LOOP SPACES AND HOMOTOPY OPERATIONS

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April 18, 1995

ABSTRACT. We describe two obstruction theories for a given topological space $\mathbf{X}$ to be a loop space, the first requiring a given H-space structure on $\mathbf{X}$, and the second not. Both are defined in terms of higher homotopy operations.

1. INTRODUCTION

An H-space is a topological space $\mathbf{X}$ with a multiplication map $m : \mathbf{X} \times \mathbf{X} \to \mathbf{X}$ and identity $* \in \mathbf{X}$. The motivating example is a topological group $\mathbf{G}$, which from the point of view of homotopy theory is just a loop space: $\mathbf{G} \simeq \Omega B\mathbf{G} = \mathbf{map}_*(\mathbf{S}^1, B\mathbf{G})$. The question of whether a given H-space $\mathbf{X}$ is, up to homotopy, a loop space, and thus a topological group (cf. [Mi1, §3]), has been studied from a variety of viewpoints – see [A, Ba, DoL, F, H, Ma2, St1, St2, Ste, Su, Z], and the surveys in [St3], [St4, §1], and [Ke, Part II]. Here we address this question from the aspect of homotopy operations.

As is well known, the homotopy groups of a space $\mathbf{X}$ have Whitehead products and composition operations defined on them; in addition, there are various higher order operations on $\pi_*\mathbf{X}$, such as Toda brackets; and the totality of these actually determine the (weak) homotopy type of $\mathbf{X}$ (cf. [Bl4, §7.17]). Therefore, they should enable us (in theory) to answer any homotopy-theoretic question about $\mathbf{X}$ – in particular, whether it is homotopy equivalent to a loop space. It is the purpose of this note to explain in what sense this can actually be done, using two possible approaches:

First, we explain how an H-space structure on $\mathbf{X}$ can be used to define the action of the primary homotopy operations on the shifted homotopy groups $G_* = \pi_{*-1}\mathbf{X}$ (which are isomorphic to $\pi_*\mathbf{Y}$ if $\mathbf{X} \simeq \Omega\mathbf{Y}$). This action will behave properly with respect to composition of operations if $\mathbf{X}$ is homotopy-associative, and will lift to a topological action of the monoid of all maps between spheres if and only if $\mathbf{X}$ is a loop space (see Proposition 5.6 below for the precise statement). The obstructions to having such a topological action may be stated in terms of the obstruction theories for realizing Π-algebras and their morphisms described in [Bl4].

A more concrete approach, which does not require a given H-space structure on $\mathbf{X}$, yields the following:

Theorem A: *If $\mathbf{X}$ is a CW complex such that all Whitehead products vanish in $\pi_*\mathbf{X}$, then $\mathbf{X}$ is homotopy equivalent to a loop space if and only if all the higher homotopy*

1991 *Mathematics Subject Classification.* Primary 55P45; Secondary 55Q35.
Key words and phrases. loop space, topological group, H-space, Π-algebra, homotopy operations, secondary operation, Toda bracket.

operations $\langle\langle\psi\rangle\rangle$ *defined in* §7.10 *vanish coherently.*

The higher homotopy operations in question depend only on maps between wedges of spheres, and take value in homotopy groups of spheres. They are constructed by means of a certain collection of convex polyhedra, defined in §7.1 below, which may be of independent interest.

1.1. notation and conventions. $\mathcal{T}_*$ will denote the category of pointed CW complexes with base-point preserving maps, and by a *space* we shall always mean an object in $\mathcal{T}_*$, which will be denoted by a boldface letter: $\mathbf{X}$, $\mathbf{S}^n$, and so on. The base-point will be written $* \in \mathbf{X}$. The full subcategory of 0-connected spaces will be denoted by $\mathcal{T}_0$.

The space of Moore loops on $\mathbf{Y} \in \mathcal{T}_0$ will be denoted by $\Omega\mathbf{Y}$. This is homotopy equivalent to the usual loop space: that is, the space of pointed maps $\mathbf{map}_*(\mathbf{S}^1, \mathbf{Y})$ (see [W, III, Corollary 2.19]). The reduced suspension of $\mathbf{X}$ is denoted by $\Sigma\mathbf{X}$.

$\mathcal{S}_*$ will denote the category of pointed simplicial sets and pointed maps; its objects will be denoted by boldface letters $\mathbf{K}, \mathbf{L}, \dots$. The subcategory of fibrant simplicial sets (Kan complexes) will be denoted by $\mathcal{S}_*^k$, and that of reduced Kan complexes by $\mathcal{S}_0^k$. $|\mathbf{K}| \in \mathcal{T}_*$ will denote the geometric realization of a simplicial set $\mathbf{K} \in \mathcal{S}_*$, while $S\mathbf{X} \in \mathcal{S}_*^k$ will denote the singular simplicial set associated to a space $\mathbf{X} \in \mathcal{T}_*$. $\mathcal{G}$ denotes the category of simplicial groups. (See [Ma1, §§3,14,15,17] for the definitions).

For each of the categories $\mathcal{C} = \mathcal{T}_*, \mathcal{T}_0, \mathcal{S}_*^k, \mathcal{S}_0^k$, or $\mathcal{G}$, we will denote by $[\mathbf{X}, \mathbf{Y}]_{\mathcal{C}}$ (or simply $[\mathbf{X}, \mathbf{Y}]$, if there is no danger of confusion) the set of pointed homotopy classes of maps $\mathbf{X} \to \mathbf{Y}$ (cf. [Ma1, §5] and [K1, §3]). The constant pointed map will be written c_*, or simply $*$. The homotopy category of $\mathcal{C}$, whose objects are those of $\mathcal{C}$, and whose morphisms are homotopy classes of maps in $\mathcal{C}$, will be denoted by $ho\mathcal{C}$. The adjoint functors S and $|\cdot|$ induce equivalences of categories $ho\mathcal{T}_* \approx ho\mathcal{S}_*^k$; similarly $ho\mathcal{S}_0^k \approx ho\mathcal{G}$ under the adjoint functors $G, \bar{W}$ (see §5.1 below).

For any $\mathbf{X} \in \mathcal{T}_*$, we denote by $P^n\mathbf{X}$ the n-th *Postnikov approximation* to $\mathbf{X}$, so that $\pi_i\mathbf{X} \cong \pi_i P^n\mathbf{X}$ for $i \leq n$ and $\pi_i P^n\mathbf{X} = 0$ for $i > n$.

Definition 1.2. A *H-space structure* for a space $\mathbf{X} \in \mathcal{T}_*$ is a choice of an *H-multiplication map* $m : \mathbf{X} \times \mathbf{X} \to \mathbf{X}$ such that $m \circ i = \nabla$, where $i : \mathbf{X} \vee \mathbf{X} \hookrightarrow \mathbf{X} \times \mathbf{X}$ is the inclusion, and $\nabla : \mathbf{X} \vee \mathbf{X} \to \mathbf{X}$ is the fold map (induced by the identity on each wedge summand). If $\mathbf{X}$ may be equipped with such an m, we say that $\langle \mathbf{X}, m\rangle$ (or just $\mathbf{X}$) is an *H-space*. Note that if we only have $m \circ i \sim \nabla$, we can find a homotopic map $m' \sim m$ such that $m' \circ i = \nabla$ (since $\mathbf{X}$ is assumed to be well-pointed).

An H-space $\langle \mathbf{X}, m\rangle$ is *homotopy-associative* if $m \circ (m, id_X) \sim m \circ (id_X, m) : \mathbf{X} \times \mathbf{X} \times \mathbf{X} \to \mathbf{X}$. It is an *$H$-group* if it is homotopy-associative and has a (two-sided) *homotopy inverse* $\iota : \mathbf{X} \to \mathbf{X}$ with $m \circ (\iota \times id_X) \circ \Delta \sim c_* \sim m \circ (id_X \times \iota) \circ \Delta$, (where $\Delta : \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ is the diagonal). In fact, any connected homotopy-associative H-space is an H-group (cf. [W, X, Theorem 2.2]).

If $\langle \mathbf{X}, m\rangle$ and $\langle \mathbf{Y}, n\rangle$ are two H-spaces, a map $f : \mathbf{X} \to \mathbf{Y}$ is called an *H-map* if $n \circ (f \times f) \sim f \circ m : \mathbf{X} \times \mathbf{X} \to \mathbf{Y}$. The set of pointed homotopy classes of H-maps $\mathbf{X} \to \mathbf{Y}$ will be denoted by $[\mathbf{X}, \mathbf{Y}]_H$.

One similarly defines H-simplicial sets and simplicial H-maps in the category $\mathcal{S}_*$.

1.3. organization. In section 2 we review some background material on Π-algebras; in section 3 we explain how an H-space structure on $\mathbf{X}$ determines the Π-algebra structure of its potential delooping; and in section 4 we show how the obstruction theory of [Bl4] for Π-algebras and their morphisms may be used to determine whether $\mathbf{X}$ is a loop space. In section 5 we show, in the context of simplicial groups, that the Π-algebra structure on $\pi_{*-1}\mathbf{X}$ can be made "topological" if and only if $\mathbf{X}$ is a loop space (Proposition 5.6).

In section 6 we define a certain cosimplicial simplicial space up-to-homotopy, which can be rectified if and only if $\mathbf{X}$ is a loop space, and prove a technical result (Theorem 6.4), which also yields a simplification of the general theory for realizing Π-algebras described in [Bl4]. Finally, in section 7 we construct a certain collection of *face-codegeneracy polyhedra*, which are used to define the higher homotopy operations refered to in Theorem A (=Theorem 7.12). We also show (Example 7.15) how the theorem may be used in reverse to calculate a certain tertiary operations in $\pi_*\mathbf{S}^7$.

In an appendix (section 8) we describe a specific construction of a simplicial resolution of the Π-algebra $G_* = \pi_{*-1}\mathbf{X}$.

2. Π-algebras

In this section we recall some known facts on the primary homotopy operations and their relation to the H-space question. First, some definitions and notation:

Definition 2.1. A Π-*algebra* is a graded group $G_* = \{G_k\}_{k=1}^{\infty}$ (abelian in degrees > 1), together with an action on G_* of the primary homotopy operations (i.e., compositions and Whitehead products, including the "π_1-action" of G_1 on the higher G_n's, as in [W, X, §7]), satisfying the usual universal identities. See [Bl1, §3] or [Bl2, §2.1] for a more explicit description. A *morphism of* Π-*algebras* is a homomorphism of graded groups $\phi : G_* \to G'_*$ which commutes with all the operations. Π-algebras form a category, which will be denoted Π-Alg.

It will sometimes be convenient to denote G_k, the k-degree of the Π-algebra G_*, by D_kG_*.

Definition 2.2. We say that a space $\mathbf{X}$ *realizes* an (abstract) Π-algebra G_* if there is an isomorphism of Π-algebras $G_* \cong \pi_*\mathbf{X}$. (There may be non-homotopy equivalent spaces realizing the same Π-algebra – cf. [Bl4, §7.18]). Similarly, an abstract morphism of Π-algebras $\phi : \pi_*\mathbf{X} \to \pi_*\mathbf{Y}$ (between realizable Π-algebras) is *realizable* if there is a map $f : \mathbf{X} \to \mathbf{Y}$ such that $\pi_*f = \phi$.

Definition 2.3. The *free* Π-algebras are those isomorphic to $\pi_*\mathbf{W}$, for some (possibly infinite) wedge of spheres $\mathbf{W}$: More precisely, let L be a graded set $\{L_k\}_{k=1}^{\infty}$, and let $\mathbf{W} = \bigvee_{k=1}^{\infty} \bigvee_{x\in L_k} \mathbf{S}^k_x$, where each $\mathbf{S}^k_x$ is a k-sphere. Then we say that $\pi_*\mathbf{W}$ is the *free* Π-*algebra generated by* L. We shall consider each element $x \in L_k$ to be an element of $\pi_*\mathbf{W}$, by identifying it with that generator of $\pi_k\mathbf{W}$ which represents the inclusion $\mathbf{S}^k_x \hookrightarrow \mathbf{W}$. The free Π-algebra $\pi_*\mathbf{W}$ is called *finite* if L is finite.

Fact 2.4. If we let Π denote the homotopy category of wedges of spheres, and $\mathcal{F} \subset$ Π-Alg the full subcategory of free Π-algebras, then the functor $\pi_* : \Pi \to \mathcal{F}$ is an equivalence of categories. Thus any Π-algebra morphism $\phi : G_* \to G'_*$ is uniquely realizable, if G_* and G'_* are free Π-algebras – and in fact only G_* need be free.

Definition 2.5. Let $T :$ Π-$Alg \to$ Π-Alg be the "free Π-algebra" comonad (cf. [M, VI, §1]), defined $TG_* = \coprod_{k=1}^{\infty} \coprod_{g\in G_k\setminus\{0\}} \pi_*\mathbf{S}^k_{(g)}$. The counit $\varepsilon = \varepsilon_{G_*} : TG_* \twoheadrightarrow G_*$

is defined by $\iota^k_{(g)} \mapsto g$ (where $\iota^k_{(g)}$ is the canonical generator of $\pi_*\mathbf{S}^k_{(g)}$), and the comultiplication $\vartheta = \vartheta_{G_*} : TG_* \hookrightarrow T^2G_*$ is induced by the natural transformation $\bar{\vartheta} : id_{\mathcal{F}} \to T|_{\mathcal{F}}$ defined by $x_k \mapsto \iota^k_{(x_k)}$.

Definition 2.6. An *abelian* Π-algebra is one for which all Whitehead products vanish.

These are indeed the abelian objects of Π-Alg – see [Bl2, §2]. If $\mathbf{X}$ is an H-space, then $\pi_*\mathbf{X}$ is an abelian Π-algebra (cf. [W, X, (7.8)]).

3. Secondary Π-algebra structure

We now describe how an H-space structure on X determines the Π-algebra structure of a (potential) classifying space.

3.1. the James construction. For any $\mathbf{X} \in \mathcal{T}_*$, let $J\mathbf{X}$ be the James reduced product construction, with $\lambda : J\mathbf{X} \to \Omega\Sigma\mathbf{X}$ the homotopy equivalence of [W, VII, (2.6)], and $j_X : \mathbf{X} \hookrightarrow J\mathbf{X}$ and $i_X : \mathbf{X} \hookrightarrow \Omega\Sigma\mathbf{X}$ the natural inclusions.

If $\langle \mathbf{X}, m\rangle$ is an H-space, there is a retraction of spaces $\bar{m} : J\mathbf{X} \to \mathbf{X}$ (with $\bar{m} \circ j_X = id_X$), defined

(3.2) $$\bar{m}(x_1, x_2, \ldots , x_n) = m(\ldots m(m(x_1, x_2), x_3), \ldots , x_n)$$

(cf. [J1, Theorem 1.8]).

Definition 3.3. Let $\mathbf{X}$ be an H-space. Given homotopy classes of maps $\alpha \in [\Sigma\mathbf{A}, \Sigma\mathbf{B}]$ and $\beta \in [\mathbf{B}, \mathbf{X}]$, we define the *derived composition* $\alpha \star \beta \in [\mathbf{A}, \mathbf{X}]$ as follows:

Choose representatives $f : \Sigma\mathbf{A} \to \Sigma\mathbf{B}$ and $g : \mathbf{B} \to \mathbf{X}$ for α, β respectively, and let $\lambda^{-1} : \Omega\Sigma\mathbf{X} \to J\mathbf{X}$ be any homotopy inverse to λ. Then $\alpha \star \beta$ is represented by the composite

$$\mathbf{A} \xrightarrow{i_A} \Omega\Sigma\mathbf{A} \xrightarrow{\Omega\alpha} \Omega\Sigma\mathbf{B} \xrightarrow{\lambda^{-1}} J\mathbf{B} \xrightarrow{J\beta} J\mathbf{X} \xrightarrow{\bar{m}} \mathbf{X}.$$

Fact 3.4. Note that if $\alpha = \Sigma\bar{\alpha}$ for some $\bar{\alpha} : \mathbf{A} \to \mathbf{B}$, then $\alpha \star \beta = \bar{\alpha}^{\#}\beta$ (this is well-defined, because $\mathbf{X}$ is an H-space).

We shall be interested in the case where $\mathbf{B}$ is a wedge of spheres and $\mathbf{A} = \mathbf{S}^r$, so $\star$ associates a class $\omega \star (\beta_1, \ldots , \beta_k) \in \pi_n\mathbf{X}$ to any k-ary homotopy operation $\omega^{\#} : \pi_{n_1+1}(-) \times \ldots \times \pi_{n_k+1}(-) \to \pi_{n+1}(-)$ and collection of elements $\beta_i \in \pi_{n_i}\mathbf{X}$ $(i = 1, \ldots , k)$.

In particular, if $\omega : \mathbf{S}^{p+q+1} \to \mathbf{S}^{p+1} \vee \mathbf{S}^{q+1}$ represents the Whitehead product, one may define a "Samelson product" $\omega\star(-,-) : \pi_p\mathbf{X}\times\pi_q\mathbf{X} \to \pi_{p+q}\mathbf{X}$ for any H-space $\mathbf{X}$, even without assuming associativity or the existence of a homotopy inverse (compare [W, X, §5]).

However, in general this $\omega \star (-,-)$ need not enjoy any of the usual properties of the Samelson product (bi-additivity, graded-commutativity, Jacobi identity – cf. [W, X, Theorems 5.1 & 5.4]). To ensure that they hold, one needs further assumptions on $\mathbf{X}$.

First, we note the following homotopy version of [W, VII, Theorem 2.5], which appears to be folklore:

Lemma 3.5. *If* $\langle \mathbf{X}, n\rangle$ *is a homotopy-associative* H*-space, then any map* $f : \mathbf{A} \to \mathbf{X}$ *extends to an* H*-map* $\hat{f} : J\mathbf{A} \to \mathbf{X}$, *which is unique up to homotopy.*

Proof. Given $f : \mathbf{A} \to \mathbf{X}$, define $\hat{f} : J\mathbf{A} \to \mathbf{X}$ by

$$\hat{f}(x_1, \dots, x_r) = m(\dots m(m(f(x_1), f(x_2)), f(x_3)), \dots, f(x_r)).$$

This is an H-map by [N, Lemma 1.4]. Now let $\hat{g} : J\mathbf{A} \to \mathbf{X}$ be another H-map, with a homotopy $H : f \simeq g \stackrel{Def}{=} \hat{g} \circ j_A$. Since $\hat{g}$ is an H-map, there is a homotopy $G : n \circ (\hat{g} \times \hat{g}) \simeq \hat{g} \circ m$ (where $m : J\mathbf{A} \times J\mathbf{A} \to J\mathbf{A}$ is the H-multiplication). Moreover, by [N, Lemma 1.3(a)] we may assume G is stationary on $J\mathbf{A} \vee J\mathbf{A}$.

For each $r \geq 0$, let $J_r\mathbf{A}$ denote the r-th stage in the construction of $J\mathbf{A}$, with $j^s_r : J_s\mathbf{A} \hookrightarrow J_r\mathbf{A}$ and $j^s : J_s\mathbf{A} \hookrightarrow J\mathbf{A}$ the inclusions, starting with $J_0\mathbf{A} = *$ and $J_1\mathbf{A} = \mathbf{A}$. We define $T_r\mathbf{A}$ to be the pushout in the following diagram:

$$\begin{array}{ccc} J_{r-1}\mathbf{A} & \xrightarrow{i_1} & J_{r-1}\mathbf{A} \times \mathbf{A} \\ {\scriptstyle j^{r-1}_r}\downarrow & \boxed{\text{PO}} & \downarrow{\scriptstyle \bar{q}_r} \\ J_r\mathbf{A} & \xrightarrow[\iota]{} & T_r\mathbf{A} \end{array}$$

for $r \geq 1$ (so $T_1\mathbf{A} = \mathbf{A} \vee \mathbf{A}$); then $J_{r+1}\mathbf{A}$ is the pushout in:

$$\begin{array}{ccc} T_r\mathbf{A} & \xrightarrow{\psi_r = (i_1, j^{r-1}_r \times id)} & J_r\mathbf{A} \times \mathbf{A} \\ {\scriptstyle \varphi_r = (id, \bar{q}_r)}\downarrow & \boxed{\text{PO}} & \downarrow{\scriptstyle \bar{q}_{r+1}} \\ J_r\mathbf{A} & \xrightarrow[j^r_{r+1}]{} & J_{r+1}\mathbf{A} \end{array}$$

Now let $\hat{f}_r = \hat{f}|_{J_r\mathbf{A}}$ and $\hat{g}_r = \hat{g}|_{J_r\mathbf{A}}$; we shall extend $H : f \simeq g$ to a homotopy $\hat{H} : \hat{f} \simeq \hat{g}$ by inductively constructing homotopies $\hat{H}_r : \hat{f}_r \simeq \hat{g}_r$ (starting with $\hat{H}_1 = H$) such that $\hat{H}_r|_{J_{r-1}\mathbf{A}} = \hat{H}_{r-1}$: let $n_r : \mathbf{X}^r \to \mathbf{X}$ denote the n-fold multiplication $n_r(x_1, \dots, x_r) = n(\dots n(x_1, x_2), \dots), x_r)$ and $q_r : \mathbf{A}^r \to J_r\mathbf{A}$ the quotient map, so that $n^r \circ f^r = \hat{f}_r \circ q_r$.

As a first approximation, define $\bar{H}_{r+1} : \hat{f}_r \times f \simeq \hat{g}_r \times g$ on $J_r\mathbf{A} \times \mathbf{A}$ in the above pushout to be the sum of homotopies $\bar{H}_{r+1} = n \circ (\hat{H}_r \times H) + G \circ (j^r \times j_A)$. This does not quite agree with $\hat{H}_r \circ \varphi_r$ on $T_r\mathbf{A}$, but since G is stationary on $J\mathbf{A} \vee J\mathbf{A}$ we have $\bar{H}_{r+1}|_{J_r\mathbf{A}} = n \circ (\hat{H}_r \times id) + (\text{stationary}) = \hat{H}_r + (\text{stationary})$ and $\bar{H}_{r+1}|_{J_{r-1}\mathbf{A} \times \mathbf{A}} = n \circ (\hat{H}_{r-1} \times H) + G \circ (j^{r-1} \times j_A) = \bar{H}_r$.

Since $\hat{H}_1 = H$, we see that $\bar{H}_2|_{T_2\mathbf{A}} = (H + (\text{stationary}), H)$, while $H \circ \varphi_1 = (H, H)$. Thus we may assume by induction that there is a homotopy of homotopies $F : \bar{H}_{r+1}|_{T_r\mathbf{A}} \simeq \hat{H}_r \circ \varphi_r$. Since $T_r\mathbf{A} \hookrightarrow J_r\mathbf{A} \times \mathbf{A}$ is a cofibration, the inclusion

$$T_r\mathbf{A} \times I^2 \cup (J_r\mathbf{A} \times \mathbf{A}) \times (\{0,1\} \times I \cup I \times \{0\}) \hookrightarrow (J_r\mathbf{A} \times \mathbf{A}) \times I^2$$

is a trivial cofibration, and thus we may use the homotopy extension property to obtain a new homotopy $\tilde{F}$ on $(J_r\mathbf{A} \times \mathbf{A}) \times I^2$ which restricts to $\tilde{H}_{r+1} : \hat{f}_r \times f \simeq \hat{g}_r \times g$ on

$J_r\mathbf{A} \times \mathbf{A} \times I \times \{1\}$, such that $\tilde{H}_{r+1}$ extends $\hat{H}_r \circ \varphi_r$, and thus may be combined with $\hat{H}_r$ to define a homotopy $\hat{H}_{r+1}$ as required. □

Corollary 3.6. *If* $\mathbf{X}$ *is a homotopy-associative H-space, then for any* $\mathbf{A} \in \mathcal{T}_*$ *the inclusion* $j_A : \mathbf{A} \to J\mathbf{A}$ *induces a bijection* $j_A^* : [J\mathbf{A}, X]_H \xrightarrow{\cong} [\mathbf{A}, \mathbf{X}]_{\mathcal{T}_*}$.

Proof. Since $\mathbf{X}$ is homotopy-associative H-space, the retraction $\bar{n} = \widehat{id}_X : J\mathbf{X} \to \mathbf{X}$ is an H-map, by the Lemma, so we may define $\phi : [\mathbf{A}, \mathbf{X}]_{\mathcal{T}_*} \to [J\mathbf{A}, X]_H$ by $\phi([f]) = [\bar{m} \circ J(f)]$, and clearly $j_A^*(\phi([f]) = [\bar{m} \circ J(f) \circ j_A] = [f]$. On the other hand, given an H-map $g : J\mathbf{A} \to \mathbf{X}$ we have $\bar{m} \circ J(g \circ j_{\mathbf{A}}) \circ j_A \simeq g \circ j_A$, which implies that $\bar{m} \circ J(g \circ j_{\mathbf{A}}) \simeq g$ by Lemma 3.5. Thus also $\phi(j_A^*([g])) = [g]$. □

3.7. notation. If $\mathbf{X}$ is a homotopy-associative H-space, we shall write $\pi_t^H\mathbf{X}$ for $[\Omega\mathbf{S}^t, \mathbf{X}]_H = [J\mathbf{S}^{t-1}, \mathbf{X}]_H \cong \pi_{t-1}\mathbf{X}$.

Proposition 3.8. *If* $\mathbf{X}$ *is a homotopy-associative H-space, then* $\alpha \star (\beta \star \gamma) = (\alpha^\# \beta) \star \gamma$ *for any* $\alpha \in [\Sigma\mathbf{A}, \Sigma\mathbf{B}]$, $\beta \in [\Sigma\mathbf{B}, \Sigma\mathbf{C}]$, *and* $\gamma \in [\mathbf{C}, \mathbf{X}]$.

Proof. It suffices to consider $\alpha = id_{\Sigma B}$, and so to show that

$$\begin{array}{ccccc}
\Omega\Sigma\mathbf{B} & \xrightarrow{\Omega\beta} & \Omega\Sigma\mathbf{C} & \xrightarrow{\Omega\Sigma\gamma} & \Omega\Sigma\mathbf{X} \\
 & \searrow^{\Omega\Sigma(\beta\star\gamma)} & & & \downarrow{\scriptstyle \hat{m}} \\
 & & \Omega\Sigma\mathbf{X} & \xrightarrow[\hat{m}]{} & \mathbf{X}
\end{array}$$

commutes up to homotopy (where $\hat{m}$ is the composite $\Omega\Sigma\mathbf{X} \xrightarrow{\lambda^{-1}} J\mathbf{X} \xrightarrow{\bar{m}} \mathbf{X}$) – or, since $\beta\star\gamma$ is defined to be the composite $\hat{m} \circ \Omega\Sigma\gamma \circ \Omega\beta \circ i_B$, that the two composites $\phi = \hat{m} \circ \Omega\Sigma\gamma \circ \Omega\beta$ and $\psi = \hat{m} \circ \Omega\Sigma\hat{m} \circ (\Omega\Sigma)^2\gamma \circ \Omega\Sigma\Omega\beta \circ \Omega\Sigma i_B$ are homotopic.

Now if $\mathbf{X}$ is a homotopy-associative H-space, then $\hat{m}$ is an H-map by Lemma 3.5, so $\phi, \psi : \Omega\Sigma\mathbf{B} \to \mathbf{X}$ are H-maps. By Corollary 3.6 it suffices to check that $\phi \circ i_B \sim \psi \circ i_B$ – i.e., that $\hat{m} \circ \Omega\Sigma\gamma \circ \Omega\beta \circ i_B$ is homotopic to the composition of

$$\mathbf{B} \xrightarrow{i_B} \Omega\Sigma\mathbf{B} \xrightarrow{\Omega\Sigma i_B} (\Omega\Sigma)^2\mathbf{B} \xrightarrow{\Omega\Sigma\Omega\beta} (\Omega\Sigma)^2\mathbf{C} \xrightarrow{(\Omega\Sigma)^2\gamma} (\Omega\Sigma)^2\mathbf{X} \xrightarrow{\Omega\Sigma\hat{m}} \Omega\Sigma\mathbf{X} \xrightarrow{\hat{m}} \mathbf{X}.$$

But $\Omega\Sigma\gamma \circ \Omega\beta \circ i_B$ is adjoint to $(\Sigma\gamma) \circ \beta$, while the composition of

$$\mathbf{B} \xrightarrow{i_B} \Omega\Sigma\mathbf{B} \xrightarrow{\Omega\Sigma i_B} (\Omega\Sigma)^2\mathbf{B} \xrightarrow{\Omega\Sigma\Omega\beta} (\Omega\Sigma)^2\mathbf{C} \xrightarrow{(\Omega\Sigma)^2\gamma} (\Omega\Sigma)^2\mathbf{X} \xrightarrow{\Omega\Sigma\hat{m}} \Omega\Sigma\mathbf{X}$$

is adjoint to $\Sigma(\hat{m} \circ \Omega\Sigma\gamma \circ \Omega\beta \circ i_B)$ which is equal to $\Sigma(\hat{m} \circ \widetilde{(\Sigma\gamma) \circ \beta})$, (where $\tilde{f}$ denotes the adjoint of f). Since for any $f : \mathbf{Y} \to \mathbf{Z}$ the adjoint of Σf is $\Omega\Sigma f \circ i_Y$, we see $\hat{m} \circ \widetilde{\Sigma f} \sim f$, which completes the proof. □

It is readily verified that when $\mathbf{X} \simeq \Omega\mathbf{Y}$, the secondary composition is the adjoint of the usual composition in $\pi_*\mathbf{Y}$; Thus we have:

Corollary 3.9. *If* $\mathbf{X}$ *is an H-group, then the graded abelian group* G_*, *defined by* $G_k = \pi_k^H\mathbf{X} \cong \pi_{k-1}\mathbf{X}$ *(with* $\bar{\gamma} \in G_k$ *corresponding to* $\gamma \in \pi_{k-1}\mathbf{X}$*), has a* Π*-algebra structure defined by the derived compositions: that is, if* $\psi \in \pi_k(\mathbf{S}^{t_1} \vee \ldots \vee \mathbf{S}^{t_n})$ *and* $\bar{\gamma}_j \in G_{t_j}$ *for* $1 \le j \le n$, *then* $\psi^\#(\bar{\gamma}_1, \ldots, \bar{\gamma}_n) \stackrel{Def}{=} \overline{\omega \star (\gamma_1, \ldots, \gamma_n)} \in G_k$. *If* $\mathbf{X} \simeq \Omega\mathbf{Y}$, *then* G_* *is isomorphic as a* Π*-algebra to* $\pi_*\mathbf{Y}$. □

Definition 3.10. For any H-group $\langle \mathbf{X}, m\rangle$, the Π-algebra G_* of Corollary 3.9 will be called the *delooping* of $\pi_*\mathbf{X}$, and denoted by $\Omega^{-1}\pi_*\mathbf{X}$ (so in particular $\Omega^{-1}\pi_*\Omega\mathbf{Y} \cong \pi_*\mathbf{Y}$).

Remark 3.11. Note that Corollary 3.9 provides us with an algebraic obstruction to delooping a space $\mathbf{X}$: if there is no way of putting a Π-algebra structure on the graded abelian group $G_* = \pi_{*-1}\mathbf{X}$ which is consistent with Fact 3.4, then $\mathbf{X}$ is not a loop space, or even a homotopy-associative H-space. (This is of course assuming that the Π-algebra $\pi_*\mathbf{X}$ is abelian – otherwise $\mathbf{X}$ cannot even be an H-space.)

Example 3.12. Consider the Π-algebra G_* defined by $G_2 = \mathbb{Z}\langle x\rangle$, (i.e., x generates the cyclic group G_2), $G_3 = \mathbb{Z}/2\langle \eta_2^{\#}x\rangle$, $G_4 = \mathbb{Z}/2\langle \eta_3^{\#}\eta_2^{\#}x\rangle$, and $G_5 = \mathbb{Z}/2\langle \eta_4^{\#}\eta_3^{\#}\eta_2^{\#}x\rangle$, with $G_t = 0$ for $t \neq 2,3,4,5$ and all Whitehead products zero.

There can be no homotopy-associative H-space $\mathbf{X}$ with $\pi_*\mathbf{X} \cong G_*$, since the Π-algebra $G'_* = \Omega^{-1}G_*$ cannot be defined consistently: we would have $G'_3 = \mathbb{Z}\langle \bar{x}\rangle$, $G'_4 = \mathbb{Z}/2\langle \eta_3^{\#}\bar{x}\rangle$, $G'_5 = \mathbb{Z}/2\langle \eta_4^{\#}\eta_3^{\#}x\rangle$, and $G'_6 = \mathbb{Z}/2\langle \eta_5^{\#}\eta_4^{\#}\eta_3^{\#}\bar{x}\rangle$ by Fact 3.4; but $\pi_6\mathbf{S}^3 = \mathbb{Z}/12\langle\alpha\rangle$ with $6\alpha = \eta_5^{\#}\eta_4^{\#}\eta_3$, and thus $\alpha^{\#}\bar{x} \in G'_6$ cannot be defined consistently with the fact that $(6\alpha)^{\#}\bar{x} \neq 0$.

(We do not claim that G_* is realizable; but the obstructions to realizing G_* by a space $\mathbf{X} \in \mathcal{T}_*$ require secondary (or higher order) information, while the obstructions to its realization by an H-group are primary.)

4. Simplicial Π-algebras and spaces

We next recall some background on simplicial Π-algebras and spaces and bisimplicial groups:

Definition 4.1. A *simplicial object* over any category $\mathcal{C}$ is a sequence of objects $\{X_n\}_{n=0}^{\infty}$ in $\mathcal{C}$, equipped with *face maps* $d_i : X_n \to X_{n-1}$ and *degeneracies* $s_j : X_n \to X_{n+1}$ $(0 \leq i,j \leq n)$, satisfying the usual simplicial identities ([Ma1, §1.1]). The category of simplicial objects over $\mathcal{C}$ is denoted by $s\mathcal{C}$. An *augmented* simplical object $X_\bullet \to A$ over $\mathcal{C}$ is a simplicial object $X_\bullet \in s\mathcal{C}$, together with an augmentation $\varepsilon : X_0 \to A$ in $\mathcal{C}$ such that $\varepsilon \circ d_1 = \varepsilon \circ d_0$.

Definition 4.2. A simplicial Π-algebra $A_\bullet$ is called *free* if for each $n \geq 0$ there is a graded set $T^n \subseteq A_n$ such that A_n is the free Π-algebra generated by T^n, and each degeneracy map $s_j : A_n \to A_{n+1}$ takes T^n to T^{n+1}.

A *free simplicial resolution* of a Π-algebra G_* is defined to be an augmented simplicial Π-algebra $A_\bullet \to G_*$, such that $A_\bullet$ is a free simplicial Π-algebra, the homotopy groups of the simplicial group $D_kA_\bullet$ vanish in dimensions $n \geq 1$, and the augmentation induces an isomorphism $\pi_0(D_kA_\bullet) \cong G_k$.

Such resolutions always exist, for any Π-algebra G_* – see [Q1, II, §4], or the explicit construction in [Bl1, §4.3].

4.3. simplicial spaces. Let $\mathbf{W}_\bullet \in s\mathcal{T}_*$ be a simplicial space: its *realization* (or homotopy colimit) is a space $\|\mathbf{W}_\bullet\| \in \mathcal{T}_*$ constructed by making identifications in $\coprod_{n=0}^{\infty} \mathbf{W}_n \times \Delta[n]$ according to the face and degeneracy maps of $\mathbf{W}_\bullet$ (cf. [Se, §1]).

For any simplicial space $\mathbf{W}_\bullet$, there is a first quadrant spectral sequence with

$$E^2_{s,t} = \pi_s(\pi_t\mathbf{W}_\bullet) \Rightarrow \pi_{s+t}\|\mathbf{W}_\bullet\| \tag{4.4}$$

(see [BoF, Thm B.5] and [BrL, App.]).

Definition 4.5. An augmented simplical space $\mathbf{W}_\bullet \to \mathbf{X}$ is called a *resolution of* $\mathbf{X}$ *by spheres* if each $\mathbf{W}_n$ is homotopy equivalent to a wedge of spheres, and $\pi_*\mathbf{W}_\bullet \to \pi_*\mathbf{X}$ is a free simplicial resolution of Π-algebras (Def. 4.2).

Using the above spectral sequence, we see that the natural map $\mathbf{W}_0 \to \|\mathbf{W}_\bullet\|$ then induces an isomorphism $\pi_*\mathbf{X} \cong \pi_*\|\mathbf{W}_\bullet\|$, so $\|\mathbf{W}_\bullet\| \simeq \mathbf{X}$.

Definition 4.6. A simplicial object over $ho\mathcal{T}_*$, say $\mathbf{W}_\bullet \in s(ho\mathcal{T}_*)$, is, by definition, a sequence of spaces $\mathbf{W}_0, \mathbf{W}_1, \dots$ with homotopy classes of maps: $\delta_i \in [\mathbf{W}_n, \mathbf{W}_{n-1}]$, and so on, for the face maps and degeneracies, satisfying the simplicial identities (in $ho\mathcal{T}_*$). By making choices of actual maps representing each of these homotopy classes we obtain an actual diagram over $\mathcal{T}_*$, which we shall denote by ${}^h\mathbf{W}_\bullet$; of course, the simplicial identities now hold only up to homotopy, in general. Such a ${}^h\mathbf{W}_\bullet$ will be called a *simplicial space up-to-homotopy*. Note we can apply the functor $\pi_* : \mathcal{T}_* \to \Pi\text{-}Alg$ to obtain a simplicial Π-algebra $\pi_*\mathbf{W}_\bullet = \pi_*({}^h\mathbf{W}_\bullet) \in s\Pi\text{-}Alg$.

If we can change the maps and spaces of ${}^h\mathbf{W}_\bullet$ up to homotopy in such a way as to obtain a simplicial space "on the nose", say $\mathbf{V}_\bullet \in s\mathcal{T}_*$, we call this a *rectification* of $\mathbf{W}_\bullet$ or ${}^h\mathbf{W}_\bullet$. For our purposes we need not worry over the precise definition of "changing ${}^h\mathbf{W}_\bullet$ up to homotopy" (see, e.g., [DwKS, §2.2]); all we require is that $\pi_*({}^h\mathbf{W}_\bullet)$ and $\pi_*\mathbf{V}_\bullet$ be isomorphic simplicial Π-algebras.

Similarly in $ho\mathcal{S}_*^k$ or $ho\mathcal{G}$.

4.7. rectifying simplicial resolutions. This suggests one possible approach to determining whether an H-group $\mathbf{X}$ is equivalent to a loop space:

First, choose some free simplicial Π-algebra $A_\bullet$ resolving G_* (for one possible construction, see Appendix 8). By Remark 2.4, the free simplicial Π-algebra $A_\bullet$ corresponds to a unique simplicial object $\mathbf{W}_\bullet \in s(ho\mathcal{T}_*)$ over the homotopy category, with each $\mathbf{W}_n$ homotopy equivalent to a wedge of spheres, such that $\pi_*\mathbf{W}_\bullet \cong A_\bullet$; this $\mathbf{W}_\bullet$ may be represented by a simplicial space up-to-homotopy ${}^h\mathbf{W}_\bullet$ (see §4.6). As usual, $\mathbf{W}_\bullet$ may be rectified if and only if ${}^h\mathbf{W}_\bullet$ can be made ∞-*homotopy commutative* – that is, if and only if one can find a sequence of homotopies for the simplicial identities among the face and degeneracy maps, and then homotopies between these, and so on (cf. [BoaV, Corollary 4.21 & Theorem 4.49]). An obstruction theory for this was described in [Bl4, §5-6]; see Remark 6.5 below. If the obstructions vanish, ${}^h\mathbf{W}_\bullet$ may be replaced by a (strict) simplicial space $\mathbf{V}_\bullet$, and by §4.3 we have $\pi_*\|\mathbf{V}_\bullet\| \cong G_*$, so $\mathbf{Y} \stackrel{Def}{=} \|\mathbf{V}_\bullet\|$ is a candidate for the delooping of the given H-group $\mathbf{X}$.

Now we apply the obstruction theory of [Bl4, §7] to check whether the Π-algebra isomorphism $\pi_*\Omega\mathbf{Y} \xrightarrow{\cong} \pi_*\mathbf{X}$ (cf. Corollary 3.9) may be realized as a map of spaces. If so, $\Omega\mathbf{Y} \simeq \mathbf{X}$, so our given H-group $\mathbf{X}$ is indeed a loop space; if not, we must try other rectifications of ${}^h\mathbf{W}_\bullet$.

5. A simplicial group version

For our purposes it will be convenient to work at times in the category $\mathcal{G}$ of simplicial groups. First, we recall some basic definitions and facts:

5.1. simplicial groups. Let $F : \mathcal{S}_* \to \mathcal{G}$ denote the free group functor of [Mi2, §2]; this is the simplicial version of the James construction, and in particular $|F\mathbf{K}| \simeq J|\mathbf{K}|$.

Let $G : \mathcal{S}_* \to \mathcal{G}$ be Kan's simplicial loop functor (cf. [Ma1, Def. 26.3]), with $\bar{W} : \mathcal{G} \to \mathcal{S}_0^k$ its adjoint, the Eilenberg-Mac Lane classifying space functor (cf. [Ma1, §21]).

Then $|G\mathbf{K}| \simeq \Omega|\mathbf{K}|$ and $|\mathbf{K}| \simeq |\bar{W}G\mathbf{K}|$. Moreover, unlike $\mathcal{T}_*$, where we have only a (weak) homotopy equivalence, in $\mathcal{G}$ there is a canonical isomorphism $\phi : F\mathbf{K} \cong G\Sigma\mathbf{K}$ (cf. [Cu, Prop. 4.15]), and there are natural bijections

(5.2)

$$Hom_{\mathcal{S}_*}(\Sigma\mathbf{L}, \bar{W}F\mathbf{K}) \;\cong\; Hom_{\mathcal{G}}(G\Sigma\mathbf{L}, F\mathbf{K}) \;\stackrel{\phi^*}{\to}\; Hom_{\mathcal{G}}(F\mathbf{L}, F\mathbf{K}) \;\cong\; Hom_{\mathcal{S}_*}(\mathbf{L}, F\mathbf{K})$$

for any $\mathbf{L} \in \mathcal{S}_*$ (induced by the adjunctions), and similarly for homotopy classes of maps.

Thus, we may think of $F\mathbf{S}^n$ as the simplicial group analogue of the r-sphere; in particular, if $\mathbf{K}$ is in $\mathcal{G}$, or even if $\mathbf{K}$ is just an associative H-simplicial set which is a Kan complex, we shall write $\pi_t^H\mathbf{K}$ for $[\mathcal{F}\mathbf{S}^{t-1}, \mathbf{K}]_H$ (compare §3.7). Similarly, $F\mathbf{e}^n$ is the $\mathcal{G}$ analogue of the n-disc in the sense that any nullhomotopic map $f : \mathcal{F}\mathbf{S}^{n-1} \to \mathbf{K}$ extends to $F\mathbf{e}^n$.

Remark 5.3. The same facts as in §4.3 hold also if we consider bisimplicial groups (which we shall think of as simplicial objects $\mathbf{G}_\bullet \in s\mathcal{G}$) instead of simplicial spaces. In this case the realization $\|\mathbf{W}_\bullet\|$ should be replaced by the diagonal $\mathrm{diag}(\mathbf{G}_\bullet)$, and the spectral sequence corresponding to (4.4) is due to Quillen (cf. [Q2]).

The above definitions provide us with a functorial simplicial version of the derived composition of §3.3:

Definition 5.4. If $\mathbf{K} \in \mathcal{S}_*^k$ is an H-simplicial set which is a Kan complex, one again has a retraction of simplicial sets $\bar{m} : F\mathbf{K} \to \mathbf{K}$, defined as in (3.2). Given a homomorphism of simplicial groups $f : F\mathbf{A} \to F\mathbf{B}$ and a map of simplicial sets $g : \mathbf{B} \to \mathbf{K}$, the composite $\bar{m} \circ Fg \circ f : F\mathbf{A} \to \mathbf{K}$ will be denoted by $f \star g$.

Note that if $\tilde{f} : \Sigma\mathbf{A} \to \bar{W}F\mathbf{B}$ and $\bar{f} : \mathbf{A} \to F\mathbf{B}$ correspond to f under (5.2), the composite $\bar{m} \circ Fg \circ \bar{f}$ corresponds to $f \star g$, and represents the derived composition $[\bar{f}] \star [g]$ in $[\mathbf{A}, \mathbf{K}]_{\mathcal{S}_*} \cong [|\mathbf{A}|, |\mathbf{K}|]_{\mathcal{T}_*}$.

Remark 5.5. The simplicial version of the $\star$ operation defined here is obviously functorial in the sense that $(e^*f) \star g = e^*(f \star g)$ for $e : F\mathbf{C} \to F\mathbf{A}$ in $\mathcal{G}$, and $f \star (g^*h) = (f \star g)^*h$ for any H-map $h : \langle \mathbf{K}, m\rangle \to \langle \mathbf{L}, n\rangle$ between fibrant H-simplicial sets which is strictly multiplicative (i.e., $n \circ (h \times h) = h \circ m : \mathbf{K} \times \mathbf{K} \to \mathbf{L}$).

However, Proposition 3.8 is still valid only in the homotopy category, and this is in fact the obstruction to $\mathbf{K}$ being equivalent to a loop space:

Proposition 5.6. *If* $\mathbf{K}$ *is an* H*-group in* $\mathcal{S}_*^k$ *such that*

$(*)$ $f \star (g \star h) = (f^\# g) \star h$ $\forall f : F\mathbf{A} \to F\mathbf{B}$ *and* $g : F\mathbf{B} \to F\mathbf{C}$ *in* $\mathcal{G}$ *and* $h : \mathbf{C} \to \mathbf{K}$,

then $\mathbf{K}$ *is* H*-homotopy equivalent to a simplicial group (and thus to a loop space); conversely, if* $\mathbf{K} \in \mathcal{G}$ *(in particular, if* $\mathbf{K} = G\mathbf{L}$ *for some* $\mathbf{L} \in \mathcal{S}_0$*), then* $(*)$ *holds.*

Proof. Assume that $\mathbf{K}$ is an H-group in $\mathcal{S}_*^k$ satisfying $(*)$. We shall need a simplicial variant of Chris Stover's construction of resolutions by spheres (Def. 4.5), so as in [Stv, §2], define a comonad $L : \mathcal{G} \to \mathcal{G}$ by

$$(5.7) \qquad L\mathbf{G} = \coprod_{k=0}^{\infty} \coprod_{\phi \in Hom_{\mathcal{G}}(FS^k, G)} F\mathbf{S}_\phi^k \cup \coprod_{k=0}^{\infty} \coprod_{\Phi \in Hom_{\mathcal{G}}(F\mathbf{e}^{k+1}, G)} F\mathbf{e}_\Phi^{k+1},$$

where $F\mathbf{e}_\Phi^{k+1}$, the $\mathcal{G}$-disc indexed by $\Phi : F\mathbf{e}^{k+1} \to \mathbf{G}$, is attached to $F\mathbf{S}_\phi^k$, the $\mathcal{G}$-sphere indexed by $\phi = \Phi|_{F\partial\mathbf{e}^{k+1}}$, by identifying $F\partial\mathbf{e}^{k+1}$ with $F\mathbf{S}^k$ (see §5.1 above). The coproduct here is just the (dimensionwise) free product of groups; the counit $\varepsilon : L\mathbf{G} \to \mathbf{G}$ is "evaluation of indices", and the comultiplication $\vartheta : L\mathbf{G} \hookrightarrow L^2\mathbf{G}$ is as in §2.5.

Now let $\mathbf{W} = \bigvee_{k=1}^{\infty} \bigvee_{f \in Hom_{\mathcal{G}}(S^k, K)} \mathbf{S}_f^k \cup \bigvee_{k=1}^{\infty} \bigvee_{F \in Hom_{\mathcal{S}_*}(e^{k+1}, K)} \mathbf{e}_F^{k+1}$ (the analogue for $\mathcal{S}_*$ of $L\mathbf{G}$, with the corresponding identifications), and let $z : \mathbf{W} \to \mathbf{K}$ be the counit map. As in (1) of Appendix 8, z induces an epimorphism $z_* : \pi_*\mathbf{W} \twoheadrightarrow \pi_*\mathbf{K}$ of Π-algebras. ($\mathbf{K}$ is a Kan complex, but $\mathbf{W}$ is not, so we understand $\pi_*\mathbf{W}$ to be the corresponding free Π-algebra $\cong \pi_*|\mathbf{W}|$ – cf. §2.3).

As in (2) of Appendix 8, we have an epimorphism of Π-algebras $\tilde{\zeta} : \pi_*\Sigma\mathbf{W} \twoheadrightarrow G_*$, where $G_* = \Omega^{-1}\pi_*\mathbf{K}$ is the delooping of $\pi_*\mathbf{K}$ – or equivalently, $\tilde{z}_* : \pi_*^H F\mathbf{W} \to \pi_*^H\mathbf{K}$, induced by $\tilde{z} = \bar{m} \circ Fz : F\mathbf{W} \to \mathbf{K}$ (cf. §5.4).

Let $\mathbf{M}_n = L^n F\mathbf{W}$ for $n = 0, 1, \ldots$, with face and degeneracy maps determined by the comonad structure maps ε, ϑ – except for $d_n : \mathbf{M}_n \to \mathbf{M}_{n-1}$, defined $d_n = L^{n-1}\bar{d}$, where $\bar{d} : LF\mathbf{W} \to F\mathbf{W}$, restricted to a summand $F\mathbf{A}_\alpha$ in $LF\mathbf{W}$ ($\mathbf{A} = \mathbf{S}^k, \mathbf{e}^{k+1}$), is an isomorphism onto $F\mathbf{A}_\beta \hookrightarrow F\mathbf{W}$, where $\beta : \mathbf{A} \to \mathbf{K}$ is the composite $(\alpha \star z) \circ j_A$.

Because $(*)$ holds on the nose, we may verify that $\bar{d} \circ T\bar{d} = \bar{d} \circ T\varepsilon : \mathbf{M}_2 \to \mathbf{M}_0$, so that $\mathbf{M}_\bullet$ is a simplicial object over $\mathcal{G}$ (just as in the proof of Lemma 8.3 in the Appendix). Moreover, the augmented simplicial Π-algebra $\pi_*\mathbf{M}_\bullet \xrightarrow{\tilde{\zeta}} G_*$ is acyclic, by a variant of [Stv, Prop. 2.6] and Lemma 8.5 of the Appendix. Thus by the Quillen spectral sequence (see §5.3) we have $\pi_*^H \operatorname{diag}\mathbf{L}_\bullet \simeq \pi_*^H\mathbf{K}$, and thus setting $\mathbf{L} = \operatorname{diag}\bar{W}\mathbf{L}_\bullet \cong \bar{W}\operatorname{diag}\mathbf{L}_\bullet$ we obtain a Kan complex $\mathbf{L}$ such that $\mathbf{K} \simeq G\mathbf{L}$ – i.e., $|\mathbf{K}| \simeq \Omega|\mathbf{L}|$.

The converse is clear, since if $\mathbf{K} \in \mathcal{G}$ then $j_A : \mathbf{A} \to F\mathbf{A}$ induces a one-to-one correspondence between maps $f : \mathbf{A} \to \mathbf{K}$ in $\mathcal{S}_*$ and homomorphisms $\varphi : F\mathbf{A} \to \mathbf{K}$ in $\mathcal{G}$, by the universal property of F. □

Note that in fact we need only verify that 5.6$(*)$ holds for $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in $\mathcal{S}_*$ which are (weakly) homotopy equivalent to wedges of spheres.

6. The simplicial-cosimplicial construction

One disadvantage of directly applying the obstruction theory of [Bl4] for realizing $G_* = \Omega^{-1}\pi_*\mathbf{X}$ as a method for determining if $\mathbf{X}$ is a loop space is that even the algebraic step – namely, determining the Π-algebra G_* – depends on a choice of H-space structure for $\mathbf{X}$, and thus cannot be described purely in terms of homotopy operations in the classical sense. In this section and the next, we shall describe a more explicit version of the obstruction theory, which does not presuppose such a choice, and is more in the spirit of the obstruction theories of [Bl4] and [Bl5].

Definition 6.1. A *CW-resolution* of a Π-algebra G_* is a free simplicial resolution $A_\bullet \to G_*$ as in §4.2, together with a sequence of free Π-algebras $(\bar{A}_n)_{n=0}^{\infty}$, called a *CW-basis* for $A_\bullet$, such that

$$A_n \cong \coprod_{0\leq\lambda\leq n} \coprod_{I\in\mathcal{I}_{\lambda,n}} \bar{A}_{n-\lambda} \tag{6.2}$$

where $\mathcal{I}_{\lambda,n}$ is the set of all sequences of λ non-negative integers $i_1 < i_2 < \ldots < i_\lambda$ $(i_\lambda < n)$ for $0 \leq \lambda \leq n$ (compare [Bl1, §4.5.1].

For each $I \in \mathcal{I}_{\lambda,n}$, the copy of $\bar{A}_{n-\lambda}$ indexed by I is *in the image* of the λ-fold degeneracy $s_I = s_{i_\lambda} \circ \ldots s_{i_1} \circ s_{i_0}$, in the obvious sense. The face maps $d_i : A_n \to A_{n-1}$ are determinmed by the *attaching map* $\bar{d}_0^n = d_0|_{\bar{A}_n} : \bar{A}_n \to A_{n-1}$, the simplicial identities, and the requirement that $d_i|_{\bar{A}_n} = 0$ for $i \geq 1$.

Aside from allowing one to construct minimal resolutions, which are convenient for explicit computations (compare [Bl5]), such CW-resolutions also have technical advantages for the obstruction theory of [Bl4, §5-6]:

Let $A_\bullet \to G_*$ be a CW-resolution, and let $\mathbf{W}_\bullet \in s(ho\mathcal{T}_*)$ be the simplicial object over $ho\mathcal{T}_*$ which corresponds to $A_\bullet$ under Remark 2.4 – that is,

$$\mathbf{W}_n \cong \bigvee_{0\leq\lambda\leq n} \bigvee_{I\in\mathcal{I}_{\lambda,n}} (\bar{\mathbf{W}}_{n-\lambda})_{(I)} \tag{6.3}$$

where $\bar{\mathbf{W}}_n$ is a wedge of spheres such that $\pi_*\bar{\mathbf{W}}_n \cong \bar{A}_n$ (and thus $\pi_*\mathbf{W}_n \cong A_n$ as Π-algebras, and $\pi_*^h\mathbf{W}_\bullet \cong A_\bullet$ as simplicial Π-algebras). Choose some simplicial space up-to-homotopy ${}^h\mathbf{W}_\bullet$ corresponding to $\mathbf{W}_\bullet$ (Def. 4.6). We then have the following

Theorem 6.4. *Let* ${}^h\mathbf{W}_\bullet$ *be as above; then* ${}^h\mathbf{W}_\bullet$ *may be rectified, and thus* G_* *realized as* $\pi_*\mathbf{Y}$, *if and only if all the higher homotopy operations* $\langle\langle\delta\rangle\rangle \subset [\Sigma^{k-1}\bar{\mathbf{W}}_n, \mathbf{W}_{n-k-1}]$ *of* [Bl4, §5.3] *vanish coherently.*

Remark 6.5. In [Bl4, §6] we required the coherent vanishing of an additional collection of (rather inelegant) higher homotopy operations, corresponding to the degeneracies of $\mathbf{W}_\bullet$, in order for G_* to be realizable. This requirement is eliminated by Theorem 6.4.

Proof. Assume that, for $\mathbf{W}_\bullet$ as above, all $\langle\langle\delta\rangle\rangle$ vanish coherently. By definition, this means there is a compatible collection of maps $\bar{g}^\delta : P_{n-k}(\delta) \ltimes \bar{\mathbf{W}}_n \to \mathbf{W}_{k-1}$ for $\delta \in D(n-k,n)/\sim$, where each such δ is a composite of face maps, and $P_{n-k}(\delta)$ is a suitable convex polyhedron. (See [Bl4, §5] for the notation and terminology). In particular, $g^{d_0} : \bar{\mathbf{W}}_n \to \mathbf{W}_{n-1}$ is in the homotopy class determined by $\bar{d}_0^n : \bar{A}_n \to A_{n-1}$, and $g^{d_i} \sim *$ on $\bar{\mathbf{W}}_n$ for $i \geq 1$.

We may then define a compatible collection of maps $g^\psi : P(\psi) \ltimes \mathbf{W}_n \to \mathbf{W}_m$ for all simplicial morphisms $\psi : A_n \to A_m$ (and suitable polyhedra $P(\psi)$ defined in [Bl4, §6.5]), as follows:

By (6.3), it suffices to define g^ψ on each wedge summand $(\bar{\mathbf{W}}_{n-\lambda})_{(I)}$, where $I = (i_0, \ldots, i_\lambda)$; since $s_I = s_{i_\lambda} \circ \ldots s_{i_0} : \bar{\mathbf{W}}_{n-\lambda} \to (\bar{\mathbf{W}}_{n-\lambda})_{(I)}$ is a homeomorphism, it is enough to define $\hat{g}^\psi = g^\psi \circ (Id_{P(\psi)} \ltimes s_I) : P(\psi) \ltimes \bar{\mathbf{W}}_{n-\lambda} \to \mathbf{W}_m$ for all ψ, I. But for any vertex $v \in P(\psi)$ we have a corresponding factorization $\psi = \lambda_\nu \circ \ldots \circ \lambda_1$ (compare §7.3 below), with each λ_i either a face or a degeneracy map, and $\hat{g}^\psi|_{\{v\}\times \bar{W}_{n-\lambda}}$ is required to be the corresponding composition of face and degeneracy maps of ${}^h\mathbf{W}_\bullet$, precomposed with s_I.

By definition of ${}^h\mathbf{W}_\bullet$, such a simplicial morphism may be computed on $\bar{W}_{n-\lambda}$ by using the simplicial identities $d_i s_j = s_{j-1} d_i$ $(i < j)$, $d_j s_j = d_{j+1} s_j = id$, and $d_i s_j = s_j d_{i-1}$ $(i > j+1)$ to bring $\lambda_\nu \circ \ldots \circ \lambda_1 \circ s_I$ into "semi-canonical form" $s_J \circ d_K$, where we assume $J = (j_1, \ldots, j_r)$ satisfies $j_1 < j_2 < \ldots < j_r$, but make no assumptions as to $K = (k_1, \ldots, k_s)$. We then set $\hat{g}^\psi|_{\{v\}\times \bar{W}_{n-\lambda}}$ equal to d_K (i.e., the composite of the corresponding chosen representatives for the face maps in ${}^h\mathbf{W}_\bullet$), postcomposed with the embedding $s_J : \mathbf{W}_{n-\lambda-s} \hookrightarrow \mathbf{W}_m$.

Now it is clear that the only homotopies needed for the 1-*dimensional* polyhedra $P(\psi)$ are those involving the various composite face maps d_K in the semi-canonical form of $\psi \circ s_I$, since all other simplicial identities for ${}^h\mathbf{W}_\bullet$ hold precisely; by induction on the dimension we see this is true for all $P(\psi)$, so that the compatible collection $\{g^\psi\}$ is simply the given compatible collection $\{\bar{g}^\delta\}$, post-composed with the appropriate embeddings s_J, and in fact we may collapse the polyhedra $P(\psi)$ to the face-map polyhedra, or permutohedra, $P_{n-k}(\delta)$ (cf. [Bl4, §4] and §7.1 below). □

Definition 6.6. A Δ-*cosimplicial object* $E^\bullet_\Delta$ over a category $\mathbf{C}$ is a sequence of objects $E^0, E^1, \ldots,$ together with *coface maps* $d^i : E^n \to E^{n+1}$ for $1 \leq 1 \leq n$ satisfying $d^j d^i = d^i d^{j-1}$ for $i < j$. Given a cosimplicial object $E^\bullet$ (cf. [BoK, X, 2.1]), we let $E^\bullet_\Delta$ denote the underlying Δ-cosimplicial object (obtained by forgetting the codegeneracies).

6.7. the cosimplicial James construction. Given a space $\mathbf{X} \in \mathcal{T}_*$, we define a Δ-cosimplicial space $\mathbf{U}^\bullet_\Delta = U(\mathbf{X})^\bullet_\Delta$ by setting $\mathbf{U}^n = \mathbf{X}^{n+1}$ (the Cartesian product), and $d^i(x_0, \ldots, x_n) = (x_0, \ldots, x_{i-1}, *, x_i, \ldots, x_n)$. Note that $J\mathbf{X} = \operatorname{colim} \mathbf{U}(X)^\bullet_\Delta$ and

Fact 6.8. If $\langle \mathbf{X}, m\rangle$ is a (strictly) associative H-space, we can extend $\mathbf{U}^\bullet_\Delta$ to a full cosimplicial space $\mathbf{U}^\bullet$ by setting $s^j(x_0, \ldots, x_n) = (x_0, \ldots, m(x_j, x_{j-1}), \ldots, x_n)$.

Definition 6.9. Let $A_\bullet$ be a CW-resolution of the Π-algebra $\pi_*\mathbf{X} = \pi_*\mathbf{U}^0$. We construct a Δ-cosimplicial augmented simplicial Π-algebra $(E_\bullet)^\bullet_\Delta \to \pi_*\mathbf{U}^\bullet_\Delta$, such that each $E^n_\bullet$ is a CW-resolution of $\pi_*\mathbf{U}^n = \pi_*(\mathbf{X}^{n+1})$, with CW-basis $\{\bar{E}^n_r\}_{r=0}^\infty$. We start by setting $\bar{E}^0_r = \bar{C}^0_r = \bar{A}_r$ for all $r \geq 0$, and then define $\bar{E}^n_r$ by a double induction (on $r \geq 0$ and then on $n \geq 0$) as

$$\bar{E}^n_r = \coprod_{0\leq\lambda\leq n} \coprod_{I\in\mathcal{I}_{\lambda,n}} [\bar{C}^{n-\lambda}_r]_I, \tag{6.10}$$

where $\mathcal{I}_{\lambda,n}$ is as in (6.2) and $\bar{C}^m_0 = 0 = \bar{C}^0_r$ for all $m, r \geq 0$.

The coface maps $d^i : E^{n-1}_r \to E^n_r$ are determined by the cosimplicial identities and the requirement that $d^i|_{[\bar{C}^{n-\lambda}_r]_{(i_1,\ldots,i_n)}}$ be an isomorphism onto $[\bar{C}^{n-\lambda}_r]_{(i_1,\ldots,i_n,i)}$ if $i > i_n$.

The only summand in (6.10) which is not defined is thus $[\bar{C}^n_r]_\emptyset$, which we denote simply by $\bar{C}^n_r$. We require that it be an n-th *cross-term* in the sense that $\bar{d}_0|_{\bar{C}^n_r}$ does not factor through the image of any coface map $d^i : E^{n-1}_{r-1} \to E^n_{r-1}$. Other than that, $\bar{C}^n_r$ may be any free Π-algebra which ensures that (6.10) defines a CW-basis for a CW-resolution $E^n_\bullet \to \pi_*\mathbf{U}^n$. We shall call the double sequence $((\bar{C}^n_r)^\infty_{n=1})^\infty_{r=1}$ a *cross-term basis* for $(E_\bullet)^\bullet_\Delta$.

Note that $A_\bullet$ is a retract of $E^2_\bullet$ in two different ways (under the two coface maps d^0, d^1), corresponding to the fact that $\mathbf{X}$ is a retract of $\mathbf{X}\times\mathbf{X}$ in two different ways; the presence of the cross-terms $\bar{C}^2_r$ indicates that $A_\bullet \times A_\bullet$ is a resolution of $\pi_*\mathbf{X}^2$, but not a free one, while $A_\bullet \amalg A_\bullet$ is a free simplicial Π-algebra, but not a resolution.
Similarly, $\mathbf{X}\times\mathbf{X}$ embeds in $\mathbf{X}^3$ in three different ways, and so on.

Example 6.11. For any $A_\bullet \to \pi_*\mathbf{X}$ we may set $\bar{C}^2_1 = \coprod_{S^p_x \hookrightarrow A^{(0)}_0} \coprod_{S^q_y \hookrightarrow A^{(1)}_0} S^{p+q-1}_{(x,y)}$, with $\bar{d}_0|_{S^{p+q-1}_{(x,y)}} = [\iota_x, \iota_y]$ (in the notation of §2.5). The higher cross-terms $\bar{C}^n_1 = 0$ for $n \geq 3$, since any k-th order cross-term element z in $\coprod^n_{j=0} A^{(j)}_0$ $(k \geq 3)$ is a sum of elements of the form $z = \zeta^\#[\dots[[\iota^{r_1}_{(x_1)}, \iota^{r_2}_{(x_2)}], \iota^{r_3}_{(x_3)}], \dots, \iota^{r_k}_{(x_k)}]$, and then

$$z = d_0(\zeta^\#[\dots[\iota^{r_1+r_2-1}_{(x_1,x_2)}, s_0\iota^{r_3}_{(x_3)}], \dots, s_0\iota^{r_k}_{(x_k)}]).$$

Definition 6.12. Let ${}^h(\mathbf{W}_\bullet)^\bullet_\Delta \to \mathbf{U}^\bullet_\Delta$ be the Δ-cosimplicial augmented simplicial space up-to-homotopy which corresponds to $(E_\bullet)^\bullet_\Delta \to \pi_*\mathbf{U}^\bullet_\Delta$ under Fact 2.4. Each $\mathbf{W}^n_r$ is homotopy equivalent to a wedge of spheres, and has a wedge summand $\bar{\mathbf{W}}^n_r \hookrightarrow \mathbf{W}^n_r$ corresponding to the CW-basis free Π-algebra summand $\bar{E}^n_r \hookrightarrow E^n_r$. We let $\bar{\mathbf{C}}^n_r$ denote the wedge summand of $\bar{\mathbf{W}}^n_r$ corresponding to $\bar{C}^n_r \hookrightarrow \bar{E}^n_r$.

We do not enter here into the question of whether every free simplicial Π-algebra resolution of a *realizable* Π-algebra $\pi_*\mathbf{Y}$ may be realized by a resolution of $\mathbf{Y}$ by spheres as in Definition 4.5 (but see [Bl5, 4.1(a)]). However, every space $\mathbf{Y}$ has a functorial resolution by spheres $\mathbf{V}_\bullet(Y) \to \mathbf{Y}$ by [Stv, Prop. 2.6], and one may in fact construct smaller (non-functorial) resolutions, as in [Bl5, 3.12, 4.19]. Thus we may make the following

Assumption 6.13. $(E_\bullet)^\bullet_\Delta$ maps monomorphically into $\pi_*\mathbf{V}_\bullet(U^\bullet_\Delta)$, and ${}^h(\mathbf{W}_\bullet)^\bullet_\Delta \to \mathbf{U}^\bullet_\Delta$ can be rectified so as to yield a strict Δ-cosimplicial augmented simplicial space $(\mathbf{W}_\bullet)^\bullet_\Delta \to \mathbf{U}^\bullet_\Delta$ realizing $(E_\bullet)^\bullet_\Delta \to \pi_*\mathbf{U}^\bullet_\Delta$.

Definition 6.14. Now assume that $\pi_*\mathbf{X}$ is an abelian Π-algebra (Def. 2.6) – this is the necessary Π-algebra condition in order for $\mathbf{X}$ to be an H-space – and let $\mu : \pi_*\mathbf{X} \times \pi_*\mathbf{X} \to \pi_*\mathbf{X}$ be the morphism of Π-algebras defined levelwise by the group operation (see [Bl5, §2]). This μ is of course associative, in the sense that $\mu \circ (\mu, id) = \mu \circ (id, \mu) : \pi_*(\mathbf{X}^3) \to \pi_*\mathbf{X}$, so it allows one to extend the Δ-cosimplicial Π-algebra $F^\bullet_\Delta \stackrel{Def}{=} \pi_*(\mathbf{U}^\bullet_\Delta)$ to a full cosimplicial Π-algebra $F^\bullet$, defined as in §6.8.
Since $E^n_\bullet \to F^n = \pi_*\mathbf{U}^n$ is a free resolution of Π-algebras, the codegeneracy maps $s^j : F^n \to F^{n-1}$ induce maps of simplicial Π-algebras $s^j_\bullet : E^n_\bullet \to E^{n-1}_\bullet$, unique up to simplicial homotopy, by the universal property of resolutions (cf. [Q1, I, p. 1.14 & II, §2, Prop. 5]). Note, however, that the individual maps $s^j_r : E^n_r \to E^{n-1}_r$ are not unique, in general; in fact, different choices may correspond to different H-multiplications on $\mathbf{X}$.

These maps s^j make $(E_\bullet)^\bullet_\Delta \to F^\bullet_\Delta$ into a full cosimplicial augmented simplicial Π-algebra $E^\bullet_\bullet \to F^\bullet$, and thus ${}^h\mathbf{W}^\bullet_\bullet \to \mathbf{U}^\bullet_\Delta$ into a cosimplicial augmented simplicial space up-to-homotopy (for which we may assume by 6.13 that all simplicial identities, and all the cosimplicial identities involving only the coface maps, hold precisely).

Proposition 6.15. *The cosimplicial simplicial space up-to-homotopy* ${}^{h}\mathbf{W}^{\bullet}_{\bullet}$ *of* §6.14 *may be rectified if and only if* $\mathbf{X}$ *is homotopy equivalent to a loop space.*

Proof. If $\mathbf{X}$ is a loop space, it has a strictly associative H-multiplication $m : \mathbf{X} \times \mathbf{X} \to \mathbf{X}$ which induces μ on $\pi_*(-)$ (cf. [Gr, Prop. 9.9]), so $\mathbf{U}^{\bullet}_{\Delta}$ extends to a cosimplicial space $\mathbf{U}^{\bullet}$ by Fact 6.8. Applying the functorial construction of [Stv, §2] to $\mathbf{U}^{\bullet}$ yields a (strict) cosimplicial augmented simplicial space $(\mathbf{V}_{\bullet})^{\bullet}_{\Delta} \to \mathbf{U}^{\bullet}$, and since we assumed $\pi_*\mathbf{W}^{n}_{\bullet}$ embeds in $\pi_*\mathbf{V}^{n}_{\bullet}$ for each n, ${}^{h}\mathbf{W}^{\bullet}_{\bullet}$ may also be rectified.

Conversely, if $\mathbf{W}^{\bullet}_{\bullet}$ is a (strict) cosimplicial simplicial space realizing $E^{\bullet}_{\bullet}$, then we may apply the realization functor for simplicial spaces in each cosimplicial dimension $n \geq 0$ to obtain $\|\mathbf{W}^{n}_{\bullet}\| \simeq \mathbf{U}^{n} = \mathbf{X}^{n+1}$ (by (4.4)). The realization of the codegeneracy map $\|s^0\| : \|\mathbf{W}^{1}_{\bullet}\| \to \|\mathbf{W}^{0}_{\bullet}\|$ induces $\mu : \pi_*(\mathbf{X}^2) \to \pi_*\mathbf{X}$, so it corresponds to an H-space multiplication $m : \mathbf{X}^2 \to \mathbf{X}$ (see [Bl5, Prop. 2.7]).

The fact that $\|\mathbf{W}^{\bullet}_{\bullet}\|$ is a (strict) cosimplicial space means that all composite codegeneracy maps $\|s^0 \circ s^{j_1} \circ \dots s^{j_{n-1}}\| : \|\mathbf{W}^{n}_{\bullet}\| \to \|\mathbf{W}^{0}_{\bullet}\|$ are equal, and thus all possible composite multiplications $\mathbf{X}^{n+1} \to \mathbf{X}$ (i.e., all possible bracketings in (3.2)) are homotopic, with homotopies between the homotopies, and so on – in other words, the H-space $\langle \mathbf{X}, m \rangle$ is an A_∞ space (see [St3, Def. 11.2]) – so that $\mathbf{X}$ is homotopy equivalent to loop space by [St3, Theorem 11.4]. Note that we only required that the codegeneracies of ${}^{h}\mathbf{W}^{\bullet}_{\bullet}$ be rectified; after the fact this ensures that the full cosimplicial simplicial space is rectifiable. □

In summary, the question of whether $\mathbf{X}$ is a loop space reduces to the question of whether a certain diagram in the homotopy category, corresponding to a diagram of free Π-algebras, may be rectified – or equivalently, may be made ∞-homotopy commutative.

7. Polyhedra and higher homotopy operations

As in [Bl4, §4], there is a sequence of higher homotopy operations which serve as obstructions to such a rectification, and these may be described combinatorially in terms of certain polyhedra, as follows:

Definition 7.1. The *N-permutohedron* $\mathbf{P}^N$ is defined to be the convex hull in $\mathbb{R}^N$ of the points $p_\sigma = (\sigma(1), \sigma(2), \dots, \sigma(N))$, where σ ranges over all permutations $\sigma \in \Sigma_N$ (cf. [Sc]). It is $(N-1)$-dimensional.

For any two integers $0 \leq n < N$, the corresponding (N,n)*-face-codegeneracy polyhedron* $\mathbf{P}^N_n$ is a quotient of the N-permutohedron $\mathbf{P}^N$ obtained by identifying two vertices p_σ and $p_{\sigma'}$ to a single vertex $\bar{p}_\sigma = \bar{p}_{\sigma'}$ of $\mathbf{P}^N_n$ whenever $\sigma = (i, i+1)\sigma'$, where $(i, i+1)$ is an adjacent transposition and $\sigma(i), \sigma(i+1) > n$.

Since each facet A of $\mathbf{P}^N$ is uniquely determined by its vertices (see below), the facets in the quotient $\mathbf{P}^N_n$ are obtained by collapsing those of $\mathbf{P}^N$ accordingly.

Note that $\mathbf{P}^N_{N-1}$ is the N-permutohedron $\mathbf{P}^N$, and in fact the quotient map $q : \mathbf{P}^N \twoheadrightarrow \mathbf{P}^N_n$ is homotopic to a homeomorphism (though not a combinatorial isomorphism, of course) for $n \geq 1$. On the other hand, $\mathbf{P}^N_0$ is a single point. For non-trivial examples of face-codegeneracy polyhedra, see Figures 1 & 2 below.

Fact 7.2. From the description of the facets of the permutohedron given in [GG], we see that $\mathbf{P}_n^N$ has an edge connecting a vertex p_σ to any vertex of the form $p_{(i,i+1)\sigma}$ (unless $\sigma(i),\sigma(i+1) > n$, in which case the edge is degenerate).

More generally, let $\bar{p}_\sigma$ be any vertex of $\mathbf{P}_n^N$. The facets of $\mathbf{P}_n^N$ containing $\bar{p}_\sigma$ are determined as follows:

Let $\mathbb{P} = \langle 1,2,\dots,\ell_1 \mid \ell_1+1,\dots,\ell_2 \mid \dots \mid \ell_{i-1}+1,\dots,\ell_i \mid \dots \mid \ell_{r-1}+1,\dots,N\,\rangle$ be a partition of $1,\dots,N$ into r consecutive blocs, subject to the condition that for each $1 \leq j < r$ at least one of $\sigma(\ell_i)$, $\sigma(\ell_{i+1})$ is $\leq n$. Denote by n_i the number of j's in the i-th bloc (i.e., $\ell_{i-1}+1 \leq j \leq \ell_i$) such that $\sigma(j) \leq n$. Then $\mathbf{P}_n^N$ will have a subpolyhedron $Q(\mathbb{P})$ (containing p_σ) which is isomorphic to the product

$$\mathbf{P}_{n_1}^{\ell_1} \times \mathbf{P}_{n_2}^{\ell_2-\ell_1} \times \cdots \times \mathbf{P}_{n_i}^{\ell_i-\ell_{i-1}} \times \cdots \times \mathbf{P}_{n_r}^{N-\ell_{r-1}}.$$

This follows from the description of the facets of the N-permutohedron in [Bl4, §4.3].

We denote by $(\mathbf{P}_n^N)^{(k)}$ the union of all facets of $\mathbf{P}_n^N$ of dimension $\leq k$. In particular, for $n \geq 1$ we have $\partial\mathbf{P}_n^N \overset{Def}{=} (\mathbf{P}_n^N)^{(N-2)} = \mathbf{S}^{N-2}$, since the homeomorphism $\tilde{q}:\mathbf{P}^N \to \mathbf{P}_n^N$ preserves $\partial\mathbf{P}^N$.

7.3. factorizations. Given a cosimplicial simplicial object $E_\bullet^\bullet$ as in §6.14, any composite face-codegeneracy map $\psi : E_{m+\ell}^{n+k} \to E_\ell^k$ may be factored uniquely $\psi = \phi \circ \theta$, where $\theta : E_{m+\ell}^{n+k} \to E_{m+\ell}^k$ may be written $\theta = s^{j_1} \circ s^{j_2} \circ \cdots s^{j_n}$ for $0 \leq j_1 < j_2 < \ldots < j_n < n+k$ and $\phi : E_{m+\ell}^k \to E_\ell^k$ may be written $\phi = d_{i_1} \circ d_{i_2} \circ \cdots d_{i_n}$ for $0 \leq i_1 < i_2 < \ldots < i_n \leq m+\ell$.

Let $\mathcal{D}(\psi)$ denote the set of all possible factorizations of ψ as a composite of face and codegeneracy maps: $\psi = \lambda_{n+m} \circ \ldots \circ \lambda_1$. We define recursively a bijective correspondence between $\mathcal{D}(\psi)$ and the vertices of an $(n+m)$-permutohedron $\mathbf{P}^{n+m}$, as follows (compare [Bl4, Lemma 4.7]):

The canonical factorization $\psi = d_{i_1} \circ d_{i_2} \circ \cdots d_{i_n} \circ s^{j_1} \circ s^{j_2} \circ \cdots s^{j_n}$ corresponds to the vertex p_{id}. Next, assume that the factorization $\psi = \lambda_{n+m} \circ \ldots \circ \lambda_1$ corresponds to p_σ. Then the factorization corresponding to $p_{\sigma'}$, for $\sigma = (i,i+1)\sigma'$, is obtained from $\psi = \lambda_1 \circ \ldots \circ \lambda_{n+m}$ by switching λ_i and λ_{i+1}, using the identity $s^j \circ s^i = s^{i-1} \circ s^j$ for $i > j$ if λ_i and λ_{i+1} are both codegeneracies, and the identity $d_i \circ d_j = d_{j-1} \circ d_i$ for $i < j$ if they are both face maps.

Passing to the quotient face-codegeneracy polyhedron, we see that the vertices of $\mathbf{P}_n^{n+m}$ are now identified with factorizations of ψ of the form

(7.4)

$$E_{m+\ell}^{n+k} \xrightarrow{s^{j_{n_t}^t}} E_{m+\ell}^{n+k-1} \ldots E_{m+\ell}^{n_t+1} \xrightarrow{s^{j_1^t}} E_{m+\ell}^{n_t} \xrightarrow{\theta_t} E_{m_t}^{n_t} \ldots E_{m_1}^{n_1} \xrightarrow{s^{j_{n_1}^0}} \ldots E_{m_1}^{n+1} \xrightarrow{s^{j_{n_0}^0}} E_{m_1}^{n} \xrightarrow{\theta_0} E_{m}^{n},$$

where θ_i is a composite of face maps (i.e., we do not distinguish the different ways of decomposing θ_i as $d_{k_1} \circ \ldots d_{k_r}$). The collection of such factorizations of ψ will be denoted by $D(\psi)/\sim$, where $\sim$ is the obvious equivalence relation on $D(\psi)$. We shall denote the face-codegeneracy polyhedron $\mathbf{P}_n^{n+m}$ with its vertices so labelled by $\mathbf{P}_n^{n+m}(\psi)$. An example for $\psi = d_0 d_1 s^0 s^1$ appears in Figure 1.

7.5. notation. For $\psi : E_{m+\ell}^{n+k} \to E_\ell^k$ as above, we denote by $\mathcal{C}(\psi)$ the collection of all composite face-codegeneracy maps $\rho : E_{m(\rho)+\ell(\rho)}^{n(\rho)+k(\rho)} \to E_{\ell(\rho)}^{k(\rho)}$ such that ρ is of the form $\rho = \xi_t \circ \ldots \circ \xi_s$ $(1 \leq s \leq t \leq \nu)$ for some decomposition $\psi = \xi_\nu \circ \ldots \circ \xi_1 = \theta_0 \circ s^{j_{n_0}^0} \circ \ldots \circ s^{j_{n_1}^0} \circ \theta_1 \circ \ldots \circ \theta_t \circ s^{j_1^t} \circ \ldots \circ s^{j_{n_t}^t}$ of (7.4). That is, we allow only those

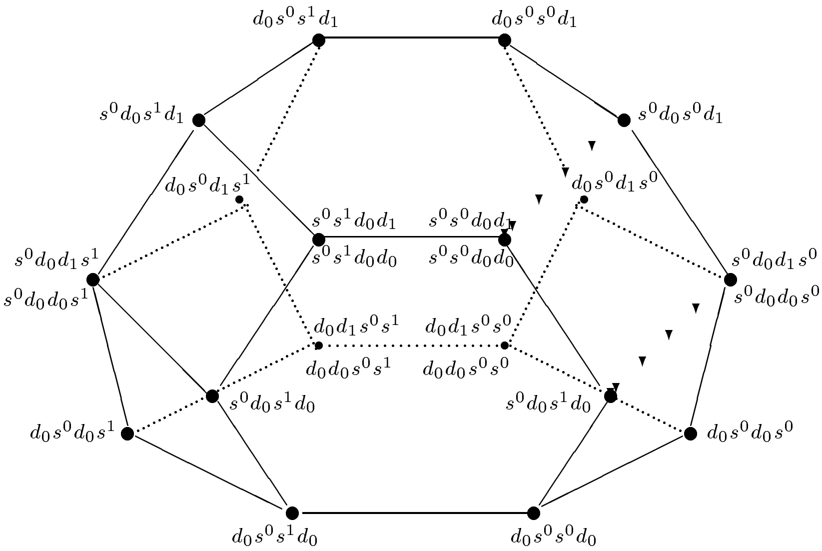

FIGURE 1. The face-codegeneracy polyhedron $\mathbf{P}^4_2(d_0d_1s^0s^1)$

subsequences $\lambda_b, \dots, \lambda_a$ of a factorization $\psi = \lambda_{n+m} \circ \dots \circ \lambda_1$ in $\mathcal{D}(\psi)$ which are compatible with the equivalence relation $\sim$ in the sense that λ_{b+1} and λ_b are not both face maps, and similarly for λ_{a-1} and λ_a. Such a ρ will be called *allowable.*

7.6. higher homotopy operations. Given a cosimplicial simplicial space up-to-homotopy ${}^h\mathbf{W}^\bullet_\bullet$ as in §6.7, we now define a certain sequence of higher homotopy operations. First recall that the *half-smash* of two spaces $\mathbf{X}, \mathbf{Y} \in \mathcal{T}_*$ is $\mathbf{X} \ltimes \mathbf{Y} \stackrel{Def}{=} (\mathbf{X} \times \mathbf{Y})/(\mathbf{X} \times \{*\})$; if $\mathbf{X}$ is a suspension, there is a (non-canonical) homotopy equivalence $\mathbf{X} \ltimes \mathbf{Y} \simeq \mathbf{X} \wedge \mathbf{Y} \vee \mathbf{X}$.

Definition 7.7. Given a composite face-codegeneracy map $\psi : \mathbf{W}^{n+k}_{m+\ell} \to \mathbf{W}^k_\ell$ as above, a *compatible collection for* $\mathcal{C}(\psi)$ and ${}^h\mathbf{W}^\bullet_\bullet$ is a set $\{g^\rho\}_{\rho \in \mathcal{C}(\psi)}$ of maps $g^\rho : \mathbf{P}^{n(\rho)+m(\rho)}_{m(\rho)}(\rho) \ltimes \mathbf{W}^{n(\rho)+k(\rho)}_{m(\rho)+\ell(\rho)} \to \mathbf{W}^{k(\rho)}_{\ell(\rho)}$ for each $\rho \in \mathcal{C}(\psi)$, satisfying the following condition:

Assume that for such a $\rho \in \mathcal{C}(\psi)$ we have some decomposition

$$\rho = \xi_\nu \circ \dots \circ \xi_1 = \theta_0 \circ s^{j^0_{n_0}} \circ \dots \circ s^{j^0_{n_1}} \circ \theta_1 \circ \dots \circ \theta_t \circ s^{j^t_1} \circ \dots \circ s^{j^t_{n_t}}$$

in $\mathcal{D}(\rho)/\sim$, as in (7.4), and let $\mathbb{P} = \langle 1, \dots, \ell_1 \mid \dots \mid \ell_{i-1}+1, \dots, \ell_i \mid \dots \mid \ell_{r-1}+1, \dots, \nu \rangle$ be a partition of $(1, \dots, \nu)$ as in §7.2, yielding a sequence of composite face-codegeneracy maps $\rho_i \in \mathcal{C}(\rho) \subseteq \mathcal{C}(\psi)$ for $i = 1, \dots, r$.

Let $Q(\mathbb{P}) \cong \mathbf{P}^{\ell_1}_{n_1}(\rho_1) \times \dots \times \mathbf{P}^{\ell_i - \ell_{i-1}}_{n_i}(\rho_i) \times \dots \times \mathbf{P}^{\nu - \ell_{r-1}}_{n_r}(\rho_r)$ be the corresponding sub-polyhedron of $\mathbf{P}^{n(\rho)+m(\rho)}_{m(\rho)}(\rho)$. Then we require that $g^\rho|_{Q(\mathbb{P}) \ltimes \mathbf{W}^{n(\rho)+k(\rho)}_{m(\rho)+\ell(\rho)}}$ be the composite of the corresponding maps g^{ρ_i} in the sense that

$$g^\rho(x_1, \dots, x_r, w) = g^{\rho_1}(x_1, g^{\rho_2}(x_2, \dots, g^{\rho_r}(x_r, w) \dots)) \tag{7.8}$$

for $x_i \in \mathbf{P}^{\ell_i - \ell_{i-1}}_{n_i}(\rho_i)$ and $w \in \mathbf{W}^{n(\rho)+k(\rho)}_{m(\rho)+\ell(\rho)}$.

We further require that if $\rho = \lambda_1$ is of length 1, then g^ρ must be in the prescribed homotopy class of the face or codegeneracy map λ_1. Thus in particular, for each vertex $\bar{p}_\sigma$ of $\mathbf{P}^{n+m}_n(\psi)$, indexed by a factorization $\psi = \xi_\nu \circ \ldots \circ \xi_1$ in $\mathcal{D}(\psi)/\sim$, the map $g^\rho|_{\{\bar{p}_\sigma\}\times \mathbf{W}^{n+\ell}_{m+k}}$ represents the class $[\xi_\nu \circ \ldots \circ \xi_1]$.

Fact 7.9. Any compatible collection of maps $\{g^\rho\}_{\rho\in\mathcal{C}(\psi)}$ for $C(\psi)$ induces a map $f = f^\psi : \partial\mathbf{P}^{n+m}_n \ltimes \mathbf{W}^{n+k}_{m+\ell} \to \mathbf{W}^k_\ell$ (since all the facets of $\partial\mathbf{P}^{n+m}_n$ are products of face-codegeneracy polyhedra of the form $\mathbf{P}^{n(\rho)+m(\rho)}_{n(\rho)}(\rho)$ for $\rho \in \mathcal{C}(\psi)$, and condition (7.8) guarantees that the maps g^ρ agree on intersections).

Definition 7.10. Given ${}^h\mathbf{W}^\bullet_\bullet$ as in §6.14, for each $k \geq 2$ and each composite face-codegeneracy map $\psi : \mathbf{W}^{n+k}_{m+\ell} \to \mathbf{W}^k_\ell$, the *k-th order homotopy operation* associated to ${}^h\mathbf{W}^\bullet_\bullet$ and ψ is a subset $\langle\langle\psi\rangle\rangle$ of the track group $[\Sigma^{n+m-2}\mathbf{W}^{n+k}_{m+\ell}, \mathbf{W}^k_\ell]$, defined as follows:

Let $S \subseteq [\partial\mathbf{P}^{n+m}_n \ltimes \mathbf{W}^{n+k}_{m+\ell},\ \mathbf{W}^k_\ell]$ be the set of homotopy classes of maps $f = f^\psi : \partial\mathbf{P}^{n+m}_n \ltimes \mathbf{W}^{n+k}_{m+\ell} \to \mathbf{W}^k_\ell$ which are induced as above by some compatible collection $\{g^\rho\}_{\rho\in\mathcal{C}(\psi)}$ for $\mathcal{C}(\psi)$.

Now choose a splitting

$$\partial\mathbf{P}^{n+m}_n(\psi) \ltimes \mathbf{W}^{n+k}_{m+\ell} \cong \mathbf{S}^{n+m-2} \ltimes \mathbf{W}^{n+k}_{m+\ell} \simeq (\mathbf{S}^{n+m-2} \wedge \mathbf{W}^k_\ell) \vee \mathbf{W}^k_\ell \tag{7.11}$$

and let $\langle\langle\psi\rangle\rangle \subseteq [\Sigma^{n+m-2}\mathbf{W}^{n+k}_{m+\ell}, \mathbf{W}^k_\ell]$ be the image of the subset S under the resulting projection.

It is clearly a necessary condition in order for the subset $\langle\langle\psi\rangle\rangle$ to be non-empty that all the lower order operations $\langle\langle\rho\rangle\rangle$ *vanish* (i.e., contain the null class) for all $\rho \in \mathcal{C}(\psi)\setminus\{\psi\}$ – because otherwise the various maps $g^\rho : \mathbf{P}^{n(\rho)+m(\rho)}_{m(\rho)}(\rho) \ltimes \mathbf{W}^{n(\rho)+k(\rho)}_{m(\rho)+\ell(\rho)} \to \mathbf{W}^{k(\rho)}_{\ell(\rho)}$ cannot even extend over the interior of $\mathbf{P}^{n(\rho)+m(\rho)}_{m(\rho)}(\rho)$. A *sufficent* condition is that the operations $\langle\langle\rho\rangle\rangle$ vanish *coherently*, in the sense that the choices of compatible collections for the various ρ be consistent on common subpolyhedra (see [Bl4, §5.7] for the precise definition, and [Bl4, §5.9] for the obstructions to coherence).

On the other hand, if ${}^h\mathbf{W}^\bullet_\bullet$ is the cosimplicial simplicial space up-to-homotopy of §6.9 (corresponding to the cosimplicial simplicial Π-algebra $(E_\bullet)^\bullet_\Delta$ with the CW-basis $\{\bar{E}^n_r\}^\infty_{r,n=0}$), then the vanishing of the homotopy operation $\langle\langle\psi|_{\bar{\mathbf{C}}^n_r}\rangle\rangle$ – with ψ restricted to the (n,r)-cross-term – implies the vanishing of $\langle\langle\psi\rangle\rangle$, for any $\psi : \mathbf{W}^{n+k}_{m+\ell} \to \mathbf{W}^k_\ell$ (assuming lower order vanishing). This is because outside of the wedge summand $\bar{\mathbf{C}}^n_r$, the map ψ is determined by the maps $\rho \in \mathcal{C}(\psi)$ and the coface and degeneracy maps of ${}^h\mathbf{W}^\bullet_\bullet$, which we may assume to ∞-homotopy commute by induction and 6.13 respectively.

We may thus sum up the results of this section, combined with Proposition 6.15, in:

Theorem 7.12. *Let $\pi_*\mathbf{X}$ be an abelian Π-algebra (for some space $\mathbf{X} \in \mathcal{T}_*$). Then $\mathbf{X}$ is homotopy equivalent to a loop space if and only if all the higher homotopy operations $\langle\langle\psi|_{\bar{\mathbf{C}}^n_r}\rangle\rangle$ defined above vanish coherently.*

Remark 7.13. As observed in §6.7, for any $\mathbf{X} \in \mathcal{T}_*$ the space $J\mathbf{X}$ is the colimit of the Δ-cosimplicial space $\mathbf{U}(X)^\bullet_\Delta$, and in fact the n-th stage of the James construction, $J_n\mathbf{X}$, is the (homotopy) colimit of the $(n-1)$-coskeleton of $\mathbf{U}^\bullet_\Delta$. Thus if we think of the sequence of higher homotopy operations "in the simplicial direction" as obstructions to the validity of 5.6(∗) (up to ∞-homotopy commutativity), then the n-th cosimplicial dimension corresponds to verifying the identity 5.6(∗) for $f \circ i_A : \mathbf{A} \to F\mathbf{B}$ of James filtration $n+1$ (cf. [J3, §2]).

In particular, if we fix $k = \ell = 0$, $n = 1$ and proceed by induction on m, we are computing the obstructions for the existence of an H-multiplication on $\mathbf{X}$, as in [Bl5]. (Thus if $\mathbf{X}$ is endowed with an H-space structure to begin with, they must all vanish.) Observe that the face-codegeneracy polyhedron $\mathbf{P}^n_1$ is an $(n-1)$-cube, as in Figure 2, rather than the $(n-1)$-simplex we had in [Bl5, §4] – so the homotopy operations we obtain here are more complicated. This is because they take value in the homotopy groups of spheres, rather than those of the space $\mathbf{X}$.

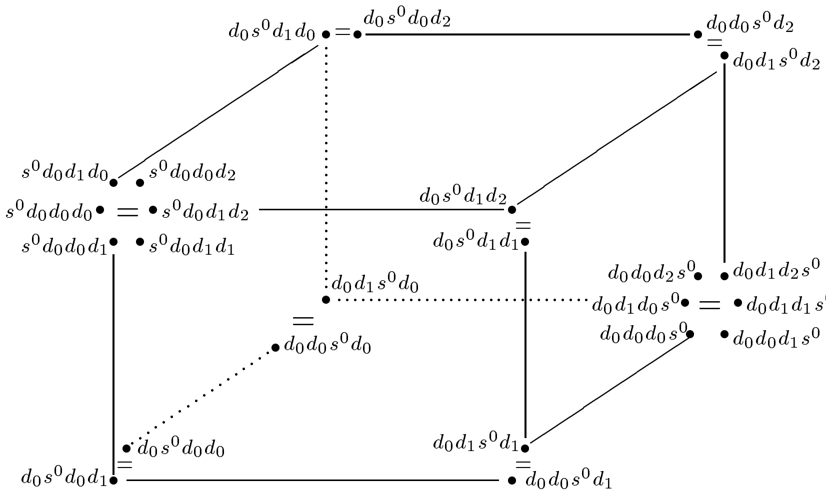

FIGURE 2. The face-codegeneracy polyhedron $\mathbf{P}^4_1(d_0d_1d_2s^0)$

As a corollary to Theorem 7.12 we may deduce the following result of Hilton (cf. [H, Theorem C]):

Corollary 7.14. *If $\langle \mathbf{X}, m\rangle$ is a $(p-1)$-connected H-space with $\pi_i\mathbf{X} = 0$ for $i \geq 3p$, then $\mathbf{X}$ is a loop space, up to homotopy*

Proof. Choose a CW-resolution of $\pi_*\mathbf{X}$ which is $(p-1)$-connected in each simplicial dimension, and let $E^\bullet_\bullet$ be as in §6.9. By definition of the cross-term Π-algebras C^n_r in §6.9, they must involve Whitehead products of elements from *all* lower order cross-terms; but since $\mathbf{X}$ is an H-space by assumption, all obstructions of the form $\langle\langle\psi|_{\bar{\mathbf{C}}^1_r}\rangle\rangle$ vanish (see §7.13). Thus, the lowest dimensional obstruction possible is a third-order operation $\langle\langle\psi|_{\bar{\mathbf{C}}^2_r}\rangle\rangle$ $(r \geq 2)$, which involves a triple Whitehead product and thus takes value in $\pi_i\mathbf{W}^k_\ell$ for $i \geq 3p$. If we apply the $(3p-1)$-Postnikov approximation functor to ${}^h\mathbf{W}^\bullet_\bullet$ in each dimension, to obtain ${}^h\mathbf{Z}^\bullet_\bullet$, all obstructions to rectification vanish, and from the spectral sequence of (4.4) we see that obvious map $\mathbf{X} = \|\mathbf{W}^1_\bullet\| \to \|\mathbf{Z}^1_\bullet\|$ induces an isomorphism in π_i for $i < 3p$. Since $\|\mathbf{Z}^1_\bullet\|$ is a loop space by Theorem 7.12, so is its $(3p-1)$-Postnikov approximation, namely $\mathbf{X}$. □

Example 7.15. The 7-sphere is an H-space (under the Cayley multiplication, for example), but none of the 120 possible H-multiplications on $\mathbf{S}^7$ are homotopy-associative; the first obstruction to homotopy-associativity is a certain "separation element" in $\pi_{21}\mathbf{S}^7$ (cf. [J2, Theorem 1.4 and Corollary 2.5]).

Since $\pi_*\mathbf{S}^7$ is a free Π-algebra, it has a very simple CW-resolution $A_\bullet \to \pi_*\mathbf{S}^7$, with $\bar{A}_0 \cong \pi_*\mathbf{S}^7$ (generated by ι^7), and $\bar{A}_r = 0$ for $r \geq 1$. A cross-term basis (§6.9) for the cosimplicial simplicial Π-algebra $E^\bullet_\bullet$ of §6.14 is then given in dimensions < 24 by:

- $\bar{C}^1_1 \cong \pi_*\mathbf{S}^{13}$, with $\bar{d}_0\iota^{13} = [d^0\iota^7, d^1\iota^7]$;
- $\bar{C}^2_2 \cong \pi_*\mathbf{S}^{19}$, with $\bar{d}_0\iota^{19} = [d^0\iota^{13}, s_0d^2d^1\iota^7] - [d^1\iota^{13}, s_0d^2d^0\iota^7] + [d^2\iota^{13}, s_0d^1d^0\iota^7]$;
- $\bar{C}^n_r$ is at least 24-connected for all other n, r.

We set $s^j_r|_{\bar{C}^n_r} = 0$ for all $n \leq 2$; this determines $E^\bullet_\bullet$ in degrees ≤ 21 and cosimplicial dimensions ≤ 2.

By Remark 7.13, the two secondary operations $\langle\langle d_0s^0|_{\bar{\mathbf{C}}^1_1}\rangle\rangle$ and $\langle\langle d_1s^0|_{\bar{\mathbf{C}}^1_1}\rangle\rangle$ must vanish; on the other hand, by Corollary 7.14 all obstructions to $\mathbf{S}^7$ being a loop space are in degrees ≥ 21, so the only relevant cross-term is $\bar{C}^2_2$, with three possible third-order operations $\langle\langle \psi|_{\bar{\mathbf{C}}^2_2}\rangle\rangle$, for $\psi = d_0d_1s^0s^1$, $d_0d_2s^0s^1$, or $d_1d_2s^0s^1$. The corresponding face-codegeneracy polyhedra $P^4_2(\psi)$ is as in Figure 1.

It is straightforward to verify that the operations $\langle\langle \psi|_{\bar{\mathbf{C}}^2_2}\rangle\rangle$ are trivial for $\psi = d_0d_2s^0s^1$ or $d_1d_2s^0s^1$ (in fact, many of the maps g^ρ, for $\rho \in C(\psi)$, may be chosen to be null). On may also show that there is a compatible collection $\{g^\rho\}_{\rho\in C(\varphi)}$ for $\varphi = d_0d_1s^0s^1$, in the sense of §7.7, so that the corresponding subset $\langle\langle \varphi|_{\bar{\mathbf{C}}^2_2}\rangle\rangle \subseteq \pi_{21}\mathbf{S}^7$ is non-empty; in fact, it contains the only possible obstruction to the 21-Postnikov approximation for $\mathbf{S}^7$ to be a loop space.

The existence of the tertiary operation $\langle\langle \varphi|_{\bar{\mathbf{C}}^2_2}\rangle\rangle$ corresponds to the fact that the element $[[\iota^7, \iota^7], \iota^7] - [[\iota^7, \iota^7], \iota^7] + [[\iota^7, \iota^7], \iota^7] \in \pi_{21}\mathbf{S}^7$ is trivial "for three different reasons": because of the Jacobi identity, because all Whitehead products vanish in $\pi_*\mathbf{S}^7$, and because of the linearity of the Whitehead product – i.e., $[0, \alpha] = 0$.

On the other hand, we know that there *is* a 3-primary obstruction to the homotopy-associativity of any H-multiplication on $\mathbf{S}^7$, namely the element $\sigma^\#_{14}\tau_7 \in \pi_{21}\mathbf{S}^7$ (see [J2, Theorem 2.6]). We deduce that $0 \notin \langle\langle \varphi|_{\bar{\mathbf{C}}^2_2}\rangle\rangle$, and in fact (modulo 3) this tertiary operation consists exactly of the elements $\pm\sigma^\#_{14}\tau_7$.

For a detailed calculation of such higher order operations using simplicial resolutions of Π-algebras, see [Bl5, §4.13].

Remark 7.16. Our approach to the question of whether $\mathbf{X}$ is a loop space is clearly based on, and closely related to, the classical approaches of Sugawara and Stasheff (cf. [St1, St2, Su]. One might wonder why Stasheff's associahedra K_i (cf. [St1, §2,6]) do not show up among the face-codegeneracy polyhedra we describe above. Apparently this is because we do not work directly with the space $\mathbf{X}$, but rather with its simplicial resolution, which may be thought of as a "decomposition" of $\mathbf{X}$ into wedges of spheres.

8. Appendix: Constructing simplicial resolutions

The obstruction theory of section 4 required a Π-algebra resolution $A_\bullet$ of $\Omega^{-1}\pi_*\mathbf{X}$ as the initial algebraic ingredient. There are of course many possible constructions of such resolutions; for practical purposes a minimal CW-resolution (as in §6.1) is the most convenient. We here describe an approach specifically geared towards $\Omega^{-1}\pi_*\mathbf{X}$, in line with section 3, because it may help to explain the simplicial group analogue used in the proof of Proposition 5.6.

Given an H-group $\mathbf{X}$, we may proceed as follows to construct a free simplicial Π-algebra resolving $G_* = \Omega^{-1}\pi_*\mathbf{X}$:

(1) Choose a space $\mathbf{W}$ which is homotopy equivalent to a wedge of spheres, and a map $z : \mathbf{W} \to \mathbf{X}$ which induces a surjection $z_\# : \pi_*\mathbf{W} \twoheadrightarrow \pi_*\mathbf{X}$. We may assume that $z_\#(\iota^k_{(\alpha)}) = \alpha \in \pi_k\mathbf{X}$, where $\iota^k_{(\alpha)} \in \pi_k\mathbf{S}^k_{(\alpha)}$ is the canonical generator of the free Π-algebra $\pi_*\mathbf{S}^k_{(\alpha)}$, and $\pi_*\mathbf{W} \cong \coprod_{k,\alpha} \pi_*\mathbf{S}^k_{(\alpha)}$ is a coproduct in Π-Alg of such free Π-algebras for various k and α.

For example, we could let $\mathbf{W} = \bigvee_{k=1}^\infty \bigvee_{\alpha\in\pi_k\mathbf{X}\setminus\{0\}} \pi_*\mathbf{S}^k_{(\alpha)}$, so that $\pi_*\mathbf{W} \cong T(\pi_*\mathbf{X})$; then by Fact 2.4 there is a map $z : \mathbf{W} \to \mathbf{X}$, unique up to homotopy, which realizes the counit $\varepsilon_{\pi_*\mathbf{X}} : T(\pi_*\mathbf{X}) \twoheadrightarrow \pi_*\mathbf{X}$.

(2) Since $\mathbf{X}$ is an H-group, we may let $G_* = \Omega^{-1}\pi_*\mathbf{X}$ denote the delooping provided by Corollary 3.9, and define a morphism of Π-algebras $\tilde{\zeta} : \pi_*\Sigma\mathbf{W} \to G_*$ by setting $\tilde{\zeta}(\bar{\iota}^{k+1}_{(\alpha)}) = \bar{\alpha} \in G_{k+1}$, where $\bar{\iota}^{k+1}_{(\alpha)}$ generates the summand $\pi_*\Sigma\mathbf{S}^k_{(\alpha)}$ in $\pi_*\Sigma\mathbf{W}$.

More explicitly, any $\beta \in \pi_k\Sigma\mathbf{W}$ is given as $\beta = \psi^\#(\bar{\iota}_{(\alpha_1)}, \dots, \bar{\iota}_{(\alpha_n)})$ where ψ is some n-ary homotopy operation and $\bar{\iota}_{(\alpha_j)} = \bar{\iota}^{m_j+1}_{(\alpha_j)}$ is as above; then

$$\tilde{\zeta}(\beta) = \psi^\#(\tilde{\zeta}(\bar{\iota}_{(\alpha_1)}), \dots, \tilde{\zeta}(\bar{\iota}_{(\alpha_n)})) = \psi^\#(\bar{\alpha}_1, \dots, \bar{\alpha}_n) = \overline{\psi \star (\alpha_1, \dots, \alpha_n)}. \tag{8.1}$$

In fact, $\tilde{\zeta}$ may be identified with $\tilde{z}_\# : \pi^H_*(J\mathbf{W}) \to \pi^H_*\mathbf{X} \cong G_*$ (using §3.7), where $\tilde{z} : J\mathbf{W} \to \mathbf{X}$ is the H-map $\bar{m} \circ J(z)$ of §3.1. Thus $\tilde{\zeta}(\beta) = \overline{\beta \star \zeta}$, where ζ denotes the homotopy class of $z : \mathbf{W} \to \mathbf{X}$.

Note that $\tilde{\zeta} : \pi_*\Sigma\mathbf{W} \twoheadrightarrow G_*$: is also surjective, since for every $\gamma \in \pi_n\mathbf{X}$ there is an $\omega \in \pi_n\mathbf{S}^k$ such that $\gamma = \omega^\#(z_*\iota^k_{(\alpha)})$ for some $\mathbf{S}^k_{(\alpha)} \hookrightarrow \mathbf{W}$ (ω is unary since $\mathbf{X}$ is an H-space), and then

$$\bar{\gamma} = \overline{\omega^\#(z_\#\iota^k_{(\alpha)})} = \overline{(\Sigma\omega) \star (z_\#\iota^k_{(\alpha)})} = (\Sigma\omega)^\# \overline{z_\#\iota^k_{(\alpha)}} = (\Sigma\omega)^\#\bar{\alpha} = \tilde{\zeta}((\Sigma\omega)^\#\bar{\iota}^{k+1}_{(\alpha)})$$

by Fact 3.4 and Corollary 3.9.

(3) Set $A_0 = \pi_*\Sigma\mathbf{W} \in \Pi$-$Alg$, and for each $n \geq 1$ let $A_n = T^nA_0$. Define face maps $d_i = d^n_i : A_n \to A_{n-1}$ by $d^n_i = T^i(\varepsilon_{T^{n-i}})$, and degeneracies $s_j = s^n_j : A_n \to A_{n+1}$ by $s^n_j = T^j(\vartheta_{T^{n-1-j}})$ for $0 \leq i, j \leq n-1$ (compare [Go, App., §3]), with $s^n_n = T^n\bar{\vartheta}$ for $\bar{\vartheta} : A_0 \to TA_0$ as in Definition 2.5.

(4) We define a morphism of Π-algebras $\bar{d} : A_1 \to A_0$ by letting $\bar{d}(\iota^k_{(\beta)}) = \bar{\iota}^k_{(\beta\star\zeta)}$ for each $\beta \in D_kA_0$ – that is, if $\beta = \psi^\#(\bar{\iota}_{(\alpha_1)}, \dots, \bar{\iota}_{(\alpha_n)})$ as above, then by (8.1) we have

$$\bar{d}(\iota^k_{(\beta)}) = \iota^k_{(\beta\star\zeta)} = \iota^k_{(\psi\star(\alpha_1,\dots,\alpha_n))}. \tag{8.2}$$

We then set $d^n_n : A_n \to A_{n-1}$ to be $T^{n-1}\bar{d}$ for all $n \geq 1$.

Lemma 8.3. $A_\bullet = \langle (A_n)^\infty_{n=0}, (d^n_i), (s^n_j) \rangle \xrightarrow{\tilde{\zeta}} G_*$ *is an augmented simplicial* Π-*algebra.*

Proof. All the simplicial identities, except for those involving d^n_n, follow as usual from the identities for the comonad $\langle T, \varepsilon, \vartheta\rangle$ (cf. [Go, App., §3] or [BoK, I, §4.1]).

Now let $\iota^k_{(\gamma)}$ be a generator for a coproduct summand $\pi_*\mathbf{S}^k_{(\gamma)}$ in $A_2 = TA_1$, where $\gamma \in D_k\mathbf{A}_1$. Then $(T\varepsilon_{A_0})\iota^k_{(\gamma)} = \iota^k_{(\varepsilon_{A_0}\circ\gamma)}$, and thus $(\bar{d}\circ T\varepsilon_{A_0})\iota^k_{(\gamma)} = \iota^k_{((\varepsilon_{A_0}\circ\gamma)\star\zeta)} \in D_kA_0$, where $(\varepsilon_{A_0}\circ\gamma)\star\zeta$ is in D_kG_*.

Write $\gamma = \omega^{\#}(\iota_{\beta_1},\dots,\iota_{\beta_m})$ for some m-ary homotopy operation $\omega^{\#}$ and $\iota_{\beta_i} = \iota^{\ell_i}_{\beta_i}$ a generator of a summand $\pi_*\mathbf{S}^{\ell_i}_{\beta_i}$ in $A_1 = TA_0$, where $\beta_i \in D_{\ell_i}A_0$ $(i = 1,\dots,m)$. Then $\varepsilon_{A_0}\circ\gamma = \omega^{\#}(\beta_1,\dots,\beta_m)$.

Therefore, if we write each β_i $(i = 1,\dots,m)$ as $\beta_i = \psi_i^{\#}(\bar{\iota}_{(\alpha_1)},\dots,\bar{\iota}_{(\alpha_n)})$ (for $\bar{\iota}_{(\alpha_j)}$ as in (2)), then $\varepsilon_{A_0}\circ\gamma = (\omega^{\#}(\psi_1,\dots,\psi_n))^{\#}(\bar{\iota}_{(\alpha_1)},\dots,\bar{\iota}_{(\alpha_n)})$, so

$$\begin{aligned}(\varepsilon_{A_0}\circ\gamma)\star\zeta &= ((\omega^{\#}(\psi_1,\dots,\psi_n))^{\#}(\bar{\iota}_{(\alpha_1)}\vee\dots\vee\bar{\iota}_{(\alpha_n)}))\star\zeta\\ &= (\omega^{\#}(\psi_1,\dots,\psi_n))\star(\alpha_1,\dots,\alpha_n) = \omega\star((\psi_1,\dots,\psi_n))\star(\alpha_1,\dots,\alpha_n))\end{aligned}$$

by (8.2), and thus

$$(8.4)\qquad (\bar{d}\circ T\varepsilon_{A_0})\iota^k_{(\gamma)} = \iota^k_{(\omega\star((\psi_1,\dots,\psi_n))\star(\alpha_1,\dots,\alpha_n))} \in D_kA_0.$$

On the other hand $(T\bar{d})\iota^k_{(\gamma)} = \iota^k_{(\bar{d}\gamma)}$ by definition of T, and

$$\bar{d}\gamma = \bar{d}(\omega^{\#}(\iota_{\beta_1},\dots,\iota_{\beta_m})) = \omega^{\#}(\bar{d}\iota_{(\beta_1)},\dots,\bar{d}\iota_{(\beta_m)})) = \omega^{\#}(\iota_{(\beta_1\star\zeta)},\dots,\iota_{(\beta_m\star\zeta)})$$

since $\bar{d}$ is a morphism of Π-algebras.

Thus $(\bar{d}\circ T\bar{d})\iota^k_{(\gamma)} = \bar{d}(\iota^k_{(\omega^{\#}(\iota_{(\beta_1\star\zeta)},\dots,\iota_{(\beta_m\star\zeta)}))}) = \iota^k_{((\omega^{\#}(\iota_{(\beta_1\star\zeta)},\dots,\iota_{(\beta_m\star\zeta)}))\star\zeta)}$ again by (8.2). Since $\beta_i = \psi_i^{\#}(\bar{\iota}_{(\alpha_1)},\dots,\bar{\iota}_{(\alpha_n)})$ for $i = 1,\dots,m$, we have $\beta_i\star\zeta = \psi_i\star(\alpha_1,\dots,\alpha_n)$, so

$$\begin{aligned}(\omega^{\#}(\iota_{(\beta_1\star\zeta)},\dots,\iota_{(\beta_m\star\zeta)}))\star\zeta &= \omega^{\#}(\psi_1\star(\alpha_1,\dots,\alpha_n),\dots,\psi_m\star(\alpha_1,\dots,\alpha_n))\\ &= \omega\star((\psi_1\vee\dots\vee\psi_m)\star(\alpha_1,\dots,\alpha_n))\end{aligned}$$

by Proposition 3.8, so by (8.4) we see that $\bar{d}\circ T\bar{d} = \bar{d}\circ T\varepsilon : A_2 \to A_0$.

Since $\tilde{\zeta}\circ\varepsilon_{A_0} = \tilde{\zeta}\circ\bar{d}$ and $\bar{d}\circ\bar{\vartheta} = id_{A_0}$ (by Fact 3.4), the remaining simplicial identities follow from the fact that ε and ϑ are natural transformations. □

Lemma 8.5. *The augmented simplicial* Π-*algebra* $A_\bullet \xrightarrow{\tilde{\zeta}} G_*$ *is acyclic.*

Proof. For any $n \geq 0$ we may represent any $x \in \pi_nD_kA_\bullet$ by a normalized cycle $\beta \in D_kA_n$ with $d_i\beta = 0$ for $0 \leq i \leq n$ (cf. [Ma1, §17]), and consider $\iota^k_{(\beta)} \in D_kA_{n+1} = D_kTA_n$:

$d_0\iota^k_{(\beta)} = \beta$, while $d_i\iota^k_{(\beta)} = T^i\varepsilon_{A_{s-i}}\iota^k_{(\beta)} = T(T^{i-1}\varepsilon_{A_{s-i}})\iota^k_{(\beta)} = \iota^k_{(T^{i-1}\varepsilon_{A_{s-i}}\beta)} = \iota^k_{(d_{i-1}\beta)} = \iota^k_0 = 0$ (see Definition 2.5) for $1 \leq i \leq n$. If $n \geq 1$ then also $d_{n+1}\iota^k_{(\beta)} = T(T^{n-1})\bar{d}\iota^k_{(\beta)} = \iota^k_{(T^{n-1}\bar{d}\beta)} = \iota^k_{(d_n\beta)} = \iota^k_0 = 0$, so β is a normalized boundary and thus $x = 0$ in $\pi_nD_kA_\bullet$.

For $n = 0$ by (2) we know $\tilde{\zeta} : A_0 \to G_*$ is surjective. Given $\beta \in D_kKer(\tilde{\zeta}) \subseteq D_kA_0$, we may assume $\beta = \omega^{\#}(\bar{\iota}_{\alpha_1},\dots,\bar{\iota}_{\alpha_n})$, and thus

$$0 = \tilde{\zeta}(\beta) = \tilde{\zeta}(\omega^{\#}(\bar{\iota}_{\alpha_1},\dots,\bar{\iota}_{\alpha_n})) = \omega^{\#}(\tilde{\zeta}(\bar{\iota}_{\alpha_1}),\dots,\tilde{\zeta}(\bar{\iota}_{\alpha_n})) = \omega^{\#}(\alpha_1,\dots,\alpha_n) = \beta\star\zeta$$

by (8.2).

Thus again $d_0\iota^k_{(\beta)} = \beta$, while $d_1\iota^k_{(\beta)} = \bar{d}\iota^k_{(\beta)} = \iota^k_{(\beta\star\zeta)} = \iota^k_0 = 0$, so $[\beta] = 0$ in $\pi_0 D_k A_\bullet$, which shows that $\tilde{\zeta}$ indeed induces an isomorphism $\pi_0 A_\bullet \cong G_*$. $\square$

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