*_{(0)} \twoheadrightarrow B^*$ is a fibration.

For the inductive step, assume given a cofibration $f : A^* \to B^*$ which is monic in cohomology, and epic in cohomology through dimension $n-1$. Let $P^n = Z_B^n \cup A^n \stackrel{i^n}{\hookrightarrow} B^n$) and $Q^n = P^n / (A^n \cup Im(\delta_B^{n-1}))$, with $t^n : P^n \to IQ^n$ the obvious composite map.

Let $\mathcal{N}$ denote the set of extensions $\nu : B^n \to IQ^n$ ($\nu \circ i^n = t^n$), and define the cochain complex $E^*$ to be equal to $B^*$ in dimensions $\le n$, zero above dimension $n+1$, with $E^{n+1} = \prod_{\nu \in \mathcal{N}} IQ^n_{(\nu)}$ and $\delta_E^{n+1} : B^n \to E^{n+1}$ determined by the $\nu$'s. Finally

let $C^*$ be the pullback of $B^* \to \tau^n B^* \leftarrow E^*$, (where $\tau^n B^*$ is again the truncated complex), so $C^i = B^i$ for $i \leq n$. The obvious maps $A^* \xrightarrow{j} C^* \xrightarrow{p} B^*$ give a factorization with $j$ a cofibration which is monic in cohomology, and epic in cohomology through dimension $n$.

Note that since the maps $p_{(n)}$ are isomorphisms in degrees $\leq n$, there is no $lim^1$ in calculating $H^i C^* = H^i(\lim C^*_{(n)})$ (cf. [Mil]). This problem did not arise in the dual case (§4.2 and Proposition 5.2), since colim is exact. $\qquad\square$

Note that $\mathcal{M} = R\text{-}Mod$ has functorial injective envelopes (constructed as in [Mac2, III, 7.4]). The construction given here is of course an example of *dual canonical factorizations* (defined dually to Def. 4.2).

*Remark* 6.5. The explicit factorization of any $f : A^\bullet \to B^\bullet$ in the category $c\mathcal{M}$ of cosimplicial objects over $\mathcal{M}$ is easily obtained from the proof of Proposition 6.4 using Proposition 6.2. It should be pointed out that in place of the truncated cochain complex $\tau^n A^*$, which vanished above dimension $n$, we must use the $n$-th coskeleton $csk^n A^\bullet \overset{Def}{=} L(\tau^n NA^\bullet)$, which is defined in cosimplicial dimensions $> n$ by (6.3) (compare the dual description in [Bl, 5.3.4]). Similarly for the construction of $E^*$ from $\tau^n A^*$ (cf. [Bl, 5.3.2]).

## 7. Cosimplicial coalegbras

In order to define right derived functors over a category of coalgebras, one would obviously like to dualize the constructions of section 4. However, there seems to be no reasonable (right) model category structure on the category of cosimplicial sets. Thus our approach here is more restricted. First, we recall the definition of a category of coalgebras:

**Definition 7.1.** (i) A *comonad* (or cotriple) $\mathfrak{S}$ on a category $\mathcal{C}$ consists of a functor $S : \mathcal{C} \to \mathcal{C}$ equipped with two natural transformations: $\varepsilon : S \to id_\mathcal{C}$ and $\nu : S \to S^2$ satisfying: $S\varepsilon \circ \nu = \varepsilon_S \circ \nu = id_\mathcal{C}$ and $\nu S \circ \nu = S\nu \circ \nu : S^3 \to S$ (cf. [EM1, §2]).

(ii) A *coalgebra* over a comonad $\mathfrak{S} = \langle S : \mathcal{C} \to \mathcal{C}, \nu, \varepsilon \rangle$ is an object $C \in \mathcal{C}$ together with a morphism $\varphi : C \to SC$ such that $S\varphi \circ \varepsilon_C = id_C$ and $S\varphi \circ \varphi = \nu \circ \varphi : C \to S^3 C$. The category of coalgebras over $\mathfrak{S}$ (with the obvious morphisms) will be denoted by $\mathcal{C}_\mathfrak{S}$.

*Remark* 7.2. For every comonad $\mathfrak{S} = \langle S : \mathcal{C} \to \mathcal{C}, \nu, \varepsilon \rangle$ there is a pair of adjoint functors $\mathcal{C} \overset{G}{\underset{V}{\rightleftarrows}} \mathcal{C}_\mathfrak{S}$ such that $S = VG$ (and conversely, every pair of adjoint functors yield a comonad). $V : \mathcal{C}_\mathfrak{S} \to \mathcal{C}$ is the faithful "underlying $\mathcal{C}$-object" functor, and $GC = (C, \nu)$.

The relation between the categorical definition and the more concrete analogue of Definition 3.6 is more problematic, since we need the underlying $\mathcal{C}$-object to be an object in an abelian category, and not just a set. (This is in order to make use of the model category structure on $c\mathcal{M}$ defined in section 6, since we do not have one on $c\mathcal{S}et$, as noted above). Thus we specialize to the case where $\mathcal{C}$ is a monoidal abelian category $\langle \mathcal{M}, \otimes \rangle$ (see [Mac1, VII,1] for the definition; the only example we shall actually need being $\mathcal{M} = R\text{-}Mod$ and $\otimes = - \otimes_R -$):

**Definition 7.3.** A category of *universal coalgebras* over $\langle \mathcal{M}, \otimes \rangle$ is a category $\mathcal{B}$, whose objects are objects $A \in \mathcal{M}$, together with an action of a fixed set of $n$-ary co-operators

$W = \cup_{n=0}^{\infty} \{\omega : A \to A^{\otimes n}\}$, satisfying a set of identities $E$; the morphisms are functions on the sets which commute with the co-operators.

**Example 7.4.** Categories of universal coalgebras include:

The category $\mathcal{C}_R$ of coalgebras over a ring $R$ ([Sw, 1.0]) ; the category $\mathcal{CC}_R$ of cocommutative coalgebras over $R$ ([Sw, 3.2]) ; and for each prime $p$, the category $\mathcal{CA}_p$ of (graded) unstable coalgebras over the mod $p$ Steenrod algebra (see [BoK2, §11.3]).

More generally, let $\mathcal{A}$ be a category of universal algebras (see §3.7), in which $U : \mathcal{A} \to \mathcal{S}et$ factors through $U' : \mathcal{A} \to R\text{-}Mod$, for some ring $R$. We may then define a *conjugate* category $\mathcal{A}^{\star}$ of universal coalgebras as follows:

Since the $n$-ary operators of $\mathcal{A}$ are in one-to-one correspondence with the elements of the set $UFX_n$, where $X_n$ is a set with $n$ elements, we let $A_n = U'FX_n$, and define the $n$-ary co-operators of $\mathcal{A}^{\star}$ to be the elements of the $R$-module *conjugate* (or $R$-dual): $A_n^{\star} \overset{Def}{=} Hom_R(A_n, R) \in Mod\text{-}R$. We assume that $A_n$ is a finitely generated $R$-module for each $n$ (of finite type, if $R$ is a graded ring).

The relations among the co-operators correspond to the elements of $Hom_R(A_n^{\star}, A_m^{\star})$, just as the relations in $\mathcal{A}$ are determined by $Hom_{\mathcal{S}et}(UFX_n, UFX_m)$.

In some cases the functor $G : \mathcal{M} \to \mathcal{B}$, which is right adjoint to the "underlying object" functor $V : \mathcal{B} \to \mathcal{M}$, has a description as a "cofree coalgebra" functor. This is true for $\mathcal{B} = \mathcal{C}_{\mathbb{F}}$, where $\mathbb{F}$ is a field; see [Sw, 6.4.1] for an explicit description. Similarly for $\mathcal{B} = \mathcal{CC}_{\mathbb{F}}$ (see [Sw, 6.4.1,6.4.4]). For $\mathcal{B} = \mathcal{CA}_p$, we have

$$GX_* = H_*(\prod_{n=1}^{\infty} K(X_n, n); \mathbb{F}_p)$$

where $\mathcal{M} = gr_+\mathbb{F}_p\text{-}Mod$ (see [BoK2, 11.4]).

It is clear that Proposition 6.2 and Theorem 4.14 (as well as their proofs, and §§4.1, 4.2, 4.7, 4.11, and 4.13) may be dualized to yield:

**Proposition 7.5.** *For any category of universal coalgebras $\mathcal{B}$ over $\mathcal{M} = R\text{-}Mod$, the functor $G$ creates all colimits and sequential colimits of epimorphisms in $\mathcal{B}$, and the pair $(G, V)$ produces all limits in $\mathcal{B}$.*

**Theorem 7.6.** *Let $\langle \mathcal{C}; \mathfrak{W}_{\mathcal{C}}, \mathfrak{C}_{\mathcal{C}}, \mathfrak{F}_{\mathcal{C}} \rangle$ be a right model category with dual canonical factorizations of type $\kappa$, and $\mathcal{C} \overset{V}{\underset{G}{\rightleftarrows}} \mathcal{D}$ a pair of adjoint functors. Assume that $V$ creates sequential limits of epimorphisms of type $\kappa$, that $\mathcal{D}$ has all finite limits, and that*

*(\*)   the derived factorization $A \overset{i}{\longrightarrow} C \overset{p}{\longrightarrow} B$ for any $f$ in $\mathcal{D}$ satisfies $Vi \in \mathfrak{C}_{\mathcal{C}} \cap \mathfrak{W}_{\mathcal{C}}$.*

*Then $(G, V)$ create a right model category structure $\langle \mathcal{D}; \mathfrak{W}_{\mathcal{D}}, \mathfrak{C}_{\mathcal{D}}, \mathfrak{F}_{\mathcal{D}} \rangle$.*

In order to see when hypothesis (\*) of the Theorem applies, let us consider the case where $\mathcal{C} = c\mathcal{B}$ and $\mathcal{D} = c\mathcal{M}$ are both categories of cosimplicial objects, over a category $\mathcal{B}$ of universal coalegbras and an abelian category $\mathcal{M}$, respectively, and the adjoint functors $\mathcal{C} \rightleftarrows \mathcal{D}$ have been prolonged (§2.16) from some pair $\mathcal{M} \overset{V}{\underset{G}{\rightleftarrows}} \mathcal{B}$.

Now the derived factorization of a map $f : A^{\bullet} \to B^{\bullet}$ in $\mathcal{C} = c\mathcal{B}$, as given by the proof of Proposition 6.4, may be described as follows:

We start with $C^{\bullet}_{(0)} \overset{Def}{=} A^{\bullet} \times GITC(B^{\bullet}) \times GIRV(B^{\bullet})$, where $G : c\mathcal{M} \to c\mathcal{B}$ is as above, $I : c\mathcal{M} \to c\mathcal{M}$ is the (prolonged) injective envelope functor, and $gr\mathcal{M} \overset{T}{\underset{C}{\rightleftarrows}} c^{*}\mathcal{M}$ are the adjoint pair of Proposition 6.4. (Here we identify $c\mathcal{M}$ with $c^{*}\mathcal{M}$ by Proposition 6.2).

In general, $C^{\bullet}_{(n)}$ is the pullback of

$$C^{\bullet}_{(n-1)} \to Gcsk^{n}C^{\bullet}_{(n)} \leftarrow GE^{\bullet}$$

(see remark 6.5), so we see that $C^{\bullet}_{(n)}$ agrees with $C^{\bullet}_{(n-1)}$ through dimension $n$, $C^{n+1}_{(n)} = C^{n+1}_{(n-1)} \times GE^{n+1}$, and $C^{i}_{(n)}$ $(i > n)$ is determined by (6.3). Thus it is clear that $A^{\bullet} \to C^{\bullet}$ will be a cofibration. To verify that the inductive cohomology conditions hold for the derived construction in $c\mathcal{B}$, note that they hold in $c^{*}\mathcal{M} \cong c\mathcal{M}$ because the composite map

$$H \hookrightarrow P^{n} \overset{i^{n}}{\longrightarrow} B^{n} = VC^{n}_{(n-1)} \overset{(\nu)}{\longrightarrow} E^{n+1}$$

is a monomorphism. Here $H \cong H^{n}B^{*}/Im(f_{*}) \cong H^{n}(VC^{\bullet}_{(n-1)})/Im(j_{(n-1)})_{*}$. In the derived factorization we need to know that the composite:

$$H \hookrightarrow VC^{n}_{(n-1)} \overset{V\widetilde{(\nu)}}{\longrightarrow} VGE^{n+1}$$

is monic, where $\widetilde{(\nu)} : C^{n}_{(n-1)} \to GE^{n+1}$ is adjoint to $(\nu)$. This follows because for any $f : VX \to Y$ in $\mathcal{M}$ we have $\eta \circ V\tilde{f} = f$ (for $\eta : VGY \to Y$ the adjoint of $id_{GY}$).

Thus even though Proposition 4.8, and thus Theorem 4.15, do not dualize usefully to our situation (because Quillen's small object argument does not dualize to limits), we have the following simplified situation where $(*)$ of Theorem 7.6 holds:

**Proposition 7.7.** *Let $c\mathcal{M}$ be the category of cosimplicial objects over an abelian category $\mathcal{M}$ with functorial injective envelopes, endowed with the model category structure given by Propositions 6.2 & 6.4, and let $\mathcal{M} \overset{V}{\underset{G}{\rightleftarrows}} \mathcal{B}$ be a pair of adjoint functors such that $V$ is faithful. Then the factorization given in the proof of Proposition 6.4 satisfies hypothesis $(*)$ of Theorem 7.6.*

*Remark* 7.8. Theorem 7.6 provides a right model category structure on $c\mathcal{B}$ for any category of universal coalgebras $\mathcal{B}$ over an abelian category $\mathcal{M}$, since the hypotheses of Proposition 7.7 will in fact hold for such a $\mathcal{B}$ (compare [BaW, 3.3, Thm. 9]). However, for the purposes of "homotopical algebra", further assumptions may be needed.

In particular, in order for the "triple derived functors" (cf. [BaB]) of $T$ to coincide with the right derived functors (as defined in §4.11), we would want $GA$ to be an injective in $\mathcal{B}$ for any $A \in \mathcal{M}$. This will be true, for example, if *all* objects in $\mathcal{M}$ are injective (e.g., if $\mathcal{M} = \mathbb{F}\text{-}Mod$ for some field $F$).

For any $B \in \mathcal{B}$ one has a cosimplicial coalgebra $C^{\bullet} \in c\mathcal{B}$ obtained by the "dual standard construction" (cf. [BaB] or [God, App., §3]), with $C^{n} = (GV)^{n+1}B$ and the coface and codegeneracy maps determined by the comonad structure maps $\varepsilon$ and $\nu$ of §7.1. Moreover, the coaugmentation $\varepsilon : B \to C^{0}$ defines a map $i : c(B)^{\bullet} \to C^{\bullet}$ (where $c(B)^{\bullet}$ is the constant cosimplicial object which is $B$ in each dimension).

Under the hypothesis that $GA$ is always an injective, it is readily verified that $i : c(B)^{\bullet} \to C^{\bullet}$ is a trivial cofibration: it is always a weak equivalence, and it is a cofibration by Fact 2.12 and the extension properties of injective objects. A similar

argument shows that if $T : B \to \mathcal{B}'$ is a functor between such categories of universal algebras (possibly trivial – that is, simply abelian categories), its prolongation $cT : c\mathcal{B} \to \mathcal{B}'$ will preserve weak equivalences between cofibrant objects (compare [Mac1, III, Thm 3.1]), so that its right derived functors are defined (§2.16). Thus in particular we have the following

**Fact 7.9.** *In the right model category structure on $c\mathcal{CA}_p$ defined by Theorem 7.6 and Proposition 6.4, we may identify the $E_2$-term of the mod $p$ Bousfield-Kan spectral sequence as the right derived functors of $\mathrm{Hom}_{\mathcal{CA}_p}(B, -)$, as in [BoK2, Thm. 12.1].*

*Remark* 7.10. It should perhaps be observed that the situation for an abelian category $\mathcal{M}$, in which both left and right derived functors may be defined, is anomolous: it arises because $\mathcal{M}$ may be viewed either as a category of universal algebras or as a category of universal coalgebras, over itself. In general, most algebraic categories will support either left or right derived functors, but not both.

Moreover, the situations are not precisely dual, because categories of simplicial universal algebras derive their left model category structure from $\mathcal{S}$, while cosimplicial universal coalgebras get their right model category structure only from the underlying abelian category (which will usually be $\mathcal{M} = \mathbb{F}\text{-}Mod$, for some field $\mathbb{F}$).

Finally, as noted above, the failure of Quillen's small object argument for limits, even in abelian categories (see [Mil]), implies that some of the pleasant properties of adjoint functors with respect to model categories do not dualize.

## REFERENCES

[BaB]     M. Barr & J. Beck, "Homology and standard constructions", in B. Eckmann, ed., *Seminar on Triples and Categorical Homology Theory*, Springer-Verlag *Lec. Notes Math.* **80**, Berlin-New York, 1969, pp. 245-330.

[BaW]     M. Barr & C.F. Wells. *Toposes, Triples and Theories*, Springer-Verlag *Grund. math. Wissens.* **278**, Berlin-New York, 1985.

[Bau]     H.J. Baues, *Algebraic Homotopy*, Camb. St. Adv. Math. **15**, Cambridge U. Press, Cambridge-New York, 1988.

[Bl]      D. Blanc, "Derived functors of graded algebras", *J. Pure Appl. Alg.* **64** (1990) No. 3, pp. 239-262.

[BlS]     D. Blanc and C.S. Stover, "A generalized Grothendieck spectral sequence", In N. Ray & G. Walker, editors, *Adams Memorial Symposium on Algebraic Topology, Vol. 1*, Lond. Math. Soc. Lec. Notes Ser. **175**, Cambridge U. Press, Cambridge, 1992, pp. 145-161.

[Bo1]     A.K. Bousfield, "The localization of spaces with respect to homology", *Topology* **14** (1975) No. 2, pp. 135-150.

[Bo2]     A.K. Bousfield, "Nice homology coalgebras", *Trans. AMS* **148** (1979), pp. 473-489.

[BoF]     A.K. Bousfield & E.M. Friedlander, "Homotopy theory of $\Gamma$-spaces, spectra, and bisimplicial sets", In M.G. Barratt & M.E. Mahowald, eds., *Geometric Applications of Homotopy Theory, II*, Springer-Verlag *Lec. Notes Math.* **658**, Berlin-New York, 1978, pp. 80-130.

[BoK1]    A.K. Bousfield & D.M. Kan, *Homotopy limits, Completions, and Localizations*, Springer-Verlag *Lec. Notes Math.* **304**, Berlin-New York, 1972.

[BoK2]    A.K. Bousfield & D.M. Kan, "The homotopy spectral sequence of a space with coefficients in a ring", *Topology* **11** (1972), pp. 79-106.

[CaG]     J.G. Cabello & A.R. Garzón, "Closed model structures for algebraic models of $n$-types", *J. Pure Appl. Alg.* (to appear).

[Co]      P.M. Cohn, *Universal Algebras*, Math. & its Applic. **6**, D. Reidel Publ. Co., Dordrecht, Holland, 1981.

[Do]      A. Dold, "Homology of symmetric products and other functors of complexes", *Ann. Math. (2)* **68** (1958), pp. 54-80.

[DoP]      A. Dold & D. Puppe, "Homologie nicht-additiver Funktoren", *Ann. Inst. Fourier* **11** (1961) pp. 201-312.

[DwHK]     W.G. Dwyer, M.J. Hopkins, & D.M. Kan. "The homotopy theory of cyclic sets", *Trans. AMS* **291** (1985), pp. 281-289.

[DK]       W.G. Dwyer & D.M. Kan, "Equivalences between homotopy theories of diagrams", In W. Browder, ed., *Algebraic Topology and Algebraic K-Theory*, Ann. Math. Studies **113**, Princeton U. Press, Princeton, NJ, 1987, pp. 180-205.

[DwS]      W.G. Dwyer & J. Spalinski, "Homotopy theories and model categories", in I.M. James, ed, *Handbook of Algebraic Topology*, North-Holland, Amsterdam-New York, 1995, pp. 73-126.

[EM1]      S. Eilenberg & J.C. Moore, "Adjoint functors and triples", *Ill. J. Math.*, **9** (1965), pp. 381-398.

[EM2]      S. Eilenberg & J.C. Moore, *Foundations of relative homological algebra*, Mem. AMS. **55**, AMS, Providence, RI, 1965.

[God]      R. Godement, *Topologie algébrique et théorie des faisceaux*, Act. Sci. & Ind. No. 1252, Publ. Inst. Math. Univ. Strasbourg **XIII**, Hermann, Paris 1964.

[Goe]      P.G. Goerss, "On the André-Quillen Cohomology of Commutative $F_2$-algebras", *Astérisque* **186** 1990.

[He]       A. Heller, *Homotopy theories*, Mem. AMS. **383**, AMS, Providence, RI, 1988.

[Hu]       P.J. Huber, "Homotopy theory in general categories", *Math. Ann.* **144** (1961) No. 5, pp. 361-385.

[L]        F.E.J. Linton, "Coequalizers in categories of algebras", In B. Eckmann, ed., *Seminar on Triples and Categorical Homology Theory*, Springer-Verlag *Lec. Notes Math.* **80**, Berlin-New York, 1969, pp. 75-90.

[Mac1]     S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag *Grad. Texts in Math.* **5**, Berlin-New York, 1971.

[Mac2]     S. Mac Lane, *Homology*, Springer-Verlag *Grund. math. Wissens.* **114**, Berlin-New York, 1963.

[May]      J.P. May, *Simplicial Objects in Algebraic Topology*, U. Chicago Press, Chicago-London, 1967.

[Mil]      J.W. Milnor, "On axiomatic homology theory", *Pac. J. Math.* **12** (1962), pp. 337-341.

[Mit]      B. Mitchell, *Theory of Categories*, Pure & Appl. Math. **17**, Academic Press, New York, 1965.

[Q1]       D.G. Quillen, *Homotopical Algebra*, Springer-Verlag *Lec. Notes Math.* **20**, Berlin-New York, 1963.

[Q2]       D.G. Quillen, "Rational homotopy theory", *Ann. Math.* **90** (1969) No. 2, pp. 205-295.

[Q3]       D.G. Quillen, "On the (co-)homology of commutative rings", In *Applications of categorical algebra*, Proc. Symp. Pure Math. **17**, AMS, Providence, RI, 1970, pp. 65-87.

[ScV]      R. Schwänzl & R.M. Vogt, "The categories of $A_\infty$- and $E_\infty$-monoids and ring spaces as closed simplicial and topological model categories", *Arch. Math.*, **56** (1991), pp. 405-411.

[St]       A. Strøm, "The homotopy category is a homotopy category", *Arch. Math.* **23** (1972), pp. 435-441.

[Sw]       M.E. Sweedler, *Hopf Algebras*, W.A. Benjamin, New York, 1969.

[TZ]       G. Takeuti & W.M. Zaring, *Introduction to Axiomatic Set Theory*, Springer-Verlag *Grad. Texts in Math.* **1**, Berlin-New York, 1971.

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