Moduli Spaces of Homotopy Theory

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This article is dedicated to the memory of Bob Brooks, colleague and friend

Abstract. The moduli spaces refered to are topological spaces whose path components parametrize homotopy types. Such objects have been studied in two separate contexts: rational homotopy types, in the work of several authors in the late 1970's; and general homotopy types, in the work of Dwyer-Kan and their collaborators. We here explain the two approaches, and show how they may be related to each other.

1. Introduction

The concept of "moduli" for a mathematical object goes back to Riemann, who used it to describe a set of parameters determining the isomorphism class of a Riemann surface of a given topological type. He also recognized that the set of all conformal equivalence classes of such surfaces can itself be given a complex structure, although this was only made precise by Teichmüller, Ahlfors, Bers, and others in the twentieth century. Similarly, the family of birational equivalence classes of algebraic curves of each genus (over $\mathbb{C}$, say) are parametrized by an algebraic variety of moduli, as Mumford showed, and there are analogues in higher dimensions.

If we fix an oriented surface S of genus g, a marked Riemann surface is a choice of an orientation-perserving diffeomorphism of S into a compact Riemann surface X, and the equivalence classes of such marked surfaces form Teichmüller space $\mathcal{T}_g$, a complex analytic variety. The mapping class group Γ_g of orientation preserving diffeomorphisms of S acts on $\mathcal{T}_g$, and the quotient is isomorphic to the *moduli space* $\mathcal{M}_g$ of isomorphism classes of complex structures on S – or equivalently, of smooth projective curves over $\mathbb{C}$. Since the action of Γ_g is virtually free, and $\mathcal{T}_g$ is contractible, $\mathcal{M}_g$ has the same rational homology as the classifying space $B\Gamma_g$.

Note that the set of all Riemann surfaces of a given genus can be obtained by deforming a given conformal (or equivalently, complex) structure on a fixed sample

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surface Σ_g – or the corresponding Fuchsian group Γ. Such deformations are typically governed by an appropriate cohomology group.

Note also that there are several levels of possible structures on a compact surface: topological, differentiable, real or complex analytic, hyperbolic, conformal, metric, and so on, which define different types of moduli space. See [**31**] or [**51**] for more details and references on the classical theory.

1.1. Moduli spaces. Thus, there are a number of common themes in the moduli problems arising in various areas of mathematics:

(a) Finding a set of parameters to describe appropriate equivalence classes of objects in a certain geometric category. These may occur at varous levels, and we distinguish between discrete parameters (such as the genus of a surface) and the finer (continuous) *moduli*.
(b) Giving the set of such equivalence classes the structure of an object in the same category, which we may call a *moduli space*.
(c) The *deformation* of a given structure, controlled by cohomology, as a means of obtaining all other possible structures.
(d) To pass between levels of structure, one can often quotient by a group action.

As we shall see, all these themes will find expression in the context of homotopy theory. One aspect of the usual moduli spaces for which we have no analogue in our setting is the important role played by compactifications.

1.2. Moduli spaces in homotopy theory. The homotopy theory of the classical moduli spaces of Riemann surfaces and the corresponding mapping class groups has been studied extensively – see, for example, [**29, 35, 53**]. However, this is not our subject; we shall be concerned here rather with the analogues of such moduli spaces in the category of topological spaces, answering to the general description given above.

The simplest examples of this approach are provided by various mapping spaces, starting with the loop space ΩX of all pointed maps from the circle into a topological space X. Here the set $\pi_0\Omega X$ of components is isomorphic to the fundamental group of X, and the higher homotopy groups are again those of X (re-indexed). More significantly, the components of space of maps from X to $B\mathrm{O}$ or $B\mathrm{U}$, the classifying spaces of the infinite orthogonal or unitary groups, correspond to equivalence classes of (real or complex) vector bundles over X, and the higher homotopy groups correspond to various reductions of structure, by Bott periodicity. Further examples of simple "moduli spaces" of topological spaces are provided by the configuration spaces of n distinct points in a manifold M, which have also been studied extensively (see [**21**] for a comprehensive survey).

1.3. Moduli spaces and homotopy types. Our focus here will be more specifically on moduli spaces of all homotopy types (suitably interpreted), which have been investigated from at least two points of view over the past twenty-five years:

- In work of Lemaire-Sigrist, Félix and Halperin-Stasheff, simply-connected rational homotopy types correspond to the components of a certain (infinite dimensional) algebraic variety over $\mathbb{Q}$ – or alternatively, the quotient of such a variety by appropriate group actions (Sec. 2).

- The "classification complex" approach of Dwyer-Kan yields a space whose components parametrize the homotopy types of all CW complexes (Sec. 6).

Of course, neither statement has much content, as stated; the point is that both approaches offer (surprisingly similar) tools for inductively analyzing the collection of all relevant homotopy types:

In each case we begin with a coarse classification of homotopy types by algebraic invariants, such as the cohomology algebra H of a space, or the corresponding structure on homotopy groups, called a Π-algebra (which reduces rationally to a graded Lie algebra over $\mathbb{Q}$ – see Sec. 3). One might also refine this initial classification using both homotopy and cohomology groups (Sec. 5).

(a) For each H, we have homological classification of spaces with this cohomology algebra. This is usually presented as an obstruction theory, stated in terms of (algebraically defined) cohomology groups of H. It can be reinterpreted as expressing the moduli space as the limit of a tower of fibrations (Sec. 7), with successive fibers described in terms of these cohomology groups (Sec. 8).

(b) In the rational case, we have a deformation theory approach, which describes all homotopy types with $H^*(X,\mathbb{Q}) \cong H$ by perturbing a canonical model. In fact, this was the main focus of the rational homotopy theory approach mentioned above.

Our main goal here is to bring out the connection between the integral and rational cases. From the first point of view, this can be made explicit at the algebraic level, comparing the two cohomology theories through appropriate spectral sequences (Sec. 9).

On the other hand, the rational deformation theory has no integral analogue, since it uses differential graded models, which exist only for simply-connected rational homotopy types. However, it turns out that the deformations can be described geometrically in terms of higher homotopy operations (Sec. 4), and these do have integral versions.

1.4. Remark. One could – with some justice – argue that the analogy between the rational and integral "spaces of homotopy types" we describe here and classical moduli spaces is rather tenuous (although it is merely incidental to the point we want to make). One aspect in particular for which we can offer no real analogue is the fact that the classical moduli spaces have the same structure (algebraic or analytic variety, etc.) as the objects being classified.

In some sense, however, a homotopy type *is* just the collection of all its homotopy invariants, suitably interpreted (see [**41**, I,4.9]). In fact, one goal of homotopy theory is to produce a manageable set of such invariants, sufficient for distinguishing between type. A starting point for such a set of invariants is the Postnikov system of a space (and its invariants); so it is perhaps fitting that our analysis of the space of all homotopy types is carried out by means of its Postnikov system, and the associated collections of homotopy invariants (higher homotopy operations, or cohomology classes) which appear in our obstruction theory. Thus one could choose to view these collections of invariants themselves – rather than the space of homotopy types – as the true "moduli object". This is certainly the best way to understand the connections between the rational and integral cases, and it also might make (somewhat far-fetched) sense of the claim that the moduli object is itself of the same kind as the things it classifies. We leave the philosophically-inclined reader to pursue this line of thought at his or her discretion.

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2. Rational homotopy and deformation theory

We first describe the deformation-theoretic approach to rational homotopy types, starting with some basic background material:

2.1. Rational homotopy. Rational homotopy theory deals with rational algebraic invariants of homotopy types – that is, it disregards all torsion in the homology and homotopy groups. Quillen and Sullivan, respectively, proposed two main algebraic models for the rational homotopy type of a simply-connected space X:

(i) A differential graded-commutative algebra, or DGA (A^*_X, d), where $(A^i_X)_{i=0}^\infty$ is a graded-commutative algebra over $\mathbb{Q}$, equipped with a differential d of degree $+1$ (which is a graded derivation with respect to the product), and $H^*(A_X, d) \cong H^*(X, \mathbb{Q})$. Since X is simply-connected, we may assume (A^*_X, d), is, too – that is, $A^0_X = \mathbb{Q}$, $A^1_X = 0$.

(ii) A differential graded Lie algebra, or DGL (L^X, ∂), with $L^X = (L^X_i)_{i=1}^\infty$ a positively-graded Lie algebra over $\mathbb{Q}$, and ∂ a graded Lie derivation of degree -1, such that $H_i(L^X, \partial) \cong \pi_i \Omega X \otimes \mathbb{Q}$ for $i \geq 1$, with the Samelson products as Lie brackets.

For more information see the survey [**25**], and the original sources [**42, 50**].

2.2. Definition. A DGA of the form $(\Lambda V, d)$, where ΛV is the free graded-commutative algebra generated by the graded vector space $V = (V^i)_{i=0}^\infty$ is called *cofibrant*. A cofibrant DGA $(\Lambda V, d)$ such that $\mathrm{Im}\,(d) \subseteq \Lambda^+ V \cdot \Lambda^+ V$ (where $\Lambda^+ V$ is the sub-algebra of elements in positive degree), is called a *minimal model*.

A DGA or DGL map which induces an isomorphism in (co)homology is called a *quasi-isomorphism*.

2.3. Proposition (see [**1**]). *Any simply-connected DGA A has a minimal model with a quasi-isomorphism* $q : (\Lambda V, d) \to (A, d_A)$, *and this is unique up to isomorphism.*

One can also define cofibrant and minimal models for DGLs, and show that they have similar properties (cf. [**25**, §21]).

2.4. Remark. Both the category $\mathcal{DGA}_1$ of simply-connected DGAs, and the category $\mathcal{DGL}_0$ of connected DGLs have model category structures (see [**41**]); thus each has a concept of homotopy, and as Quillen showed, the corresponding homotopy categories $\mathrm{ho}\,\mathcal{DGA}_1$ and $\mathrm{ho}\,\mathcal{DGL}_0$ are both equivalent to the rational homotopy category $\mathrm{ho}\,\mathcal{T}_1^{\mathbb{Q}}$ of simply-connected topological spaces (cf. [**42**]). The weak equivalences in $\mathcal{T}_1^{\mathbb{Q}}$ correspond of course to quasi-isomorphisms for DGAs or DGLs.

Note that every simply-connected graded-commutative algebra over $\mathbb{Q}$ is realizable as the rational cohomology ring of some (1-connected) space, and similarly, every positively graded Lie algebra is realizable as the rational homotopy groups of a space.

2.5. The space of rational homotopy types. We may therefore identify the collection $\mathcal{M} = \mathcal{M}^{\mathcal{T}_1^{\mathbb{Q}}}$ of all rational weak homotopy types with the set of weak homotopy types of 1-connected DGAs (over $\mathbb{Q}$) – i.e., equivalence classes under the equivalence relation generated by the quasi-isomorphisms – and similarly for DGLs. Moreover, there is a coarse classification of such types, with the discrete parametrization provided by the cohomology rings; we denote the sub-collection of $\mathcal{M}$ consisting of all DGAs with a given graded cohomology ring H by $\mathcal{M}_H$. This has a distinguished element:

2.6. Definition. A DGA A is called *formal* if it is quasi-isomorphic to $(H^*A, 0)$ (the cohomology ring of A thought of as a DGA with zero differential). Dually, a DGL L is *coformal* if it is quasi-isomorphic to $(H_*L, 0)$.

$\mathcal{M}_H$ was studied by Jean-Michel Lemaire and François Sigrist in [**34**], by Yves Félix in [**23**], and independently by Steve Halperin and Jim Stasheff in [**27**]. They showed that every rational homotopy type in $\mathcal{M}_H$ can be obtained by suitable deformations of a fixed model of the unique formal object:

2.7. Definition. The *bigraded model* (B, d) for H is the minimal model $B = \Lambda Z$ for $(H, 0)$ of Proposition 2.3, where $Z = Z_*^*$, the graded vector space of generators, is equipped with an additional homological grading (indicated by the lower index).

The differential takes Z_n^i to $\Lambda(\oplus_{k=1}^{n-1} Z_k)^{i+1}$, so one still has a homological grading on the cohomology groups of B. The vector spaces Z_n^* are defined (almost canonically) by induction on n, so that at the n-th stage we kill the cohomology in homological dimension k for $0 < k < n$. $(\Lambda Z_*, d)$ is essentially the Tate-Jozefiak resolution of a graded commutative algebra H (cf. [**52, 32**]) – i.e., a minimal "cellular" resolution. See [**27**, §3] for the details.

The deformation consists of a perturbation of the differential d to $D = d + D'$ – of course, this is done in such a way that the cohomology is unchanged, and $D \circ D = 0$. We can no longer expect D to respect both gradings, but if we think of B as being only *filtered* by $\mathcal{F}_n B := \oplus_{k-0}^{n} (\Lambda Z^*)_k$, then the perturbation is such that D' takes Z_n into $\mathcal{F}_{n-2} B$.

2.8. Theorem (cf. [**27**, Theorem 4.4]). *If (A, d_A) is any DGA with $H^*(A, d_A) \cong H$, one can deform the bigraded model (B, d) for $(H, 0)$ into a filtered model (B, D) quasi-isomorphic to (A, d_A), and this is unique up to isomorphism.*

2.9. The variety of deformations. The deformations D of d are determined by successive choices of linear transformations $D'_n : Z_n \to \mathcal{F}_{n-2} \Lambda Z^*_{<n}$, satisfying a quadratic algebraic condition (to ensure that $D^2 = 0$). Thus the collection of all possible perturbations of the given bigraded model (B, d) constitute an (infinite dimensional) algebraic variety over $\mathbb{Q}$, which we denote by $\mathcal{V}_H$. Every simply-connected rational homotopy type is represented by at least one point of $\mathcal{V}_H$; each such point actually represents an "augmented" DGA X, equipped with a specific isomorphism $\phi : H^*(X; \mathbb{Q}) \to H$.

However, one can have more than one point corresponding to each augmented rational homotopy type: the points of $\mathcal{M}_H$ correspond bijectively with the orbits of V_H under the action of an appropriate group, as in [**34**, Thm. 4] or [**23**, Ch. 1] (note that these two superificially different descriptions are in fact the same, by [**24**]):

In the approach described in [**47**], we first divide by the action of a certain pro-unipotent algebraic group G_H to obtain the set V_H/G_H of augmented rational homotopy types as above. We must then further mod out by $\mathrm{Aut}(H)$ to get the plain (i.e., unaugmented) rational homotopy types, so that $\mathcal{M}_H \cong \mathrm{Aut}(H)\backslash V_H/G_H$. In [**47**, §6] it is then shown that the quotient set V_H/G_H can also be described as the set of path components of the Cartan-Chevalley-Eilenberg "standard construction" $C(L)$ (cf. §5.2) on a certain Lie algebra L (namely, the algebra of derivations of the bigraded model for H).

Note that though $\mathcal{M}_H$ can be uncountable, it can also be small: a DGA (A,d) is called *intrinsically formal* if $\mathcal{M}_{H^*A}$ consists of a single point.

2.10. Obstruction theory. We have seen that the collection $\mathcal{M}^{\mathcal{T}_1^{\mathbb{Q}}}$ of all simply-connected rational homotopy types indeed exhibits the basic characteristics of a moduli space listed in §1.1. However, to make this useful, we need further information about its overall structure.

This is provided by Halperin and Stasheff in the form of an obstruction theory for realizing a given isomorphism of rational cohomology algebras $\phi : H^*(A, d_A) \to H^*(B, d_B)$ by a DGA map (or equivalently, by a map of the corresponding rational spaces). Since such a map induces a homotopy equivalence between the respective cofibrant models, this can be thought of as a method for distinguishing between the homotopy types in $\mathcal{M}_H$.

The obstruction theory, which only works in full generality if the DGAs in question are of finite type, was presented in [**27**, §5] in terms of a sequence of elements $\mathcal{O}_n(f) \in \mathrm{Hom}(Z_{n+1}, H(B))$ – or rather, in a quotient of this group. In [**49**, §4], these are re-interpreted as elements in $\bigoplus_p H^{p+1}(B; \pi_p(H(A)))$.

2.11. Remark. By analogy with classical deformation theory, we would expect the deformations of the bigraded model to be "controlled" by a suitable differential graded Lie algebra (L,d), or perhaps by its cohomology. In fact, Félix observed that the Halperin-Stasheff obstructions $\mathcal{O}_n(f)$ can be thought of as lying in the F-cohomology group ${}_FH^1_n(B,d)$ of the bigraded model (B,d) for $(H,0)$ (see [**23**, Ch. 5, Prop. 7]). Moreover, these groups can be identified with the tangent (or Harrison) cohomology of the graded algebra H (cf. [**23**, Ch. 3, Prop. 4]), which is the same as the André-Quillen cohomology, and this can indeed be calculated as the cohomology of a suitable DGL (cf. [**46**, §4]).

Moreover, both Schlessinger and Stasheff (see [**49**, §5]) and Félix (cf. [**23**, Ch. 4-5]) obtain results connecting formality and intrinsic formality of a DGA with its tangent cohomology, though the relation does not work both ways (see [**23**, Ch. 4, Ex. 2]).

We shall give a uniform treatment of this cohomological description in a more general context in Section 8. However, for technical reasons (inter alia, convergence of second- versus first-quadrant spectral sequences), this is done for the Eckmann-Hilton dual – that is, in terms of Π-algebras (the integral version of graded Lie algebras over $\mathbb{Q}$). The mod p version – that is, an obstruction theory for realizing unstable (co)algebras over the Steenod algebra – appears in [**6**].

3. Moduli spaces of graded Lie algebras

The description of the moduli space of rational homotopy types in terms of DGLs, rather than DGAs, has received less attention. However, as noted above, it has certain technical advantages, especially in the integral case (where we need no longer restrict to simply-connected spaces, incidentally). The dualization to DGLs is quite straightforward, but we shall describe it here from a slightly different point of view.

3.1. Deformations of DGLs. The deformation theory for DGLs proceeds precisely along the lines indicated in §2.5*ff.*: given a positively-graded Lie algebra $P = P_*$ over $\mathbb{Q}$, consider the collection $\mathcal{M}_P$ of all rational homotopy types of simply-connected spaces X – or equivalently, of DGLs (L, ∂) – such that $\pi_*\Omega X \otimes \mathbb{Q} = H_*(L, \partial) \cong P$ (as Lie algebras).

Again there is a distinguished coformal element $(P, 0)$ in $\mathcal{M}_P$ (Def. 2.6). Its DGL minimal model (B, ∂) can be given an additional homological grading, and the differential in this bigraded model can be deformed so as to obtain a filtered model for any X in $\mathcal{M}_P$. See the sketch in [**28**]; the details appear in the unpublished thesis [**40**].

3.2. The resolution model category. Since the category $\mathcal{D}GL_0$ is not abelian, from the point of view of homotopical algebra it is more natural to consider simplicial resolutions of objects in this category, rather than "chain complex" resolutions (which is what the homological grading of the bigraded model gives us). Moreover, this generalizes more naturally to the integral setting, where differential graded models are no longer available (although see [**36**] for a way around this).

Recall that a *simplicial object* $X_\bullet$ over a category $\mathcal{C}$ is a functor $X : \Delta^{op} \to \mathcal{C}$, where Δ is the category of finite sequences $\mathbf{n} = (0, 1, \ldots, n)$ $(n \in \mathbb{N})$, with order-preserving maps as morphisms. The category of simplicial objects over $\mathcal{C}$ will be denoted by $s\mathcal{C}$.

Of course, $\mathcal{C} = \mathcal{D}GL_0$ (as well as $\mathcal{D}GA_1$) is itself a model category (as shown by Quillen in [**42**]), so in considering simplicial DGLs we can take advantage of the procedure described by Dwyer, Kan, and Stover for generating a model category structure on $s\mathcal{C}$. Essentially, this consists of:

- choosing a certain set $\{M_\alpha\}_{\alpha \in A}$ of "good" objects in $\mathcal{C}$ (corresponding to the projectives of homological algebra);
- creating simplicial objects $\Sigma^n M_\alpha$ for each $n \in \mathbb{N}$ and $\alpha \in A$ by placing M_α in simplicial dimension n; and using these to
- define weak equivalences (which are detected by maps out of $\Sigma^n M_\alpha$), and
- construct the cofibrant objects (corresponding to projective resolutions), and more generally all cofibrations.

See [**19**] for more details; as extended by Bousfield (who calls it a "resolution model category") in [**14**], this procedure works in great generality. In particular, it includes the model categories for universal algebras defined by Quillen in [**41**, II, §4], as well as the graded version (see, e.g., [**12**, §2.2]). The original example considered by Dwyer, Kan, and Stover was that of simplicial spaces; this can be readily modified to cover simplicial rational spaces (equivalently, simplicial DGLs or DGAs).

3.3. Simplicial DGLs. In applying the above procedure to $\mathcal{C} = \mathcal{D}GL_0$, we have a natural candidate for the "good objects" – namely, the DGL spheres $\{\mathcal{S}^n\}_{n=2}^{\infty}$ (which are just minimal models for $\{S^n_{\mathbb{Q}}\}_{n=2}^{\infty}$ – i.e., $\mathcal{S}^n = \mathcal{S}^n_{(x)}$ is a free graded Lie algebra on a single generator x in dimension n).

This determines what we mean by a (simplicial) *resolution* of a given DGL (L, ∂) – namely, a cofibrant simplicial DGL V, with each V_n homotopy equivalent to a coproduct $\coprod_{i \in I_n} \mathcal{S}^{n_i}$, such that the augmented simplicial DGL $V \to L$ is weakly contractible. A map $\varphi : V \to X$ of simplicial DGLs is a weak equivalence if and only if after taking (internal) homology in each simplicial dimension, the resulting map of graded simplicial abelian groups $\varphi_* : H_*V \to H_*X$ induces an isomorphism $H_n(\varphi_*) : H_n(H_*(V)) \xrightarrow{\cong} H_n(H_*(X))$ for each $n \geq 0$.

By [**42**, Thm. 2.1], we know that the model category $\mathcal{D}GL_0$ is equivalent to the category $s\mathcal{L}ie_1$ of connected simplicial Lie algebras; similarly for bigraded differential Lie algebras and simplicial graded Lie algebras by [**7**, Prop. 2.9]. In fact, it turns out that there is also a one-to-one correspondence between filtered DGLs and (cofibrant) simplicial DGL resolutions (see [**7**, Prop. 5.13]), so the DGL deformation theory mentioned in §3.1 can be translated into the language of simplicial resolutions.

3.4. Moduli and realizations. One can analyze $\mathcal{M}_P$ in terms of a *realization problem*: which distinct rational homotopy types of 1-connected topological spaces have the given graded Lie algebra P as their rational homotopy groups? In view of the equivalences of categories noted above, we can replace the DGL bigraded model (B, ∂) for $(P, 0)$ by a free simplicial resolution $G \to P$ of graded Lie algebras, and observe that if P itself can be realized by a rational space X (and so by a DGL (L^X, ∂)), then G can be realized by a *simplicial* space (or DGL) V. Moreover, any weak equivalence $\varphi : V \to W$ of simplicial spaces (or DGLs) induces a weak equivalence $\varphi_\# : \pi_* V \otimes \mathbb{Q} \to \pi_* W \otimes \mathbb{Q}$, and since $\pi_* V \otimes \mathbb{Q} = G$ is a resolution of P, this is just a choice of an isomorphism between two isomorphic copies of P.

As we shall see below (§6), there is a "moduli space" associated to any model category, and the various levels of classification which are a feature of moduli spaces in general (§1.1(i)) can be understood in terms of "structure-reducing" functors between the corresponding model categories – here exemplified by $\pi_*(\ ) \otimes \mathbb{Q} : \mathcal{T}_1 \to \operatorname{gr} \mathcal{L}ie$ (extended to simplicial objects).

4. Moduli and higher homotopy operations

Ideally, we want not only an algebraic description of the components of $\mathcal{M}_H$ in terms of the cohomology of the graded algebra H, say, as in §2.10, but also a geometric interpretation of this description – with the potential for lifting to the integral case.

Note that there is an obvious candidate for geometric invariants to distinguish between inequivalent rational spaces with the same cohomology – namely, Massey products, and their higher order analogues (see [**38, 33**]). In fact, Halperin and

Stasheff's original motivation for their obstruction theory was to make precise a folk theorem stating that a space is formal if and only if all higher order Massey products – or more generally, all higher-order cohomology operations – vanish.

One difficulty is in the very definition of higher cohomology operations (see, e.g., [**37**]). This can be bypassed to some extent by re-interpreting them in terms of differentials in a spectral sequence (cf. [**27**, §7]) – at the price of losing both their geometric content, and the ability to identify the same higher operation in different contexts.

Because their Eckmann-Hilton dual – namely, higher homotopy operations – is somewhat more intuitive than the cohomology version, we shall concentrate on these. A precise definition requires some care; in the following sections, we present the version of [**10**]:

4.1. Definition. A *lattice* is a finite directed non-unital category Γ (that is, we omit the identity maps), equipped with two objects $v_{\text{init}} = v_{\text{init}}(\Gamma)$ and $v_{\text{fin}} = v_{\text{fin}}(\Gamma)$ with a unique map $\phi_{\max} : v_{\text{init}} \to v_{\text{fin}}$, and for every $w \in V := \text{Obj}\,\Gamma$, there is at least one map from v_{init} to w, and at least one from w to v_{fin}. A composable sequence of k arrows in Γ will be called a *k-chain*.

4.2. The W-construction. Given a lattice Γ, one can define a new category $W\Gamma$ enriched over cubical sets, with the same set of objects V (cf. [**13**, III, §1]):

For $u, v \in V$, let $\Gamma_{n+1}(u,v)$ be the set of $(n+1)$-chains from u to v in Γ, and

$$W\Gamma(u,v) := \coprod_{n\geq 0} \Gamma_{n+1}(u,v) \times I^n / \sim,$$

where I is the unit interval. Write

$$f_1 \circ_{t_1} f_2 \cdots f_n \circ_{t_n} f_{n+1}$$

for

$$\langle u \xrightarrow{f_{n+1}} v_n \cdots \to v_1 \xrightarrow{f_1} v \rangle \times (t_n, \ldots, t_1)$$

in $\Gamma_{n+1}(u,v) \times I^n$; then the relation $\sim$ is generated by

$$f_1 \circ_{t_1} f_2 \cdots f_n \circ_{t_n} f_{n+1} \ \sim \ f_1 \circ_{t_1} \cdots \circ_{t_{i-1}} (f_i f_{i+1}) \circ_{t_{i+1}} \cdots \circ_{t_n} \circ f_{n+1},$$

if $t_i = 0$ for $1 \leq i \leq n$, where $(f_i f_{i+1})$ denotes f_i composed with f_{i+1}.

The categorial composition in $W\Gamma$ is given by the concatenation:

$$(f_1 \circ_{t_1} \cdots \circ_{t_l} f_{l+1}) \circ (g_1 \circ_{u_1} \cdots \circ_{u_k} g_{k+1}) := (f_1 \circ_{t_1} \cdots \circ_{t_l} f_{l+1} \circ_1 g_1 \circ_{u_1} \cdots \circ_{u_k} g_{k+1}).$$

We write $\mathcal{P} := W\Gamma(v_{\text{init}}, vf)$.

4.3. Definition. The *basis* category $bW\Gamma$ for a lattice Γ is defined to be the cubical subcategory of $W\Gamma$ with the same objects, and with morphisms given by $bW\Gamma(u,v) := W\Gamma(u,v)$ if $(u,v) \neq (v_{\text{init}}, v_{\text{fin}})$, while

$$bW\Gamma(v_{\text{init}}, v_{\text{fin}}) := \bigcup \{\alpha \circ \beta \mid \beta \in W\Gamma(v_{\text{init}}, w), \alpha \in W\Gamma(w, v_{\text{fin}}), v_{\text{init}} \neq w \neq v_{\text{fin}}\},$$

so that $b\mathcal{P} := bW\Gamma(v_{\text{init}}, v_{\text{fin}})$ consists of all decomposable morphisms.

4.4. Fact ([**10**, Proposition 2.15]). *For any lattice Γ, $W\Gamma(v_{\text{init}}, v_{\text{fin}})$ is isomorphic to the cone on its basis, with vertex corresponding to the unique maximal 1-chain.*

We can use the W-construction to define a higher homotopy operations, as follows:

4.5. Definition. Given $\mathcal{A} : \Gamma \to \operatorname{ho}\mathcal{T}_*$ for a lattice Γ as above, the corresponding *higher order homotopy operation* is the subset $\langle\langle\mathcal{A}\rangle\rangle \subset [\mathcal{A}(v_{\text{init}}) \rtimes b\mathcal{P}, \mathcal{A}(v_{\text{fin}})]_{\operatorname{ho}\mathcal{T}_*}$ of the homotopy equivalence classes of maps

$$C\mathcal{A}|_{bW\Gamma(v_{\text{init}},v_{\text{fin}})} : bW\Gamma(v_{\text{init}}, v_{\text{fin}}) = b\mathcal{P} \longrightarrow \mathcal{T}_*(\mathcal{A}(v_{\text{init}}), \mathcal{A}(v_{\text{fin}}))$$

induced by all possible continuous functors $C\mathcal{A} : bW\Gamma \to \mathcal{T}_*$ such that $\pi \circ C\mathcal{A} = \mathcal{A} \circ (\varepsilon|_{bW\Gamma})$, where the half-smash is defined $X \rtimes K := (X \times K)/(\{*\} \times K) = X \wedge K_+$.

$\langle\langle\mathcal{A}\rangle\rangle$ is said to *vanish* if it contains the homotopy class of a constant map $b\mathcal{P} \longrightarrow \mathcal{T}_*(\mathcal{A}(v_{\text{init}}), \mathcal{A}(v_{\text{fin}}))$.

Given a lattice Γ, a *rectification* of a functor $\mathcal{A} : \Gamma \to \operatorname{ho}\mathcal{T}_*$ is a functor $F : \Gamma \to \mathcal{T}_*$ with $\pi \circ F$ naturally isomorphic to $\mathcal{A}$ (where $\pi : \mathcal{T}_* \to \operatorname{ho}\mathcal{T}_*$ is the obvious projection functor).

4.6. Proposition ([**10**, Thm. 3.8]). *$\mathcal{A}$ has a rectification if and only if the homotopy operation $\langle\langle\mathcal{A}\rangle\rangle$ vanishes (and in particular, is defined)*

With this at hand, we can now reformulate the deformation theory for the bigraded DGL model for $(P, 0)$ in the following form:

4.7. Theorem ([**7**, Thm. 7.14]). *For each connected graded Lie algebra P of finite type, there is a tree T_P, with each node α indexed by a cofibrant DGL L^α such that $H_*L^\alpha \cong P$, starting with $L^0 \simeq (P, 0)$ at the root 0. For each node α there is an integer $n_\alpha > 0$ such that L^α agrees with its successors in degrees $\leq n_\alpha$, with $n_0 = 1$, so that the sequential colimit $L^{(\infty)} = \operatorname{colim}_k L^{(k)}$ along any branch is well defined, and for any rational homotopy type in $\mathcal{M}_P$ there exists such a tree. Furthermore, for each two immediate successors β, γ of a node α, there is a higher homotopy operation $\langle\langle\beta, \gamma\rangle\rangle \subseteq [\mathbb{S}^n, L]$ which vanishes for $L = L^\beta$, but not for $L = L^\gamma$ (or conversely).*

Thus we have reformulated the deformation of the bigraded model as an obstruction theory for distinguishing between the various realizations of a given rational graded Lie algebra P in terms of higher homotopy operations. This can also be done integrally (cf. [**4**]), and one could identify these operations with appropriate cohomology classes in the dual version of the Halperin-Stasheff obstructions (see [**5**]).

4.8. Example of a DGL deformation. We shall not explain how the higher operations $\langle\langle\beta, \gamma\rangle\rangle$ are defined, since the construction is rather technical; see [**7**, §4] for the details. Instead, the reader might find the following example instructive:

Consider the graded Lie algebra $P = \mathbb{L}\langle a_1, b_1, c_2\rangle/I$, where $\mathbb{L}\langle T_*\rangle$ is the free graded Lie algebra on a graded set of generators T_*, (with the subscripts indicating the dimension), and I is here the Lie ideal generated by $[a, a]$ and $[[c, a], [b, a]]$.

Step I: The minimal model for the coformal DGL $(P, 0) \in \mathcal{D}GL_0$ is (B, ∂_B), where B in dimensions ≤ 7 is $\mathbb{L}\langle a_1, b_1, c_2, x_3, y_5, w_6, z_7\rangle$, with

$$\begin{aligned} \partial_B(x) &= [a, a], \\ \partial_B(y) &= 3[x, a], \\ \partial_B(w) &= [[c, a], [b, a]], \\ \partial_B(z) &= 4[y, a] + 3[x, x]. \end{aligned}$$

The bigraded model A_{**} is obtained from B by introducing an additional (homological) grading: $a, b \in A_{0,1}$, $c \in A_{0,2}$, $x \in A_{1,3}$, $w \in A_{1,6}$, $y \in A_{2,5}$, $z \in A_{3,7}$, and so on.

Step II: The free simplicial DGL resolution $C_\bullet \to L = (\mathcal{L}, 0)$ may be described in homological dimensions ≤ 3 as follows:

(1) $C_0^{(0)}$ is the DGL coproduct of

$$\begin{aligned} \mathcal{S}^1_{(a^{(0)})} &= \mathcal{S}^1_{(\langle a\rangle)}, \\ \mathcal{S}^1_{(b^{(0)})} &= \mathcal{S}^1_{(\langle b\rangle)}, \\ \mathcal{S}^2_{(c^{(0)})} &= \mathcal{S}^2_{(\langle c\rangle)}. \end{aligned}$$

(2) $C_1^{(0)} = \mathcal{S}^2_{(x^{(0)})} \amalg \mathcal{S}^6_{(w^{(0)})}$, where

$$\begin{aligned} x^{(0)} &= \langle[\langle a\rangle, \langle a\rangle]\rangle, \\ w^{(0)} &= \langle[[\langle c\rangle, \langle a\rangle], [\langle b\rangle, \langle a\rangle]]\rangle. \end{aligned}$$

(3) $C_2^{(0)}$ consists of $\mathcal{S}^5_{(y^{(0)})}$, where

$$y^{(0)} = \langle 3[\langle[\langle a\rangle, \langle a\rangle]\rangle, \langle\langle a\rangle\rangle]\rangle.$$

(4) $C_3^{(0)}$ consists of $\mathcal{S}^6_{(z^{(0)})}$, where

$$z^{(0)} = \langle 4[\langle 3[\langle[\langle a\rangle, \langle a\rangle]\rangle, \langle\langle a\rangle\rangle]\rangle, \langle\langle\langle a\rangle\rangle\rangle] + 6[\langle[\langle\langle a\rangle\rangle, \langle\langle a\rangle\rangle]\rangle, \langle\langle[\langle a\rangle, \langle a\rangle]\rangle\rangle]\rangle.$$

In analogy with the DGL sphere of §3.3, the DGL n-disk $\mathcal{D}^n_{(x)}$ is the free graded Lie algebra on two generators: x in dimension n and $\partial(x)$ in dimension $n-1$. With this notation, for $C_\bullet^{(1)}$ we need in addition:

(1) $\mathcal{D}^3_{(x^{(1)})} \hookrightarrow C_0^{(1)}$ with

$$\partial_W(x^{(1)}) = d_1(x^{(0)}) = \langle[a, a]\rangle.$$

(2) $\mathcal{D}^6_{(y^{(1)})} \hookrightarrow C_1^{(1)}$ with

$$\begin{aligned} y^{(1)} &= \langle 3[\langle x\rangle, \langle a\rangle]\rangle \\ \partial_W(y^{(1)}) = d_2(y^{(0)}) &= \langle 3[\langle[a, a]\rangle, \langle a\rangle]\rangle. \end{aligned}$$

(3) $\mathcal{D}^7_{(z^{(1)})} \hookrightarrow C_2^{(1)}$ with

$$\begin{aligned} z^{(1)} &= \langle 4[\langle 3[\langle x\rangle, \langle a\rangle]\rangle, \langle\langle a\rangle\rangle] + 6[\langle[\langle a\rangle, \langle a\rangle]\rangle, \langle\langle x\rangle\rangle]\rangle \\ \partial_W(z^{(1)}) = d_3(z^{(0)}) &= \langle 4[\langle 3[\langle[a, a]\rangle, \langle a\rangle]\rangle, \langle\langle a\rangle\rangle] + 6[\langle[\langle a\rangle, \langle a\rangle]\rangle, \langle\langle[a, a]\rangle\rangle]\rangle. \end{aligned}$$

For $C_\bullet^{(2)}$ we need in addition

(1) $\mathcal{D}^{7}_{(y^{(2)})} \hookrightarrow C^{(2)}_{0}$ with

$$\partial_W(y^{(2)}) = d_1(y^{(1)}) = \langle 3[x,a]\rangle.$$

(2) $\mathcal{D}^{8}_{(z^{(2)})} \hookrightarrow C^{(2)}_{1}$ with

$$\begin{aligned} z^{(2)} &= \langle 4[\langle y\rangle,\langle a\rangle] + 3[\langle x\rangle,\langle x\rangle]\rangle \\ \partial_W(z^{(2)}) = d_2(z^{(1)}) &= \langle 4[\langle 3[x,a]\rangle,\langle a\rangle] + 6[\langle [a,a]\rangle,\langle x\rangle]\rangle. \end{aligned}$$

For $C^{(3)}_{\bullet}$ we must add $\mathcal{D}^{9}_{(z^{(3)})} \hookrightarrow C^{(3)}_{0}$ with

$$\partial_W(z^{(1)}) = d_1(z^{(2)}) = \langle 4[y,a] + 3[x,x]\rangle.$$

Step III: So far, we have constructed the DGL simplicial resolution of the coformal DGL $(B,\partial_B) \simeq (P,0)$. Now we consider a different DGL having the same homology (i.e., a non-weakly equivalent topological space with the same rational homotopy groups):

Consider the DGL $(B',\partial_{B'})$ where $B' := \mathbb{L}\langle a_1, b_1, c_2, x_3, y_5, z_7, \ldots\rangle$, is a free Lie algebra with $\partial_{B'}(x) = [a,a]$, $\partial_{B'}(y) = 3[x,a] - [[b,a],c]$, $\partial_{B'}(z) = 4[y,a] + 3[x,x]$, and so on.

Here $P \cong H_*(B) = \mathbb{L}\langle a_1, b_1, c_2\rangle/\langle [a,a], [[c,a],[b,a]]\rangle$, so the bigraded model for $(P,0)$ is (A_{**},∂_B) of Step II above, and the filtered model is obtained from it by setting $D(y) = 3[x,a] - [[b,a],c]$ and $D(z) = 4[y,a] + 3[x,x] - 4w + 2[[x,b],c]$.

The corresponding free simplicial DGL resolution is obtained from $C_{\bullet}$ of Step II by making the following changes:

(1) Set

$$y^{(1)} := \langle 3[\langle x\rangle,\langle a\rangle] - [[\langle b\rangle,\langle a\rangle],\langle c\rangle]\rangle,$$

with

$$\partial_W(y^{(1)}) = d_2(y^{(0)}) = \langle 3[\langle [a,a]\rangle,\langle a\rangle]\rangle$$

as before (but now $\partial_W(y^{(2)}) = \langle 3[x,a] - [[b,a],c]\rangle$, of course).

(2) Set

$$\begin{aligned} z^{(1)} := \ &\langle 4[\langle 3[\langle x\rangle,\langle a\rangle]\rangle,\langle\langle a\rangle\rangle] + 6[\langle [\langle a\rangle,\langle a\rangle]\rangle,\langle\langle x\rangle\rangle] \\ &- 4\langle [[\langle c\rangle,\langle a\rangle],[\langle b\rangle,\langle a\rangle]]\rangle + 2[[\langle [\langle a\rangle,\langle a\rangle]\rangle,\langle\langle b\rangle\rangle],\langle\langle c\rangle\rangle]\rangle \end{aligned}$$

(with $\partial_W(z^{(1)})$ unchanged).

(3) Set

$$z^{(2)} := \langle 4[\langle y\rangle,\langle a\rangle] + 6[\langle x\rangle,\langle x\rangle] - 4\langle w\rangle + 2[[\langle x\rangle,\langle b\rangle],\langle c\rangle]\rangle,$$

with

$$\begin{aligned} \partial_W(z^{(2)}) = d_3(z^{(1)}) = \ &\langle 4[\langle 3[x,a] - [[b,a],c]\rangle,\langle a\rangle] \\ &+ 6[\langle [a,a]\rangle,\langle x\rangle] - 4\langle [[c,a],[b,a]]\rangle + 2[[\langle [a,a]\rangle,\langle b\rangle],\langle c\rangle]\rangle \end{aligned}$$

(4) Finally,

$$\partial_W(z^{(3)}) = \langle 4[y,a] + 3[x,x] - 4w + 2[[x,b],c]\rangle.$$

Step IV: In order to define the higher homotopy operations which distinguish B' from B, observe that the required half-smashed are defined as follows:

$$\mathcal{S}^2_{(x)} \rtimes D[1] = (\mathbb{L}\langle X_* \rangle, \partial'),$$

where

$$\begin{aligned} X_2 &= \{(x,(d_0)),(x,(d_1))\} \\ X_3 &= \{(x,(\mathrm{Id}))\}, \end{aligned}$$

and

$$\partial'(x,(\mathrm{Id})) = (x,(d_0)) - (x,(d_0)).$$

Similarly,

$$\mathcal{S}^3_{(y)} \rtimes D[2] = (\mathbb{L}\langle Y_* \rangle, \partial'),$$

where

$$\begin{aligned} Y_3 &= \{(y,(d_0d_1)),(y,(d_0d_2)),(y,(d_1d_2))\}, \\ Y_4 &= \{(y,(d_0)),(y,(d_1)),(y,(d_2))\}, \\ Y_5 &= \{(y,(\mathrm{Id}))\}, \end{aligned}$$

with

$$\begin{aligned} \partial'(y,(\mathrm{Id})) &= -(y,(d_0)) + (y,(d_1)) - (y,(d_2)), \\ \partial'(y,(d_0)) &= -(y,(d_0d_1)) + (y,(d_0d_2)), \\ \partial'(y,(d_1)) &= -(y,(d_0d_1)) + (y,(d_1d_2)), \\ \partial'(y,(d_2)) &= -(y,(d_0d_2)) + (y,(d_1d_2)). \end{aligned}$$

Step V: For the DGL B' of Step III, with $C_\bullet$ as in Step II, we define $h_0 = f_0 : C_{0,*} \to B'$ by setting $f_0(\langle a \rangle) = a$, $f_0(\langle b \rangle) = b$, $f_0(\langle c \rangle) = c$, and $f_0 = 0$ for all other disks in $C_{0,*}$ (so for example $f_0(\langle [a,a] \rangle) = 0$).

Thus on $\mathcal{S}^2_{(x^{(0)})} \rtimes D[1]$ we have

$$\begin{aligned} h_1(x^{(0)},(d_0)) &= [a,a], \\ h_1(x^{(0)},(d_1)) &= 0, \end{aligned}$$

(see [**7**, Def. 4.7]), so we must choose $h_1(x^{(0)},(\mathrm{Id})) = x \in B'_3$.

Now on $\mathcal{S}^3_{(y^{(0)})} \rtimes D[2]$ we have

$$\begin{aligned} h_2(y^{(0)},(d_0d_1)) = h_2(y^{(0)},(d_0d_2)) = h_2(y^{(0)},(d_1d_2)) &= 0, \\ h_2(y^{(0)},(d_0)) = (h_1 \circ d_0)(y^{(0)},(d_0)) &= 3[x,a], \\ h_2(y^{(0)},(d_1)) = h_2(y^{(0)},(d_2)) &= 0. \end{aligned}$$

Thus in Definition 4.5 we shall be interested in a lattice Γ corresponding to the face maps of a simplicial object, with $bW\Gamma \cong \partial D[2]$. We have just defined a continuous functor $C\mathcal{A} : bW\Gamma \to \mathcal{T}_*$ for $C_\bullet$, as required there, and the resulting secondary operation is $\langle\langle 2, y^{(0)} \rangle\rangle = \{\langle 3[x,a] \rangle]\} \subseteq H_4B'$ (remember that the homology of a DGL corresponds to the rational homotopy groups of the associated space). Since $3[x,a]$ does not bound in B', $\langle\langle 2 \rangle\rangle$ does not vanish, and we have found an obstruction to the coformality of B'.

Unfortunately, DGL higher homotopy operations such as $\langle\langle 2, y^{(0)} \rangle\rangle$ are unsatisfactory, in as much as one cannot translate them canonically into integral homotopy operations. One way to avoid this difficulty is to use more general non-associative algebras, rather than Lie algebras, as our basic models. Thus, consider the category $\mathcal{DGN}$ of non-associative differential graded algebras. A DGN whose homology happens to be a graded Lie algebra will be called a *Jacobi* algebra. Every

DGL has a free Jacobi model, and these can be used to resolve any DGL in $s\mathcal{DGN}$ (see [**7**, §7]).

Step VI: Consider a Jacobi model G for the coformal DGL P of Step I. We construct the minimal Jacobi resolution $J_\bullet \to G$ corresponding to the bigraded model for L (embedded in the canonical Jacobi resolution $U_\bullet \to G$) by modifying the free resolution $C_\bullet \to L$ of Step II, as follows:

For each $n \geq 0$ define the Jacobi algebras $J_n^{(0)}$ $(n = 0, 1, \dots)$ to be the coproducts

$$\begin{aligned} J_0^{(0)} &= \mathcal{S}^1_{(\langle a\rangle)} \amalg \mathcal{S}^1_{(\langle b\rangle)} \amalg \mathcal{S}^2_{(\langle c\rangle)} \\ J_1^{(0)} &= \mathcal{S}^2_{(x^{(0)})} \amalg \mathcal{S}^6_{(w^{(0)})} \\ J_2^{(0)} &= \mathcal{S}^5_{(y^{(0)})} \\ J_3^{(0)} &= \mathcal{S}^6_{(z^{(0)})}, \end{aligned}$$

and so on.

The face maps d_i $(i = 0, 1, \dots, n)$ are defined as follows: write $\langle x\rangle \in F(B)$ for the generator corresponding to an element $x \in B'_*$, then recursively a typical DGL generator for $W_n = W_{n,*}$ (in the canonical Stover resolution $W_\bullet(B)$, defined in [**7**, §5.3]) is $\langle\alpha\rangle$, for $\alpha \in W_{n-1}$, so an element of W_n is a sum of iterated Lie products of elements of B'_*, arranged within $n+1$ nested pairs of brackets $\langle\langle\cdots\rangle\rangle$. With this notation, the i-th face map of $W_\bullet$ is "omit i-th pair of brackets", and the j-th degeneracy map is "repeat j-th pair of brackets". We assume the bracket operation $\langle -\rangle$ is linear – i.e., that $\langle \alpha x + \beta y\rangle = \alpha\langle x\rangle + \beta\langle y\rangle$ for $\alpha, \beta \in \mathbb{Q}$ and $x, y \in B$.

1. For $y^{(0)} = \langle 3[\langle[\langle a\rangle, \langle a\rangle]\rangle, \langle\langle a\rangle\rangle]\rangle \in J_2^{(0)}3$, we have

$$\begin{aligned} &d_0(y^{(0)}) = 3[\langle[\langle a\rangle, \langle a\rangle]\rangle, \langle\langle a\rangle\rangle] = 3[x^{(0)}, a^{(0)}], \\ &d_2(y^{(0)}) = \langle 3[\langle [a,a]\rangle, \langle a\rangle]\rangle, \\ &d_1(y^{(0)}) = \langle 3[[\langle a\rangle, \langle a\rangle], \langle a\rangle]\rangle. \end{aligned}$$

This no longer vanishes as in Step II, since the Jacobi identity does not hold in $\mathcal{DGN}$, but we have an element $y^{(1;1)} := \langle\lambda_3(\langle a\rangle \otimes \langle a\rangle \otimes \langle a\rangle)\rangle \in J_1^{(1)}4$ (in the notation of [**7**, Example 7.8], with $\partial(y^{(1;1)}) = d_1(y^{(0)})$.

On the other hand, we also have an element $y^{(1;2)} := \langle 3[\langle x\rangle, \langle a\rangle]\rangle \in J_1^{(1)}4$ (which we denoted simply by $y^{(1)}$ in Step II, with $\partial(y^{(1;2)}) = d_2(y^{(0)})$. The simplicial identity $d_1 d_2 = d_1 d_1$ implies that

$$d_1(y^{(1;2)}) - d_1(y^{(1;1)}) = \langle 3[x, a] - \lambda_3(a \otimes a \otimes a)\rangle$$

is a ∂_J-cycle, so we have $y^{(2;1,2)} \in J_0^{(2)}5$ with

$$\partial_J(y^{(2;1,2)}) = \langle 3[x, a] - \lambda_3(a \otimes a \otimes a)\rangle.$$

2. For $z^{(0)} = \langle 4[\langle 3[\langle[\langle a\rangle, \langle a\rangle]\rangle, \langle\langle a\rangle\rangle]\rangle, \langle\langle\langle a\rangle\rangle\rangle] + 6[\langle[\langle\langle a\rangle\rangle, \langle\langle a\rangle\rangle]\rangle, \langle\langle[\langle a\rangle, \langle a\rangle]\rangle\rangle]\rangle$ in $J_3^{(0)}4$:

(a) $z^{(1;1)} := \langle \lambda_3(x^{(0)} \otimes \langle\langle a\rangle\rangle \otimes \langle\langle a\rangle\rangle)\rangle$, with

$$d_1(z^{(0)}) = \langle 6(2[[x^{(0)}, \langle\langle a\rangle\rangle], \langle\langle a\rangle\rangle], +[[\langle\langle a\rangle\rangle, \langle\langle a\rangle\rangle], x^{(0)}])\rangle = \partial_J(z^{(1;1)}),$$

(b) $z^{(1;2)} := \langle 4[\lambda_3(\langle a\rangle \otimes \langle a\rangle \otimes \langle a\rangle), \langle\langle a\rangle\rangle]\rangle$, with

$$d_2(z^{(0)}) = \langle 4[[[a^{(0)}, a^{(0)}], a^{(0)}], \langle\langle a\rangle\rangle]\rangle = \partial_J(z^{(1;2)}) \quad \text{since} \quad [x^{(0)}, x^{(0)}] = 0).$$

(c) $z^{(1;3)} = \langle 4[\langle 3[\langle [a,a]\rangle, \langle a\rangle]\rangle, \langle\langle a\rangle\rangle] + 6[\langle[\langle a\rangle, \langle a\rangle]\rangle, \langle\langle [a,a]\rangle\rangle]\rangle \in J_2^{(1)}5$, with

$$d_3(z^{(0)}) = \partial_J(z^{(1;3)}).$$

3. Next, in $J_\bullet^{(2)}$:

(a) The simplicial identity $d_1 d_1 = d_1 d_2$ implies that

$$\begin{aligned} d_1(z^{(1;1)}) - d_1(z^{(1;2)}) = \langle 4[&\lambda_3(\langle a\rangle \otimes \langle a\rangle \otimes \langle a\rangle), \langle a\rangle] \\ &- 6\lambda_3([\langle a\rangle, \langle a\rangle] \otimes \langle a\rangle \otimes \langle a\rangle)\rangle \end{aligned}$$

is a ∂_J-cycle – and indeed we have

$$z^{(2;1,2)} := \langle \lambda_4(\langle a\rangle \otimes \langle a\rangle \otimes \langle a\rangle \otimes \langle a\rangle)\rangle \in J_2^{(2)}6$$

with

$$\partial(z^{(2;1,2)}) = d_1(z^{(1;1)}) - d_1(z^{(1;2)}).$$

(b) Similarly,

$$\begin{aligned} d_1(z^{(1;3)}) - d_2(z^{(1;1)}) = \langle &12[[\langle x\rangle, \langle a\rangle], \langle a\rangle] + 6[[\langle a\rangle, \langle a\rangle], \langle x\rangle] \\ &- 6\lambda_3(\langle [a,a]\rangle \otimes \langle a\rangle \otimes \langle a\rangle)\rangle \end{aligned}$$

is a ∂_J-cycle, so we have

$$z^{(2;1,3)} := \langle 6\lambda_3(\langle x\rangle \otimes \langle a\rangle \otimes \langle a\rangle)\rangle \in J_2^{(2)}6$$

with

$$\partial(z^{(2;1,3)}) = d_1(z^{(1;3)}) - d_2(z^{(1;1)}).$$

(c)
$$\begin{aligned} d_2(z^{(1;3)}) - d_2(z^{(1;2)}) = \ &\langle 4[\langle 3[x,a]\rangle, \langle a\rangle] + 6[\langle [a,a]\rangle, \langle x\rangle] \\ &- 4[\langle \lambda_3(a \otimes a \otimes a)\rangle, \langle a\rangle]\rangle \end{aligned}$$

is a ∂_J-cycle, hit by

$$z^{(2;2,3)} := \langle 4[\langle y\rangle, \langle a\rangle] + 3[\langle x\rangle, \langle x\rangle]\rangle \in J_2^{(2)}6.$$

4. Finally, we have

(a) For $d_1 z^{(2;1,2)} = \langle \lambda_4(a \otimes a \otimes a \otimes a)\rangle$ we have

$$\begin{aligned} \partial_J(d_1 z^{(2;1,2)}) &= d_1 d_2(z^{(1;2)}) - d_1 d_2(z^{(1;1)}) \\ &= \langle 4[\lambda_3(a \otimes a \otimes a), a] - 6\lambda_3([a,a] \otimes a \otimes a)\rangle \end{aligned}$$

(b) For $d_1 z^{(2;1,3)} = \langle 6\lambda_3(x \otimes a \otimes a)\rangle$ we have

$$\begin{aligned} \partial_J(d_1 z^{(2;1,3)}) &= d_1 d_2(z^{(1;1)}) - d_1 d_1(z^{(1;3)}) \\ &= \langle 6\lambda_3([a,a] \otimes a \otimes a) - 12[[x,a],a] - 6[[a,a],x]\rangle \end{aligned}$$

(c) For $d_1 z^{(2;2,3)} = \langle 4[y,a] + 3[x,a]\rangle$ we have

$$\begin{aligned} \partial_J(d_1 z^{(2;2,3)}) &= d_1 d_2(z^{(1;2)}) - d_1 d_1(z^{(1;3)}) \\ &= \langle 4[\lambda_3(a \otimes a \otimes a), a] - 12[[x,a],a] - 6[[a,a],x]\rangle, \end{aligned}$$

So there is an element $z^{(3;1,2,3)} \in J_0^{(3)}7$ with

$$\partial_J(z^{(3;1,2,3)}) = \langle \lambda_4(a \otimes a \otimes a \otimes a) - 6\lambda_3(x \otimes a \otimes a) + 4[y,a] + 3[x,a] \rangle.$$

This defines the second-order homotopy operation we are interested in (which also has an integral version).

5. Refined moduli spaces

Since neither the cohomolgy algebra nor the homotopy Lie algebra of a rational space determine the other, we can refine the coarse partition of the rational moduli space $\mathcal{M}$ by specifying both:

5.1. Definition. For each 1-connected graded-commutative algebra H and graded Lie algebra P over $\mathbb{Q}$, let $\mathcal{M}_{H,P}$ denote the collection of all simply-connected rational homotopy types of spaces X with $H^*(X;\mathbb{Q}) \cong H$ and $\pi_*(X) \otimes \mathbb{Q} \cong P$.

Note that we cannot simply identify this as a quotient variety with $\mathcal{M}_H \cap \mathcal{M}_P$ in any natural way, since points in $\mathcal{M}_H$ are represented by filtered DGA models, while those in $\mathcal{M}_P$ are represented by filtered DGA models.

The set $\mathcal{M}_{H,P}$ may, of course, be empty; but Lemaire and Sigrist have shown that it can also be infinite (see [**34**, §3]). In order to analyze this refined moduli space, we shall need one additional ingredient.

5.2. The Quillen equivalences. The fact that $\operatorname{ho}\mathcal{D}GL_0$ and $\operatorname{ho}\mathcal{D}GA_1$ are equivalent to each other, leads us to expect a direct algebraic relationship between the corresponding model categories, which in fact exists, and may be described as follows:

For simplicity, restrict attention to simply-connected spaces of finite type. We may therefore assume that A^i_X is finite dimensional for each $i \geq 0$ (and of course $A^0_X = \mathbb{Q}$, $A^1_X = 0$). Taking the vector space dual of A^*_X, we obtain a 1-connected differential graded-cocommutative coalgebra (C^X_*, δ), whose homology is $H_*(X;\mathbb{Q})$, with the usual coalgebra structure (dual to the ring structure in cohomology). Let $\mathcal{D}GC$ denote the category of such coalgebras.

Quillen defined a pair of adjoint functors

$$\mathcal{D}GL_0 \underset{\mathcal{L}}{\overset{\mathcal{C}}{\rightleftarrows}} \mathcal{D}GC$$

as follows:

I: Given a DGC (C_*, d), let $\mathcal{L}(C_*) := (\operatorname{Prim}(\Omega C_*), \partial)$ denote the graded Lie algebra of primitives in the cobar construction of C_*, constructed as follows:

If $C_* \cong \mathbb{Q} \oplus \bar{C}_*$ (where $\bar{C}_* = C_{\geq 2}$, in our case), and $\Sigma^{-1}V$ is the graded vector space V shifted downwards (so that $(\Sigma^{-1}V)_i := V_{i+1}$, with $\sigma^{-1}v \leftrightarrow v$), let $\Omega C_* := T(\Sigma^{-1}\bar{C}_*)$ denote the tensor algebra on $\Sigma^{-1}\bar{C}_*$ with

$$\partial(\sigma^{-1}x) := -\sigma^{-1}(dx) + \frac{1}{2}\sum_i (-1)^{|c'_i|}[\sigma^{-1}c'_i, \sigma^{-1}c''_i] \ ,$$

where $\bar{\Delta}c := \sum_i c'_i \otimes c''_i$ is the (reduced) comultiplication in $\bar{C}_*$, and $[\ ,\ ]$ is the commutator in $T(\Sigma^{-1}\bar{C}_*)$.

II: Given a DGL (L_*, ∂), let $\mathcal{C}(L_*, d)$ be the DGC $(\Lambda(\Sigma L_*), d)$, where ΣL_* is L_* shifted upwards, ΛV denotes the cofree graded coalgebra cogenerated by the graded vector space V, and the coderivation ∂ is defined by $d = d' - d''$, where

(5.1)

$$d'(\sigma x_1 \wedge \ldots \wedge \sigma x_n) := \sum_{i=1}^{n} (-1)^{i+\sum_{j<i}|x_j|} \sigma x_1 \wedge \ldots \wedge \sigma\partial x_i \wedge \ldots \wedge \sigma x_n),$$

$$d''(\sigma x_1 \wedge \ldots \wedge \sigma x_n) := \sum_{1\leq i<j\leq n} (-1)^{|x_i|}(\pm)\sigma[x_i, x_j] \wedge \sigma x_1 \ldots \widehat{\sigma x_i} \ldots \widehat{\sigma x_j} \ldots \sigma x_n) ,$$

and the sign $(\pm)$ is determined by

$$\sigma x_1 \wedge \ldots \wedge \sigma x_n = (\pm)\sigma x_i \wedge \sigma x_j \wedge \sigma x_1 \wedge \ldots \widehat{\sigma x_i} \ldots \widehat{\sigma x_j} \ldots \wedge \sigma x_n) .$$

See [**42**, App. B] or [**25**, §22] for the details.

5.3. Joint deformations. This suggests the following approach to the refined moduli space for a graded Lie algebra P and graded cocommutative coalgebra H, where we assume H and P are both of finite type, H is n-connected and P is $(n-1)$-connected $(n \geq 1)$, and $H_{n+1} \cong P_n$:

Let (B, ∂) be the bigraded model for $(P, 0)$, and $(C, d) = (C_*B, d' - d'')$ the corresponding coalgebra (§5.2 II). Note that (C, d) is cofibrant, but not necessarily minimal. As we deform (B, ∂) to produce all of $\mathcal{M}_P$, we change only d' in the coalgebra model, obtaining $(\hat{B}, \hat{\partial})$, say. At the same time, we have the deformations $(\tilde{A}, \tilde{d})$ say, of the formal DGC $(H, 0)$; and for each pair $\langle \hat{B}, \tilde{A}\rangle$, we have the variety $V_{\langle \hat{B}, \tilde{A}\rangle}$ of all DGC maps $\phi : (\hat{B}, \hat{\partial}) \to (\tilde{A}, \tilde{d})$ between them. Since both are cofree, these are determined by maps of graded vector spaces. The condition that ϕ be a quasi-isomorphism translates into a series of rank conditions, so that we get a semi-algebraic set parametrizing all such pairs which are equipped with a quasi-isomorphism between them – and in particular, the requisite homology H and homotopy groups P.

6. Nerves and moduli spaces

Since ordinary (integral) homotopy types do not have any known differential graded models, there is no hope of generalizing the deformation approach of sections 2-5 to cover them, too. However, in [**18**], Dwyer and Kan suggested an approach to such "moduli problems" based on the concept of nerves, which has proved useful conceptually in a number of contexts.

6.1. Definition. The *nerve* $\mathcal{N}C$ of a small category C is a simplicial set whose k-simplices are the sequences of k composable arrows in C, with d_i="delete i-th object and compose", s_j="insert identity arrow after j-th object". The geometric realization of the nerve is called the *classifying space* of $\mathcal{C}$, written $BC := |\mathcal{N}C|$.

The nerve was originally defined by Segal in [**48**], based on ideas of Grothendieck; Quillen, in [**43**] helped clarify the close connection between nerves and homotopy theory, as evinced in the following properties:

(1) The functor $\mathcal{N} : \mathcal{C}at \to \mathcal{S}$ takes natural transformations to homotopies (cf. [**43**, §2, Prop. 2]), so that

(2) if a functor F has a left or right adjoint, then $\mathcal{N}F$ is a homotopy equivalence; more general conditions are provided by [**43**, §2, Thm. A]). If C has an initial or final object, then $\mathcal{N}C \simeq *$.
(3) The nerve of a functor is not often a fibration; conditions when $\mathcal{N}F$ fits into a quasi-fibration sequence are provided by [**43**, §2, Thm. B].
(4) Similarly, $\mathcal{N}C$ is not usually a Kan complex, unless C is a groupoid (cf. [**26**, I, Lemma 3.5]).

6.2. Definition. A *classification complex* in a model category $\mathcal{C}$ (see [**18**, §2.1]) is the nerve of some subcategory $\mathcal{D}$ of $\mathcal{C}$, all of whose maps are weak equivalences, and which includes all weak equivalences whose source or target is in $\mathcal{D}$. The category $\mathcal{D}$ is only required to be *homotopically* small (cf. [**16**, §2.2]) – that is, $\mathcal{N}\mathcal{D}$ has a set of components, and its homotopy groups (at each vertex) are small.

This construction has two main properies:
(a) The components of $\mathcal{N}\mathcal{D}$ are in one-to-one correspondence with the weak homotopy types of $\mathcal{D}$ in $\mathcal{C}$.
(b) The component of $\mathcal{N}\mathcal{D}$ corresponding to an object $X \in \mathcal{C}$ is weakly homotopy equivalent to the classifying space of the monoid $\operatorname{haut} X$ of self weak equivalences of X (cf. [**18**, Prop. 2.3]).

6.3. Remark. Taking the model category $\mathcal{C} := \mathcal{T}_0$ of connected pointed spaces, with $\mathcal{D} = \mathcal{W}_{\mathcal{C}}$ the subcategory of *all* weak homotopy equivalences in $\mathcal{C}$, $\mathcal{M}^{\mathcal{C}} := \mathcal{N}\mathcal{W}_{\mathcal{C}}$ is the candidate suggested by the Dwyer-Kan approach for the moduli space of pointed homotopy types. There is also an unpointed version, of course; on the other hand, allowing for non-connected spaces merely complicates the combinatorics of $\mathcal{M}$, without adding any new information.

One might ask in what sense this qualifies as a moduli space (aside from having the right components). Even though the analogy with the classical examples mentioned in the introduction is not clear-cut, note that $\mathcal{M}^{\mathcal{T}_0}$ as defined using the nerve (which is natural choice for a space to associate to a category) is related to $\mathcal{M}^{\mathcal{T}_1^{\mathbb{Q}}}$ of §2.5, and the latter does exhibit many of the attributes listed in §1.1. Moreover, as we shall see in the next section, this construction can be used to interpret the obstruction theory of §2.10, as well as its integral analogue, and also to describe $\mathcal{M}$ as the limit of a tower of fibrations, which give increasingly accurate approximations to $\mathcal{M}$.

Finally, the monoid $\operatorname{haut} X$ of self equivalences corresponds to the mapping class group of self diffeomorphisms of a surface (Sec. 1), whose classifying space is closely related (or even homotopy equivalent) to the moduli space.

6.4. The model categories. In order to provide a uniform treatment in different model categories, and in particular to allow for the comparisons mentioned in §3.4, it is useful to consider a resolution model category structure (§3.2) on a category $\mathcal{C} = s\mathcal{E}$ of simplicial objects over another category $\mathcal{E}$. In fact, one can do this at two different levels, and in some sense the comparison between these is the heart of this approach:

I. The *topological* level – where $\mathcal{E}$ can take several forms:

(a) The category $\mathcal{T}_0$ of connected pointed spaces.

(b) The subcategory $\mathcal{T}_0^{\mathbb{Q}}$ of pointed connected spaces having rational universal covers (but arbitrary fundamental group).

Note that one can approximate any pointed connected space $X \in \mathcal{T}_0$ by its fibrewise rationalization $X'_{\mathbb{Q}}$ (cf. [**15**, I, 8.2]), which lies in $\mathcal{T}_0^{\mathbb{Q}}$: if $\tilde{X} \to X \to B(\pi_1 X)$ is the universal covering space fibration for X, then $X'_{\mathbb{Q}}$ fits into a functorial fibration sequence $\widetilde{X_{\mathbb{Q}}} \to X'_{\mathbb{Q}} \to B(\pi_1 X)$, in which $\widetilde{X_{\mathbb{Q}}}$ is the usual rationalization of the universal cover. However, algebraic DGL, DGA, or DGC models do not generally extend to this case (unless $\pi_1 X$ is finite and acts nilpotently on the higher groups – see, e.g., [**54**]).

(c) The subcategory $\mathcal{T}_1^{\mathbb{Q}}$ of 1-connected pointed rational spaces (in $\mathcal{T}_0^{\mathbb{Q}}$). This can be replaced by the Quillen equivalent model categories $\mathcal{D}GL_0$ or $\mathcal{D}GA_1$.

(d) Other variants are possible – for instance, we could consider functor categories over $\mathcal{E}$ – that is, diagrams $D \to \mathcal{E}$ for a fixed small category D (see [**9**]).

II. The *algebraic* level – where $\mathcal{E}$ is correspondingly:

(a) The category Π-$\mathcal{A}lg$ of Π-*algebras* – that is, positively graded groups G_*, abelian in dimensions ≥ 2, equipped with an action of the primary homotopy operations (Whitehead products, compositions, and G_1-action) satisfying the usual identities (see [**2**] for a more explicit definition).

(b) The subcategory Π-$\mathcal{A}lg^{\mathbb{Q}}$ of *rational* Π-*algebras*, which are (positively) graded Lie algebras equipped with a "fundamental group action" of an arbitrary group π, satisfying the usual identities (see [**30**]).

(c) When $\pi = 0$, we obtain the subcategory Π-$\mathcal{A}lg_1^{\mathbb{Q}} \cong \mathrm{gr}\,\mathcal{L}ie$ of simply-connected rational Π-algebras, which are just graded Lie algebras over $\mathbb{Q}$.

(d) There is also a concept of Π-algebras for arbitrary diagrams (again, see [**9**]).

7. Approximating classification complexes

The classification complex of a model category $\mathcal{C}$, as defined in §6.2, may appear to be a somewhat artificial marriage of a traditional moduli space, whose set of components correspond to the (weak) homotopy types of $\mathcal{C}$, and the individual components, which are classifying spaces of the form $B\,\mathrm{haut}\,X$. It is not clear at first glance why such complexes might be useful. To understand this, we show how $\mathcal{M}^{\mathcal{C}}$ can be approximated by a tower of fibrations, in a way that elucidates the obstruction theory of §2.10:

7.1. Postnikov systems and Eilenberg-Mac Lane objects. Recall from [**17**, §1.2] that functorial Postnikov towers

$$X \to \ldots \to P_n X \to P_{n-1} X \to \ldots \to P_0 X$$

may be defined in categories of the form $\mathcal{C} = s\mathcal{E}$ using the matching space construction of [**15**, X,§4.5]. By considering the fibers of successive Postnikov sections, we can define the analogue of homotopy groups in each such category, and find that they are corepresented, as one might expect, by appropriate suspensions of the "good objects" in $\mathcal{E}$. For both $\mathcal{E} = \mathcal{T}_0$ and $\mathcal{E} = \Pi\text{-}\mathcal{A}lg$, we find that the natural "homotopy groups" $\hat{\pi}_n X$ of any $X \in s\mathcal{E}$ (the *bigraded groups* of [**20**]) take values in $\Pi\text{-}\mathcal{A}lg$, with the obvious modifications for the rational variants.

Note that for a simplicial space $X \in s\mathcal{T}_0$, applying π_* in each simplicial dimension yields a simplicial Π-algebra $\pi_* X$, and the two sequences of Π-algebras $(\hat{\pi}_n X)_{n=0}^{\infty}$ and $(\pi_n \pi_* X)_{n=0}^{\infty}$ fit into a "spiral long exact sequence":

(7.2)

$$\dots \pi_{n+1}\pi_* X \xrightarrow{\partial^\star_{n+1}} \Omega\hat{\pi}_{n-1} X \xrightarrow{s_n} \hat{\pi}_n X \xrightarrow{h_n} \pi_n \pi_* X \xrightarrow{\partial^\star_n} \cdots \hat{\pi}_0 X \xrightarrow{h_0} \pi_0 \pi_* X \to 0$$

(see [**20**, 8.1]), in which each term is not only a Π-algebra, but a module over $\hat{\pi}_0 X \cong \pi_0 \pi_* X$ under a "fundamental group action", for $n \geq 1$. Here $\Omega\Lambda$ denotes the abelian Π-algebra obtained from a Π-algebra Λ by re-indexing and suspending the operations (so that in particular $\Omega\pi_* X \cong \pi_* \Omega X$ for any space X).

Furthermore, one can construct *classifying objects* $\mathbf{B}\Lambda \in s\mathcal{T}_0$ for any Π-algebra Λ (with $\hat{\pi}_0 \mathbf{B}\Lambda = \Lambda$ and $\hat{\pi}_i \mathbf{B}\Lambda = 0$ otherwise); *Eilenberg-Mac Lane* objects $\mathbf{B}(\Lambda, n)$; and *twisted* Eilenberg-Mac Lane objects $\mathbf{B}_\Lambda(M, n)$ for any Π-algebra Λ, Λ-module M, and $n \geq 1$, with

$$\hat{\pi}_i \mathbf{B}_\Lambda(M, n) \cong \begin{cases} \Lambda & \text{if } i = 0 \\ M & \text{(as a } \Lambda\text{-module) if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

This is true more generally for resolution model categories, under reasonable assumptions. To simplify the notation we denote by K_Λ the analogous simplicial Π-algebra (with $\pi_n K_\Lambda \cong \Lambda$ for $n = 0$, and 0 otherwise), and similarly the Eilenberg-Mac Lane objects $K(\Lambda, n)$, and twisted Eilenberg-Mac Lane objects $K_\Lambda(M, n)$ in $s\Pi\text{-}\mathcal{A}lg$. We shall use boldface in general to indicate constructions in $s\mathcal{T}_0$, as opposed to $s\Pi\text{-}\mathcal{A}lg$.

Finally, one can define natural k-invariants for the Postinkov system of a simplicial object $X \in s\mathcal{E}$, as in [**8**, Prop. 6.4], and these take values in appropriate André-Quillen cohomolgy groups (represented by the twisted Eilenberg-Mac Lane objects). These groups are denoted respectively by $H^n(X/\Lambda; M) := [X, \mathbf{B}_\Lambda(M, n)]_{\mathbf{B}\Lambda}$, for a simplicial space X equipped with a map to $\mathbf{B}\Lambda$, and $H^n(G/\Lambda; M) := [G, K_\Lambda(M, n)]_{K_\Lambda}$, for a simplicial Π-algebra G equipped with a map to K_Λ.

It turns out that for any simplicial space X there is a natural isomorphism

$$H^n(X/\Lambda; M) \xrightarrow{\cong} H^n(\pi_* X/\Lambda; M) \tag{7.3}$$

for every $n \geq 1$ (cf. [**8**, Prop. 8.7]).

7.4. Relating classification complexes. We shall now show how when $\mathcal{E} = \mathcal{T}_0$ – and more generally – the classification complex $\mathcal{M}^{\mathcal{T}_0}$ can be exhibited as the homotopy limit of a tower of fibrations, where the successive fibers have a cohomological description showing the relationship between $\pi_0 \mathcal{M}^{\mathcal{T}_0}$ and the higher homotopy groups. The tower in question is constructed essentially by taking successive Postnikov sections. The idea is an old one, and is useful even in analyzing the self-equivalences of a single space X (see, for example, [**55**]).

7.5. Definition. For a given Π-algebra Λ, we denote by $\mathcal{D}(\Lambda) = \mathcal{D}^{s\mathcal{T}_0}(\Lambda)$ the category of simplicial spaces $X \in s\mathcal{T}_0$ such that $\pi_* X \simeq \mathbf{B}\Lambda$ (in $s\Pi\text{-}\mathcal{A}lg$) (that is, $\pi_n\pi_*X \cong \Lambda$ for $n = 0$, and $\pi_n\pi_*X - 0$ otherwise). The nerve of $\mathcal{D}^{s\mathcal{T}_0}(\Lambda)$ will be denoted by $\mathcal{M}_\Lambda$.

The "pointed" version is the nerve of the category $\mathcal{R}(\Lambda)$ of pairs (X, ρ), where $X \in s\mathcal{T}_0$ and $\rho : \mathbf{B}\Lambda \to \pi_*X$ is a specified weak equivalence in $s\Pi$-$\mathcal{A}lg$ (again with weak equivalences as morphisms).

Although $\mathcal{M}_\Lambda$ is the more natural object of interest in our context, we actually study $\mathcal{R}(\Lambda)$. As noted in [**8**, §1.1], there is a fibration sequence

$$\mathcal{N}\mathcal{R}(\Lambda) \to \mathcal{M}_\Lambda \to B\,\mathrm{Aut}(\Lambda)\ , \tag{7.6}$$

where $\mathrm{Aut}(\Lambda)$ is the group of automorphisms of the Π-algebra Λ.

7.7. Definition. For each $n \geq 1$, let $\mathcal{R}_n(\Lambda)$ denote the category of n-Postnikov sections under K_Λ – that is, the objects of $\mathcal{R}_n(\Lambda)$ are pairs $(X\langle n\rangle, \rho)$, where $X\langle n\rangle \in s\mathcal{T}_0$ is a simplicial space such that $\mathbf{P}_nX\langle n\rangle \simeq X\langle n\rangle$ (Postnikov sections in $s\mathcal{T}_0$), and $\rho : P_nK_\Lambda \to P_n\pi_*X\langle n\rangle$ is a weak equivalence. The morphisms of $\mathcal{R}_n(\Lambda)$ are weak equivalences of simplicial spaces compatible with the maps ρ up to weak equivalence of simplicial Π-algebras.

7.8. A tower of realization spaces. Given Λ, the Postnikov section functors $\mathbf{P}_n : s\mathcal{T}_0 \to s\mathcal{T}_0$ of §7.1 induce compatible functors $\Phi_n : \mathcal{R}(\Lambda) \to \mathcal{R}_n(\Lambda)$ and $F_n : \mathcal{R}_{n+1}(\Lambda) \to \mathcal{R}_n(\Lambda)$; and as in [**8**, Thm. 9.4] and [**18**, Thm. 3.4], these in turn induce a weak equivalence

$$\mathcal{N}\mathcal{R}(\Lambda) \to \mathrm{holim}_n\,\mathcal{N}\mathcal{R}_n(\Lambda). \tag{7.9}$$

Combining (7.6) and (7.9), we may try to obtain information about the space of realizations $\mathcal{M}_\Lambda$ by studying the successive stages in the tower

$$\ldots\mathcal{N}\mathcal{R}_{n+1}(\Lambda) \xrightarrow{\mathcal{N}F_n} \mathcal{N}\mathcal{R}_n(\Lambda) \xrightarrow{\mathcal{N}F_{n-1}} \ldots \to \mathcal{N}\mathcal{R}_1(\Lambda). \tag{7.10}$$

8. Analyzing the tower

The first step in analyzing the tower (7.10) is to understand when the successive fibers are non-empty, and if so, to count their components. In the rational case, too, our main task was identifying the components of the space of rational homotopy types, and a partial ordering on the components was induced by the successive deformations. The problem of empty fibers did not arise there, since all DGLs (or DGAs) are realizable by rational spaces. However, the fact that we have ordered the successive choices in a tower, rather than a tree as in Theorem 4.7, suggests that we can describe them by means of an obstruction theory, as follows:

Assume given a point $(X\langle n\rangle, \rho)$ in $\mathcal{N}\mathcal{R}_n(\Lambda)$, so that $X\langle n\rangle \in s\mathcal{T}_0$ is a simplicial space which is an n-Postnikov stage (for some putative simplicial space Y realizing the given Π-algebra Λ), and $\rho : \mathbf{K}_\Lambda \to P_n\pi_*X\langle n\rangle$ is a choice of a weak equivalence. We can then reinterpret [**8**, Prop. 9.11] as saying:

8.1. Proposition. *$X\langle n\rangle \in \mathcal{R}_n(\Lambda)$ extends to an $(n+1)$-Postnikov stage in $\mathcal{R}_{n+1}(\Lambda)$ if an only if the $n+1$-st k-invariant for the simplicial Π-algebra π_*X vanishes in $H^{n+3}(\Lambda; \Omega^{n+1}\Lambda)$.*

The spiral long exact sequence (7.2) actually determines the homotopy "groups" of the simplicial Π-algebra $\pi_* X\langle n\rangle$ completely:

$$\pi_k \pi_* X\langle n\rangle \cong \begin{cases} \Lambda & \text{for } k=0, \\ \Omega^{n+1}\Lambda & \text{for } k=n+2 \\ 0 & \text{otherwise.} \end{cases} \tag{8.2}$$

However, (8.2) in itself does not imply that $\pi_* X\langle n\rangle$ is an twisted Eilenberg-Mac Lane object $K_\Lambda(\Omega^{n+1}\Lambda, n+2)$ in $s\Pi\text{-}\mathcal{A}lg$ – for that to happen, the map $\pi_* X\langle n\rangle \to K_\Lambda$ must have a section (whose existence is equivalent to the vanishing of the k-invariant in Proposition 8.1).

8.3. Remark. It may help to understand why if one considers a simplicial group $K \in \mathcal{G}$ (as a model for a connected topological space). The fundamental group then appears as $\Gamma = \pi_0 K$, the set of path components of K (all homotopy equivalent to each other) – which happens to have a group structure. When $K = K^\Gamma(M,n) = K(M,n) \ltimes \Gamma$ is a twisted Eilenberg-Mac Lane object, the choice of the section $s : B\Gamma \to K$ is what distinguishes K from a disjoint union (of cardinality $|\Gamma|$) of copies of ordinary Eilenberg-Mac Lane objects $K(M,n)$ – that is, $L := \coprod_{|\Gamma|} K(M,n)$. Even though we can put a group structure on $\pi_0 L$ so as to make it abstractly isomorphic to Γ, we will not get the right action of Γ on M (which, in the case of K, appears in the usual way by conjugation with any representative of $\gamma \in \Gamma = \pi_0 K$).

8.4. Distinguishing between liftings. Since the weak homotopy types of the realizations of a given Π-algebra Λ (that is, components of $\mathcal{M}_\Lambda^{\mathcal{T}_0}$) are in one-to-one correspondence with the weak homotopy types of the simplicial spaces Y realizing K_Λ in $s\mathcal{T}_0$ (if any), we should be able to describe them inductively in terms of the simplicial-space k-invariants of Y, and thus of the successive Postnikov approximations $X\langle n\rangle$.

So, one naturally expects that there will be a correspondence between the sections s_n and the k-invariants $\mathbf{k}_n$ in $s\mathcal{T}_0$. However, by [**8**, Prop 9.11] (using the identification of (7.3)) we know that the map $\phi : K_\Lambda(\Omega^{n+1}\Lambda, n+2) \to K_\Lambda(\Omega^{n+1}\Lambda, n+2)$ corresponding to $\mathbf{k}_n$ must be a weak equivalence, in our case – that is, it is within the indeterminacy of the $\mathbf{k}$-invariants, which is certainly contained in $\mathrm{Aut}_\Lambda(\Omega^{n+1}\Lambda)$.

Thus, the relevant information for constructing successive stages in a Postnikov tower for the simplicial space Y with $\pi_* Y \simeq K_\Lambda$ (if it exists), aside from the Π-algebra Λ, is *not* its $\mathbf{k}$-invariants, nor the k-invariants for the simplicial Π-algebras $\pi_* \mathbf{P}_n \mathbf{B}\Lambda$ $(n = 1, 2, \dots)$. Instead, it is the seemingly innocuous choice of the section $s_n : K_\Lambda \to K_\Lambda(\Omega^{n+1}\Lambda, n+2)$, with which any twisted Eilenberg-Mac Lane object (of simplicial spaces or Π-algebras) is equipped:

8.5. Proposition. *If $X\langle n\rangle \in \mathcal{R}_n(\Lambda)$ can be lifted to $\mathcal{R}_n(\Lambda)$, the possible choices for the $(n+1)$-st Postnikov stage $X\langle n+1\rangle$ are determined by the choices of sections $s_n : K_\Lambda \to \pi_* X\langle n\rangle$, which correspond to elements in $H^{n+2}(\Lambda; \Omega^{n+1}\Lambda)$.*

8.6. The components. As noted above, the Dwyer-Kan concept of classification complexes has one advantage over the traditional moduli spaces, in that the topology of each component encodes further information about the corresponding homotopy type: namely, the topological group of self-equivalences.

Such groups are generally hard to analyze, as one might expect from the topological analogue (see, e.g., [**44**]). One advantage of the tower (7.10) is that the successive fibers are generalized Eilenberg Mac-Lane spaces:

8.7. PROPOSITION ([**8**, Prop. 9.6*ff.*]). *For any* Π*-algebra* Λ *and* $n \geq 0$, *the fiber of* $\mathcal{R}_{n+1}(\Lambda) \to \mathcal{R}_n(\Lambda)$ *is either empty, or has all components weakly equivalent to* $\prod_{i=1}^{n+1} K(H^{n+3-i}_{\Lambda}(\Lambda, \Omega^{n+1}\Lambda), i)$.

9. Comparing the rational and integral versions

The constructions of the rational moduli space as an infinite-dimensional algebraic variety over $\mathbb{Q}$, and of the integral moduli space as a classification complex, are not related in any evident way. However, given a Π-algebra $\Lambda \in \Pi\text{-}\mathcal{A}lg$ and its rationalization $\Lambda^{\mathbb{Q}} \in \Pi\text{-}\mathcal{A}lg^{\mathbb{Q}}$ (defined $\Lambda^{\mathbb{Q}}_1 := \Lambda_1$ and $\Lambda^{\mathbb{Q}}_i := \Lambda_i \otimes \mathbb{Q}$ for $i \geq 2$), there is a connection between the set of components of $\mathcal{M}_{\Lambda^{\mathbb{Q}}}$ and that of $\mathcal{M}_{\Lambda}$, and also between the respective towers (7.10).

9.1. The preimage of rationalization. Clearly, Λ is realizable only if $\Lambda^{\mathbb{Q}}$ is (which is automatic in the simply-connected case), but the converse need not hold (as the example of a non-realizable torsion Π-algebra in [**3**, Prop. 4.3.6] shows). Therefore, the number of components of $\mathcal{M}_{\Lambda^{\mathbb{Q}}}$ need have no relation to that of $\mathcal{M}_{\Lambda}$, since there seems to be no known method for determining all the (integral) homotopy types X whose fibrewise rationalizations $X'_{\mathbb{Q}}$ (§6.4I(b)) are homotopy equivalent. However, the obstruction theories of §2.10 and Propositions 8.1-8.5, respectively, do provide a framework for studying the preimage of the (fibrewise) rationalization:

Given a rationalized Π-algebra A, we have

(i) an algebraic question: which Π-algebras Λ have $\Lambda^{\mathbb{Q}} \cong A$?

(ii) a homotopy-theoretic question: what are the components of $\mathcal{M}_{\Lambda}$ (though different components can map to the same component of $\mathcal{M}_A$ under the fibrewise rationalization functor)?

For the first question, restrict attention to the simply-connected case, so that A is just a (connected) graded Lie algebra over $\mathbb{Q}$. In this case we must first consider which connected graded Lie algebras over $\mathbb{Z}$ have $L \otimes \mathbb{Q} \cong A$. Given such an L, we would like to classify all possible Π-algebra structures on L; one possible approach to this problem is given by the obstruction theory of [**11**], in terms of certain relative cohomology groups (where “relative” involves comparing the same object in different categories via a forgetful functor).

Theoretically, the obstruction theory of Section 8 provides an answer to the second question, although it does not tell us how to identify all integral homotopy types having weakly equivalent rationalizations.

9.2. Comparing cohomology theories. In view of the obstruction theories of Sections 2 and 8, a first step towards understanding the relation between the integral and rational classifications is to compare the cohomology theories that house the respective obstructions. There are two main cases to consider:

I. We can compare the cohomology of a Π-algebra Λ, with coefficients in some Λ-module M (say, $M = \Omega^n\Lambda$) with the cohomology of the rationalization $\Lambda^{\mathbb{Q}}$, with rationalized coefficients in $M^{\mathbb{Q}}$ (which is a Λ-module, so in particular a $\Lambda^{\mathbb{Q}}$-module).

In this case the exact sequence of Λ-modules $0 \to K \xrightarrow{i} M \xrightarrow{q} M^{\mathbb{Q}} \to C \to 0$ (for $K := \mathrm{Ker}\,(q)$ and $C := \mathrm{Coker}\,(q)$) yields two long exact sequences in cohomology, in the usual way. Moreover, the functors

$$\Pi\text{-}\mathcal{A}lg \xrightarrow{T} \Pi\text{-}\mathcal{A}lg^{\mathbb{Q}} \xrightarrow{S} \mathcal{A}bGp,$$

where $T(-) := (-)^{\mathbb{Q}}$ and $S(-) := \mathrm{Hom}_{\Lambda}(-, M^{\mathbb{Q}})$, satisfy the conditions of [**12**, Thm. 4.4], since a rationalized free Π-algebra is free (and so H-acyclic) in $\Pi\text{-}\mathcal{A}lg^{\mathbb{Q}}$. Thus we have a generalized Grothendieck spectral sequence with

$$E^2_{s,t} = (L_s\bar{S}_t)(L_*T)\Lambda \;\Rightarrow\; H^{s+t}(\Lambda^{\mathbb{Q}}; M^{\mathbb{Q}}) \tag{9.3}$$

converging to the cohomology in the category $\Pi\text{-}\mathcal{A}lg^{\mathbb{Q}}$). Here L_s denotes the s-th left derived functor, and (for simply-connected Λ) $\bar{S} : \mathrm{bg}\,\mathcal{L}ie \to \mathrm{gr}\,\mathcal{A}bGp$ is the functor induced by S, which exists because the homotopy groups of any simplicial graded Lie algebra over $\mathbb{Q}$ actually take value in the category $\mathrm{bg}\,\mathcal{L}ie$ of bigraded Lie algebras.

For general $\Lambda \in \Pi\text{-}\mathcal{A}lg$, by [**12**, Prop. 3.2.3] we have instead a functor $\bar{S}$: $(\Pi\text{-}\mathcal{A}lg^{\mathbb{Q}})\text{-}\Pi\text{-}\mathcal{A}lg \to \mathrm{gr}\,\mathcal{A}bGp$ whose domain is the analogue for $\Pi\text{-}\mathcal{A}lg^{\mathbb{Q}}$ of the category of Π-algebras for spaces. Its objects are bigraded groups, endowed with an action of all primary homotopy operations which exist for the homotopy groups of a simplicial rational Π-algebra. As in the simply-connected case, these include the bigraded Lie bracket mentioned above, and presumably others.

II. Alternatively, one could start with a rationalized Π-algebra A – for simplicity, a graded Lie algebra – and try to compare its cohomology (with coefficients in an A-module M) taken in the category of Lie algebras with that obtained by thinking of it as an ordinary Π-algebra. This means taking the derived functors of the composite of

$$\Pi\text{-}\mathcal{A}lg^{\mathbb{Q}} \xrightarrow{I} \Pi\text{-}\mathcal{A}lg \xrightarrow{S} \mathcal{A}bGp,$$

where I is the inclusion, and $S(-) := \mathrm{Hom}_{\Pi\text{-}\mathcal{A}lg/A}(-, M) = \mathrm{Hom}_{\Pi\text{-}\mathcal{A}lg/I(A)}(-, M)$.

These do not satisfy the conditions of [**12**, Thm. 4.4], since a rationalized free Π-algebra is not a free Π-algebra. However, we can take a simplicial resolution $V \to A$ in the category $s\Pi\text{-}\mathcal{A}lg^{\mathbb{Q}}$ of simplicial graded Lie algebras, and then resolve each V_n functorially in $s\Pi\text{-}\mathcal{A}lg$ to get a bisimplicial free Π-algebra $W_{\bullet\bullet}$, with $Xd := \mathrm{diag}\,W_{\bullet\bullet}$ a free Π-algebra resolution of $I(A)$. If $\mathcal{A}b(-)$ denotes the abelianzation in the over category $\Pi\text{-}\mathcal{A}lg/A$, then $\mathcal{A}bW_{\bullet\bullet}$ is a bisimplicial abelian object, or equivalently, a double complex in $(\Pi\text{-}\mathcal{A}lg/A)_{\mathrm{ab}}$, and the bicosimplicial abelian group $SW_{\bullet\bullet}$ is $\mathrm{Hom}_{\Pi\text{-}\mathcal{A}lg/A}(\mathcal{A}bW_{\bullet\bullet}, M)$. Since

$$\begin{aligned}
\mathrm{Hom}_{\Pi\text{-}\mathcal{A}lg/A}(X, M) &= \mathrm{Hom}_{(\Pi\text{-}\mathcal{A}lg/A)_{\mathrm{ab}}}(\mathrm{diag}\,\mathcal{A}bW_{\bullet\bullet}, M)\\
&\simeq \mathrm{Tot}\,\mathrm{Hom}_{(\Pi\text{-}\mathcal{A}lg/A)_{\mathrm{ab}}}(\mathcal{A}bW_{\bullet\bullet}, M)\\
&= \mathrm{Tot}\,\mathrm{Hom}_{\Pi\text{-}\mathcal{A}lg/A}(W_{\bullet\bullet}, M)
\end{aligned}$$

by the generalized Eilenberg-Zilber Theorem, we see that both the spectral sequences for the bicomplex $SW_{\bullet\bullet}$ converge to the cohomology of $I(A)$. Since by definition

$$\pi^t_v SW_{n\bullet} \cong H^t(IQ_n, M) = H^t(L_nI(-), M)(A),$$

we obtain a cohomological spectral sequence with

$$E_2^{s,t} = (\pi^s H^t(-, M))(L_* I)A \ \Rightarrow\ H^{s+t}(I(A), M). \tag{9.4}$$

This is less useful than (9.3), since we cannot identify the E_2-term explicitly as a derived functor.

9.5. Remark. As was pointed out in Section 4, another way to relate the integral and rational moduli spaces is geometrically, using higher homotopy operations. This was the motivations behind [**7**] (in conjunction with [**4**]). The main difficulty in using homotopy operations in any systematic way for such a purpose is the lack of an appropriate taxonomy. The most promising way to overcome this would be by establishing a clear correspondence between higher operations and suitable cohomology groups.

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