HOMOTOPY OPERATIONS AND THE OBSTRUCTIONS TO BEING AN H-SPACE

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Abstract. We describe an obstruction theory for a given topological space $\boldsymbol{X}$ to be an H-space, in terms of higher homotopy operations, and show how this theory can be used to calculate such operations in certain cases.

1. introduction

An H-space is a topological space $\boldsymbol{X}$ equipped with a multiplication map $m : \boldsymbol{X} \times \boldsymbol{X} \to \boldsymbol{X}$ and an identity element $* \in \boldsymbol{X}$. The question of whether a given space $\boldsymbol{X}$ possesses such an H-space structure has been studied from a variety of viewpoints – cf. [C, SW, St1, Su1, Su2, W1]. Here we address this question from the aspect of homotopy operations.

As is well known, the homotopy groups $\pi_*\boldsymbol{X}$ of any space $\boldsymbol{X}$ have Whitehead products and composition operations defined on them, satisfying certain identities; we summarize this by saying they constitute a Π-*algebra* – see §2.1 below. In addition, there are various higher order operations on $\pi_*\boldsymbol{X}$, such as Toda brackets; and the totality of these actually determine the (weak) homotopy type of $\boldsymbol{X}$ (cf. [Bl5, §7.17]): therefore, they should enable us – in theory – to answer any homotopy-theoretic question about $\boldsymbol{X}$, including that of possessing an H-space structure. It is the purpose of this note to explain in what sense this can actually be done, using the obstruction theory for realizing Π-algebra morphisms described in [Bl5, §7]. The approach presented here generalizes a result of Andrews and Arkowitz in rational homotopy theory (see [AA, Prop. 6.9]).

1.1. **notation and conventions.** $\mathcal{T}_*$ will denote the category of pointed connected CW complexes with base-point preserving maps, and by a *space* we shall always mean an object in $\mathcal{T}_*$, which will be denoted by a boldface letter: $\boldsymbol{X}$, $\boldsymbol{S}^n$, and so on. The basepoint will be written $* \in \boldsymbol{X}$. $\boldsymbol{\Delta}[n]$ is the standard topological n-simplex in $\mathbb{R}^{n+1}$. The homotopy category of $\mathcal{T}_*$ is denoted by $ho\mathcal{T}_*$, and $[\boldsymbol{X}, \boldsymbol{Y}]$ means the set of pointed homotopy classes of maps $\boldsymbol{X} \to \boldsymbol{Y}$. The constant pointed map will be written c_*, or simply $*$.

$\mathcal{A}b$ denotes the category of abelian groups, and $gr\mathcal{A}b$ the category of graded abelian groups.

Definition 1.2. A *strict H-space structure* for a space $\boldsymbol{X} \in \mathcal{T}_*$ is a choice of a *multiplication map* $m : \boldsymbol{X} \times \boldsymbol{X} \to \boldsymbol{X}$, making the following diagram commute:

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$$\begin{tikzcd} \boldsymbol{X} \vee \boldsymbol{X} \arrow[d, "i"'] \arrow[dr, "\nabla"] & \\ \boldsymbol{X} \times \boldsymbol{X} \arrow[r, dotted, "m"] & \boldsymbol{X} \end{tikzcd}$$

where $i = \langle i_1, i_2 \rangle : \boldsymbol{X} \vee \boldsymbol{X} \to \boldsymbol{X} \times \boldsymbol{X}$ is the inclusion of the wedge in the product and $\nabla : \boldsymbol{X} \vee \boldsymbol{X} \to \boldsymbol{X}$ is the fold map (induced by the identity on each wedge summand). If $\boldsymbol{X}$ may be equipped with such an m, we say that it is a *strict H-space*.

We say $\boldsymbol{X}$ is an *H-space* if there is a map $m : \boldsymbol{X} \times \boldsymbol{X} \to \boldsymbol{X}$ making the above diagram commute up to homotopy. Since $\boldsymbol{X}$ is assumed to be well-pointed, it is then homotopy equivalent to a strict H-space (cf. [St2, §1]), so the two definitions coincide in the homotopy category.

1.3. **organization:** In section 2 we review some background on Π-algebras, and reformulate the question of the existence of an H-space structure on $\boldsymbol{X}$ in terms of Π-algebras (Proposition 2.7). In section 3 we give some more background on simplicial Π-algebras and spaces, together with a computational example. In section 4 we describe the higher order homotopy operations which serve as obstructions to realizing Π-algebra morphisms, and thus to obtaining an H-space structure on $\boldsymbol{X}$, and formulate our main result, namely:

Theorem 4.18: *A space $\boldsymbol{X}$ has an H-space structure if and only if*

(a) $\pi_*\boldsymbol{X}$ is an abelian Π-algebra (Def. 2.4); and

(b) The sequence of higher homotopy operations $\{\langle\langle n \rangle\rangle\}_{n=2}^{\infty} \subseteq \{\pi_j \boldsymbol{X}\}_{j=n}^{\infty}$ associated to the Π-algebra morphism $\mu : \pi_*\boldsymbol{X} \times \pi_*\boldsymbol{X} \to \pi_*\boldsymbol{X}$ (defined in §4.11 and 4.16) vanishes coherently.

We then use this theorem to make a sample calculation for $\mathbb{C}P^2$ (§4.19). In section 5 we describe the *torsion Whitehead product*, which may be thought of as "the first higher order obstruction" to being an H-space, and give another example (§5.10).

2. Π-algebras

In this section we recall some known facts on the primary homotopy operations and their relation to the H-space question. First, some definitions and notation:

Definition 2.1. A Π-*algebra* is a graded group $G_* = \{G_k\}_{k=1}^{\infty}$ (abelian in degrees > 1), together with an action on G_* of the primary homotopy operations (i.e., compositions and Whitehead products, including the "π_1-action" of G_1 on the higher G_n's, as in [W, X, §7]), satisfying the usual universal identities. See [Bl3, §3] or [Bl1, §2.1] for a more explicit description. A *morphism of* Π-*algebras* is a homomorphism of graded groups $\phi : G_* \to G'_*$ which commutes with all the operations. Π-algebras form a category, which will be denoted Π-*Alg*.

Definition 2.2. We say that a space $\boldsymbol{X}$ *realizes* an (abstract) Π-algebra G_* if there is an isomorphism of Π-algebras $G_* \cong \pi_*\boldsymbol{X}$. (There may be non-homotopy equivalent spaces realizing the same Π-algebra – cf. [Bl5, §7.18]). Similarly, an abstract morphism of Π-algebras $\phi : \pi_*\boldsymbol{X} \to \pi_*\boldsymbol{Y}$ (between realizable Π-algebras) is *realizable* if there is a map $f : \boldsymbol{X} \to \boldsymbol{Y}$ such that $\pi_* f = \phi$.

Definition 2.3. The *free* Π-algebras are those isomorphic to $\pi_* \boldsymbol{W}$, for some (possibly infinite) wedge of spheres $\boldsymbol{W}$: More precisely, let T be a graded set $\{T_k\}_{k=1}^{\infty}$, and let $\boldsymbol{W} = \bigvee_{k=1}^{\infty} \bigvee_{x \in T_k} \boldsymbol{S}_x^k$, where each $\boldsymbol{S}_x^k$ is a k-sphere. Then we say that $\pi_* \boldsymbol{W}$ is the *free* Π-*algebra generated by* T. We shall consider each element $x \in T_k$ to be an element of $\pi_* \boldsymbol{W}$, by identifying it with that generator of $\pi_k \boldsymbol{W}$ which represents the inclusion $\boldsymbol{S}_x^k \hookrightarrow \boldsymbol{W}$.

Definition 2.4. An *abelian* Π-algebra is one for which all Whitehead products vanish; these are indeed the abelian objects of Π-Alg – see [Bl1, §2].

Remark 2.5. If we let Π denote the homotopy category of wedges of spheres, and $\mathcal{F} \subset \Pi$-Alg the full subcategory of free Π-algebras, then the functor $\pi_* : \Pi \to \mathcal{F}$ is an equivalence of categories. Thus any Π-algebra morphism $\phi : G_* \to G'_*$ is uniquely realizable, if G_* and G'_* are free Π-algebras – and in fact only G_* need be free.

2.6. **the primary obstruction.** The *primary* condition for $\boldsymbol{X}$ to be an H-space – that is, the condition in terms of the Π-algebra $\pi_* \boldsymbol{X}$ – is simply that this Π-algebra be abelian:

Indeed, if $\boldsymbol{X}$ is an H-space, then all Whitehead products vanish in $\pi_* \boldsymbol{X}$, by [W, X, (7.8)]. On the other hand, any product map $m : \boldsymbol{X} \times \boldsymbol{X} \to \boldsymbol{X}$ induces the group operation

$$\pi_k(m) : \pi_k(\boldsymbol{X} \times \boldsymbol{X}) = \pi_k \boldsymbol{X} \times \pi_k \boldsymbol{X} \to \pi_k \boldsymbol{X}$$

for each $k \geq 1$ (cf. [G, Prop 9.9]). Thus if the map of spaces m exists, the morphism of graded groups $\mu : \pi_* \boldsymbol{X} \times \pi_* \boldsymbol{X} \to \pi_* \boldsymbol{X}$ (which is defined levelwise by the group operation) must be a morphism of Π-algebras; but then for any $\alpha \in \pi_p \boldsymbol{X}$ and $\beta \in \pi_q \boldsymbol{X}$ we have

$$0 = \mu([(\alpha, 0), (0, \beta)]) = [\mu(\alpha, 0), \mu(0, \beta)] = [\alpha, \beta] \in \pi_{p+q-1} \boldsymbol{X}$$

by [W, X, (7.7)], so $\pi_* \boldsymbol{X}$ is abelian. Thus we may summarize the "primary" answer to our question in

Proposition 2.7. *A space* $\boldsymbol{X}$ *has an* H*-space structure if and only if*

(a) $\pi_* \boldsymbol{X}$ *is an abelian* Π*-algebra; and*

(b) The Π*-algebra morphism* $\mu : \pi_* \boldsymbol{X} \times \pi_* \boldsymbol{X} \to \pi_* \boldsymbol{X}$, *defined by the group operation, is realizable* (*Def. 2.2*).

Proof. If μ is realizable by $m : \boldsymbol{X} \times \boldsymbol{X} \to \boldsymbol{X}$, then in particular the maps $f_1 = m \circ i_1 : \boldsymbol{X} \to \boldsymbol{X}$ and $f_2 = m \circ i_2 : \boldsymbol{X} \to \boldsymbol{X}$ realize $\mu \circ (i_j)_\# = id : \pi_* \boldsymbol{X} \to \pi_* \boldsymbol{X}$ $(j = 1, 2)$, so f_1, f_2, are self-homotopy equivalences of $\boldsymbol{X}$, and $\bar{m} = m \circ (f_1^{-1} \times f_2^{-1}) : \boldsymbol{X} \times \boldsymbol{X} \to \boldsymbol{X}$ satisfies $\bar{m} \circ i \simeq \nabla$ and thus $\boldsymbol{X}$ is an H-space by definition 1.2. □

It should be pointed out that $\pi_* \boldsymbol{X}$ being abelian is a necessary condition for $\boldsymbol{X}$ to be an H-space, but it is certainly not sufficient, as there are examples of spaces whose Whitehead products all vanish, but which can be shown (usually by means of their homology) to support no H-space structure – see, for example, [Ag, BJS, BG, C, IKM, P2].

3. Simplicial Π-algebras and spaces

We now recall the background on simplicial Π-algebras and spaces needed to descibe our obstruction theory for the realization of Π-algebra morphism in the next section:

Definition 3.1. We let $\mathbf{\Delta}$ denote the category of ordered sequences $\boldsymbol{n} = \langle 0, 1, \ldots, n\rangle$ ($n \in \mathbb{N}$), with order-preserving maps, and $\mathbf{\Delta}^{op}$ is the opposite category.

A *simplicial object* over any category $\mathcal{C}$ is a functor $X : \mathbf{\Delta}^{op} \to \mathcal{C}$ – i.e., a sequence of objects $\{X_n\}_{n=0}^{\infty}$ in $\mathcal{C}$, equipped with *face maps* $d_i : X_n \to X_{n-1}$ and *degeneracies* $s_j : X_n \to X_{n+1}$, satisfying the simplicial identities ([M, §1.1]):

$$
\begin{array}{lllll}
 & (i) & d_i \circ d_j = & d_{j-1} \circ d_i & \text{for } i < j \\
(3.2) & (ii) & d_i \circ s_j = & \begin{cases} s_{j-1} \circ d_i & \text{if } i < j \\ id & \text{if } i = j, j+1 \\ s_j \circ d_{i-1} & \text{if } i > j+1 \end{cases} & \\
 & (iii) & s_j \circ s_i = & s_i \circ s_{j-1} & \text{for } j > i
\end{array}
$$

We let $s\mathcal{C}$ denote the category of simplicial objects over $\mathcal{C}$.

An *augmented* simplical object $X_\bullet \to A$ over $\mathcal{C}$ is a simplicial object $X_\bullet \in s\mathcal{C}$, together with an augmentation $\varepsilon : X_0 \to A$ in $\mathcal{C}$ such that

$$(iv) \qquad \varepsilon \circ d_0 = \varepsilon \circ d_1$$

Next, we must consider resolutions of a given Π-algebra; for more details, see [Q1, II, §4] or [BS, §2]:

Definition 3.3. A simplicial Π-algebra $A_\bullet$ is called *free* if for each $n \geq 0$ there is a graded set $T^n \subseteq A_n$ such that A_n is the free Π-algebra generated by T^n, and each degeneracy map $s_j : A_n \to A_{n+1}$ takes T^n to T^{n+1}.

Definition 3.4. A *free simplicial resolution* of a Π-algebra G_* is defined to be an augmented simplicial Π-algebra $A_\bullet \to G_*$, such that:

(a) $A_\bullet$ is a free simplicial Π-algebra;
(b) the homotopy groups of the simplicial group $(A_\bullet)_k$ vanish in dimensions $n \geq 1$;
(c) the augmentation induces an isomorphism $\pi_0((A_\bullet)_k) \cong G_k$.

Here $\{(A_n)_k\}_{k=1}^{\infty}$ denotes the underlying graded group of A_n.

Such resolutions always exist, for any Π-algebra G_* – see [Q1, II, §4], or the explicit construction in [Bl3, §4.3].

Example 3.5. Write $S^k_x \cong \pi_* \boldsymbol{S}^k$ for the free Π-algebra generated by a graded set having a single element x in degree k (§2.3), and let $\eta_r^{\#}$ denote the composition with the suspended Hopf map $\eta_r \in \pi_{r+1}\boldsymbol{S}^r$. We may then describe certain free simplicial Π-algebras – namely, those with a CW basis, in the sense of [Bl3, §4.1] – by specifying $\bar{A}_n \subseteq A_n$, which consists of the non-degenerate S^k's in simplicial dimension n, and the face map $d_0|_{\bar{A}_n}$ setting $d_i|_{\bar{A}_n} = 0$ for $i > 0$. A_n is then obtained from $\bar{A}_n$ by adding degeneracies of the free Π-algebras $\bar{A}_k$ ($k < n$), as explained in [Bl3, §4.5.1].

For example, let G_* be a Π-algebra with $G_2 \cong \mathbb{Z}$ and $G_i = 0$ otherwise. In this setup a free simplicial resolution $A_\bullet \to G_* \times G_*$ is given in degrees ≤ 5 by:

- $\bar{A}_0 = S^2_{\alpha^1} \amalg S^2_{\alpha^2}$, with $\varepsilon : (\bar{A}_0)_2 \cong G_2 \times G_2 = \mathbb{Z} \oplus \mathbb{Z}$.
- $\bar{A}_1 = S^3_{\beta^1} \amalg S^3_{\beta^2} \amalg S^3_{\zeta}$, with $d_0|_{S^3_{\beta^i}} = \eta_2^{\#}\alpha^i$ for $i = 1, 2$ and $d_0|_{S^3_{\zeta}} = [\alpha^1, \alpha^2]$.

- $\bar{A}_2 = S^3_{\gamma^1} \amalg S^3_{\gamma^2} \amalg S^4_{\theta} \amalg S^4_{\kappa}$, with $d_0|_{S^3_{\gamma^i}} = [\beta^i, s_0\alpha^i]$ $(i=1,2)$,
$$\begin{aligned} d_0|_{S^4_{\theta}} &= [\zeta, s_0\alpha^1] + [\beta^1, s_0\alpha^2] + \eta_3^{\#}\zeta, \\ d_0|_{S^4_{\kappa}} &= [\zeta, s_0\alpha^2] + [\beta^2, s_0\alpha^1] + \eta_3^{\#}\zeta. \end{aligned}$$
- $\bar{A}_3 = S^5_{\delta^1} \amalg S^5_{\delta^2} \amalg S^5_{\lambda} \amalg S^5_{\mu} \amalg S^5_{\nu}$, with
$$\begin{aligned} d_0|_{S^5_{\delta^i}} &= [\gamma^i, s_1s_0\alpha^1] + [s_0\beta^i, s_1\beta^i] + \eta_4^{\#}\gamma^i \qquad (i=1,2), \\ d_0|_{S^5_{\lambda}} &= [\kappa, s_1s_0\alpha^1] + [\gamma^1, s_1s_0\alpha^2] + [s_1\zeta, s_0\beta^1] - [s_0\zeta, s_1\beta^1], \\ d_0|_{S^5_{\mu}} &= [\theta, s_1s_0\alpha^2] + [\gamma^2, s_1s_0\alpha^1] + [s_1\zeta, s_0\beta^2] - [s_0\zeta, s_1\beta^2] \text{ and} \\ d_0|_{S^5_{\mu}} &= [\theta, s_1s_0\alpha^1] + [\kappa, s_1s_0\alpha^2] + [s_1\zeta, s_0\zeta] + \eta_4^{\#}\theta + \eta_4^{\#}\kappa + [s_0\beta^2, s_1\beta^1] - [s_1\beta^2, s_0\beta^1]. \end{aligned}$$
- $\bar{A}_n = 0$ for $n \geq 4$ (in degrees ≤ 3).

The only non-trivial Π-algebra identity needed to check this is:

$$[\eta_2^{\#}\alpha, \beta] \;=\; \eta_{q+1}^{\#}[\alpha,\beta] \;-\; [[\alpha,\beta],\alpha] \tag{3.6}$$

for $\alpha \in \pi_2\boldsymbol{X}$, $\beta \in \pi_q\boldsymbol{X}$ – which follows from [Ba3, II, (2.4) & (3.4)] and [W, X, (7.14) & (8.1)].

Remark 3.7. Note that if we rationalize the category of simply-connected Π-algebras to obtain the category of graded Lie algebras over $\mathbb{Q}$ (cf. [Q3]), we have a similar description for the resolution of the abelian Lie algebra $K(\mathbb{Q},2)$ in degrees ≤ 5 – all we have to do is replace $\eta_2^{\#}x$ by $1/2[x,x]$, and omit the torsion terms $\eta_r^{\#}x$ for $r \geq 3$.

Definition 3.8. For any Π-algebra G_*, let $I(G_*) \subseteq G_*$ denote the sub-Π-algebra generated by all non-trivial primary homotopy operation (i.e., compositions and Whitehead products). The graded abelian group $Q(G_*) = G_*/I(G_*)$ is called the *module of indecomposables* of G_* (cf. [Bl3, §2]).

If $A_\bullet \to G_*$ is any free simplicial resolution of Π-algebras, and $T : \Pi\text{-}Alg \to \mathcal{A}b$ is any functor into the category of abelian groups, then the *n-th left derived functor* of T applied to G_*, written $(L_nT)G_*$, is defined to be the n-th homotopy group of the simplicial abelian group $TA_\bullet$ (see [Q1, I,§4] or [BS, §2.2.4] for more details). If $\phi : G_* \to G'_*$ is any morphism of Π-algebras, one has an induced morphism $(\phi)_\bullet : A_\bullet \to B_\bullet$ between their respective resolutions, which allows one to define the *relative* n-th derived functor of T applied to $\phi : G_* \to G'_*$, written $(L_nT)\phi$ (cf. [Bl2, §4.1]).

Definition 3.9. In particular, the n-th derived functor of the indecomposables functor $Q : \Pi\text{-}Alg \to gr\mathcal{A}b$ of G_* is called the n-th (graded) *homology module* of G_*, written $H_n(G_*)$ (see [DK, §5.1] for a more general definition). Similarly, if $\phi : G_* \subseteq G'_*$ is an morphism of Π-algebras, we denote $(L_nQ)\phi$ by $H_n(\phi)$ (or simply $H_n(G'_*, G_*)$ if ϕ is evident from the context), and call it the n-th *relative homology module* for $\phi : G_* \to G'_*$.

3.10. **simplicial spaces.** Let $\boldsymbol{W}_\bullet \in s\mathcal{T}_*$ be a simplicial space: its *realization* (or homotopy colimit) is a space $\boldsymbol{X} = \|\boldsymbol{W}_\bullet\|$ constructed by making identifications in $\coprod_{n=0}^{\infty} \boldsymbol{W}_n \times \boldsymbol{\Delta}[n]$ according to the face and degeneracy maps of $\boldsymbol{W}_\bullet$ (cf. [Se, §1]).

For any simplicial space $\mathbf{W}_\bullet$, there is a first quadrant spectral sequence with

$$E^2_{s,t} = \pi_s(\pi_t \mathbf{W}_\bullet) \Rightarrow \pi_{s+t}\|\mathbf{W}_\bullet\| \tag{3.11}$$

(see [BF, Thm B.5] or [Q2]).

In particular, if $\mathbf{W}_\bullet \to \mathbf{X}$ is an augmented simplical space such that $\pi_*\mathbf{W}_\bullet \to \pi_*\mathbf{X}$ is a free simplicial resolution of Π-algebras (Def. 3.4), we see that the natural map $\mathbf{W}_0 \to \|\mathbf{W}_\bullet\|$ (cf. [BK, XII, 2.3]) induces an isomorphism $\pi_*\mathbf{X} \cong \pi_*\|\mathbf{W}_\bullet\|$, so $\|\mathbf{W}_\bullet\| \simeq \mathbf{X}$.

We shall assume all our simplicial spaces – i.e., objects in $s\mathcal{T}_*$ – are *proper*, in the sense that the degeneracy maps are inclusions of subcomplexes (so in particular cofibrations).

Example 3.12. Let $\mathbf{X}$ be any space with $\pi_2\mathbf{X} \cong \mathbb{Z}$ and $\pi_1\mathbf{X} = \pi_3\mathbf{X} = 0$ (e.g., $\mathbf{X} = K(\mathbb{Z}, 2)$).

Let $\iota_{X^1}, \iota_{X^2} : \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ denote the two inclusions, respectively, and assume that we have chosen once and for all fixed representatives $\alpha : \mathbf{S}^2 \to \mathbf{X}$ for a generator of $\pi_2\mathbf{X}$ (so that $\alpha^i = \iota_{X^i} \circ \alpha$ $(i = 1, 2)$ represent the two generators for $\pi_2(\mathbf{X} \times \mathbf{X})$); similarly, representatives $\eta_2 : \mathbf{S}^3 \to \mathbf{S}^2$ for the Hopf map, (with $\eta_3 : \mathbf{S}^4 \to \mathbf{S}^3$ its suspension), and $w = [\iota_1, \iota_2] : \mathbf{S}^3 \to \mathbf{S}^2 \vee \mathbf{S}^2$ for the Whitehead product map.

In addition, choose nullhomotopies $H : \mathbf{e}^3 \to \mathbf{X}$ of $\alpha \circ \eta_2$ and $G : \mathbf{e}^4 \to \mathbf{X} \times \mathbf{X}$ of $[\alpha^1, \alpha^2]$ in $\mathbf{X} \times \mathbf{X}$ (see [W, X, (7.7)]). If ι^2_a, ι^2_b respectively denote the two inclusions $\mathbf{S}^2 \hookrightarrow \mathbf{S}^2_a \vee \mathbf{S}^2_b$, we may define a map $k : \mathbf{S}^3 \to \mathbf{S}^2_a \vee \mathbf{S}^2_b$ by

$$k \;=\; [\iota^2_b, \iota^2_a] \circ \eta_3 + [\iota^2_a \circ \eta_2, \iota^2_b] + [[\iota^2_b, \iota^2_a], \iota^2_a],$$

and let $K : \mathbf{e}^4 \to \mathbf{S}^2_a \vee \mathbf{S}^2_b$ be some nullhomotopy of k (which exists by (3.6)).

Now set $\mathbf{V} = \mathbf{S}^2_a \vee \mathbf{S}^2_b \vee \mathbf{e}^4_c \vee \mathbf{e}^4_d$, and let $\iota^2_a : \mathbf{S}^2_b \hookrightarrow \mathbf{V}$, $\iota^2_b : \mathbf{S}^2_b \hookrightarrow \mathbf{V}$, $\iota^3_c : \partial\mathbf{e}^4_c \hookrightarrow \mathbf{V}$, and $\iota^3_d : \partial\mathbf{e}^4_d \hookrightarrow \mathbf{V}$ be the inclusions. Then we may define $\ell : \mathbf{S}^4 \to \mathbf{V}$ by $\ell = \iota^3_c \circ \eta_3 + [\iota^3_d, \iota^2_b] + [\iota^3_c, \iota^2_a]$, and let $L : \mathbf{e}^5 \to \mathbf{V}$ denote a nullhomotopy of ℓ, which exists by [W, X (7.2)].

We now define an augmented simplicial space $\mathbf{W}_\bullet \to \mathbf{X} \times \mathbf{X}$ (see [Bl4, §4.2] for an explanation of the notation, and compare Example 3.5 above): we adopt the convention that $\iota_f : \mathbf{S}^r \to \mathbf{S}^r_f \hookrightarrow \mathbf{W}_n$ denotes a homeomorphism of the r-sphere onto the wedge summand $\mathbf{S}^r_f$ in $\mathbf{W}_n$ $(n \geq 0)$, $\xi_F : \mathbf{e}^r \hookrightarrow \mathbf{e}^r_F \hookrightarrow \mathbf{W}_n$ a homeomorphism onto the $(r+1)$-disc $\mathbf{e}^{r+1}_F$, and $\iota_F : \mathbf{S}^r \hookrightarrow \mathbf{e}^{r+1}_F \hookrightarrow \mathbf{W}_n$ a fixed embedding into the $(r+1)$-disc $\mathbf{e}^{r+1}_F$. Then $\mathbf{W}_\bullet$ is defined by:

- $\mathbf{W}_0 = \mathbf{S}^2_{\alpha^1} \vee \mathbf{S}^2_{\alpha^2} \vee \mathbf{e}^4_{H^1} \vee \mathbf{e}^4_G \vee \mathbf{e}^5_{K^1} \cup_{\mathbf{S}^4_\varphi} \mathbf{e}^5_{L^1}$, where $\mathbf{e}^5 \cup_{\mathbf{S}^4} \mathbf{e}^5$ denotes the pushout of $\mathbf{e}^5 \hookleftarrow \mathbf{S}^4 \hookrightarrow \mathbf{e}^5$ (homeomorphic to $\mathbf{S}^5$).

The augmentation $\varepsilon : \mathbf{W}_0 \to \mathbf{X} \times \mathbf{X}$ is defined by $\varepsilon|_{\mathbf{S}^2_{\alpha^i}} = \alpha^i = \iota_{X^i} \circ \alpha$ for $i = 1, 2$; $\varepsilon|_{\mathbf{e}^4_{H^1}} = \iota_{X^1} \circ H$, $\varepsilon|_{\mathbf{e}^4_G} = G$, $\varepsilon|_{\mathbf{S}^4_\varphi} = \varphi = (\alpha^1 \vee \alpha^2) \circ k$, $\varepsilon|_{\mathbf{e}^5_{K^1}} = K^1 = (\alpha^1 \vee \alpha^2) \circ K$, and $\varepsilon|_{\mathbf{e}^5_{L^1}} = L^1 = (\alpha^1 \vee \alpha^2 \vee H^1 \vee G) \circ L$.

Note that $(\alpha^1 \vee \alpha^2 \vee H^1 \vee G) \circ \ell = (\alpha^1 \vee \alpha^2) \circ k$, so ε is well-defined on $\mathbf{e}^5_{K^1} \cup_{\mathbf{S}^4_{\varphi^1}} \mathbf{e}^5_{L^1} \cong \mathbf{S}^5_\psi$. Moreover, it is not hard to see (by considering each factor of $\mathbf{X} \times \mathbf{X}$

separately) that $\psi = \varepsilon|_{\boldsymbol{S}^5_\psi}$ is nullhomotopic, so that in fact we could further embed $\boldsymbol{S}^5_\psi \hookrightarrow \boldsymbol{e}^6_\Psi \hookrightarrow \boldsymbol{W}_0$ and extend $\varepsilon|_{\boldsymbol{S}^5_\psi}$ to a nullhomotopy $\Psi : \boldsymbol{e}^6_\Psi \to \boldsymbol{X} \times \boldsymbol{X}$.

• $\boldsymbol{W}_1 = s_0\boldsymbol{W}_0 \vee \boldsymbol{S}^3_{\beta^1} \vee \boldsymbol{S}^3_\zeta \vee \boldsymbol{e}^5_K \vee \boldsymbol{e}^5_L$, where $s_0\boldsymbol{W}_0$ is a copy of $\boldsymbol{W}_0$, and $s_0 : \boldsymbol{W}_0 \to \boldsymbol{W}_1$ is a homeomorphism onto this copy (so $d_0|_{s_0\boldsymbol{W}_0} = d_1|_{s_0\boldsymbol{W}_0}$ are both equal to the inverse homeomorphism, by (3.2)(ii)). The face maps on the rest of $\boldsymbol{W}_0$ are given by:

– $d_0|_{\boldsymbol{S}^3_{\beta^1}} = \iota_{\alpha^1} \circ \eta_2$ and $d_1|_{\boldsymbol{S}^3_{\beta^1}} = \iota_{H^1}$.
– $d_0|_{\boldsymbol{S}^3_\zeta} : \boldsymbol{S}^3_\zeta \to \boldsymbol{S}^2_{\alpha^1} \vee \boldsymbol{S}^2_{\alpha^2}$ is $[\iota_{\alpha^2}, \iota_{\alpha^1}]$ and $d_1|_{\boldsymbol{S}^3_\zeta} : \boldsymbol{S}^3_\zeta \cong \partial\boldsymbol{e}^4_G$;
– $d_0|_{\boldsymbol{e}^5_K}$ is the nullhomotopy $(\iota_{\alpha^1} \vee \iota_{\alpha^2}) \circ K$ of $[\iota_{\alpha^2}, \iota_{\alpha^1}] \circ \eta_3 + [\iota_{\alpha^1} \circ \eta_2, \iota_{\alpha^2}] + [[\iota_{\alpha^2}, \iota_{\alpha^1}], \iota_{\alpha^1}]$, and $d_1|_{\boldsymbol{e}^5_K} = \xi_{K^1}$;
– $d_0|_{\boldsymbol{e}^5_L}$ is the nullhomotopy $(\iota_{\alpha^1} \vee \iota_{\alpha^2} \vee \xi_{H^1} \vee \xi_G) \circ L$ of $\iota_G \circ \eta_3 + [\iota_{H^1}, \iota_{\alpha^2}] + [\iota_G, \iota_{\alpha^1}]$, and $d_1|_{\boldsymbol{e}^5_L} = \xi_{L^1}$.

• $\boldsymbol{W}_2 = \boldsymbol{S}^4_\theta \vee s\boldsymbol{W}_1$, where $s\boldsymbol{W}_1 = s_0\boldsymbol{W}_1 \cup_{s_1 s_0 \boldsymbol{W}_0} s_1\boldsymbol{W}_1$ is the "degenerate part" of $\boldsymbol{W}_2$, the union of the images of $s_0 : \boldsymbol{W}_1 \to \boldsymbol{W}_2$ and $s_1 : \boldsymbol{W}_1 \to \boldsymbol{W}_2$ (with the identifications forced by the identity (3.2)(iii)), so that $d_i|_{s\boldsymbol{W}_1}$ for $i = 0, 1, 2$ is determined by (3.2)(ii).

$d_0|_{\boldsymbol{S}^4_\theta} = \iota_\zeta \circ \eta_3 + [\iota_{\beta^1}, \iota_{s_0\alpha^2}] + [\iota_\zeta, \iota_{s_0\alpha^1}]$, $d_1|_{\boldsymbol{S}^4_\theta} = \iota_K$, and $d_2|_{\boldsymbol{S}^4_\theta} = \iota_L$.

• For $n \geq 3$ $\boldsymbol{W}_n = s\boldsymbol{W}_{n-1}$ is defined as above by (3.2)(iii), and the face maps on $\boldsymbol{W}_n$ are thus determined by (3.2)(ii) – see [Bl3, §4.5.1] or [M, p. 95(i)].

4. Obstructions to realizing Π-algebra morphisms

We now recall the obstruction theory for the realization of Π-algebra morphisms defined in [Bl5, §7].

4.1. **realizing Π-algebra morphisms.** Given two spaces $\boldsymbol{Y}$, $\boldsymbol{X}$ and a Π-algebra morphism $\phi : \pi_*\boldsymbol{Y} \to \pi_*\boldsymbol{X}$ which we wish to realize – in our case $\boldsymbol{Y} = \boldsymbol{X} \times \boldsymbol{X}$ and $\phi = \mu$ – we proceed as follows:

(a) Choose any augmented simplicial space $\boldsymbol{V}_\bullet \to \boldsymbol{Y}$ such that $\pi_*\boldsymbol{V}_\bullet \to \pi_*\boldsymbol{Y}$ is a free simplicial resolution of Π-algebras. This may be called a simplicial *resolution of $\boldsymbol{Y}$ by wedges of spheres*; in particular, each space $\boldsymbol{V}_n \simeq \bigvee_{k=1}^\infty \bigvee_{x \in T_{n,k}} \boldsymbol{S}^k_x$ (cf. §2.3).

See [Stv1, §1] for a functorial construction of such resolutions. According to [Stv2], *any* free simplicial Π-algebra resolution $A_\bullet \to \pi_*\boldsymbol{Y}$ may be realized topologically by such a simplical space $\boldsymbol{V}_\bullet$, in the sense that $\pi_*\boldsymbol{V}_\bullet \cong A_\bullet$.

(b) By 3.2(i)&(iv), all the compositions $\varepsilon \circ d_{i_1} \circ \dots d_{i_n} : \boldsymbol{n} \to \boldsymbol{-1}$ $(0 \leq i_j \leq j$, $j = 1, \dots, n)$ are equal in $\boldsymbol{\Delta}^{op} \cup \{\boldsymbol{-1}\}$ (i.e., for any augmented simplicial object). Thus we get a unique Π-algebra morphism

$$\psi_n = \phi \circ (\varepsilon_\#) \circ (d_{i_1})_\# \circ \dots (d_{i_n})_\# : \pi_*\boldsymbol{V}_n \to \pi_*\boldsymbol{X}$$

for each $n \geq 0$ (where $(d_{i_j})_\# : \pi_*\boldsymbol{V}_j \to \boldsymbol{V}_{j-1}$ is just the i_j-th face map of the simplicial Π-algebra $\pi_*\boldsymbol{V}_\bullet$).

(c) Since by assumption $\pi_*\boldsymbol{V}_n$ is a free Π-algebra, each $\psi_n : \pi_*\boldsymbol{V}_n \to \pi_*\boldsymbol{X}$ is realizable by a map $g_n : \boldsymbol{V}_n \to \boldsymbol{X}$, unique up to homotopy (see Remark 2.5). Since $\psi_{n-1} \circ (d_i)_\# = \psi_n : \pi_*\boldsymbol{V}_n \to \pi_*\boldsymbol{X}$, we have

$$g_{n-1} \circ d_i \sim g_n : \boldsymbol{V}_n \to \boldsymbol{X} \qquad \text{for all } 0 \leq i \leq n. \tag{4.1}$$

and in fact we may set $g_n = g_{n-1} \circ d_0$ inductively for all $n \geq 1$, say.

(d) If we had equality $g_{n-1} \circ d_i = g_n$ in (4.1) for all n, i, then $\{g_n\}_{n=0}^\infty$ would induce a map of simplicial spaces $\hat{g}_\bullet : \boldsymbol{V}_\bullet \to c(\boldsymbol{X})_\bullet$ (where $c(\boldsymbol{X})_\bullet$ is the constant simplicial space which has $\boldsymbol{X}$ in each simplicial dimension, and all face and degeneracy maps equal to the identity).

But since by construction $\|\boldsymbol{V}_\bullet\| \simeq \boldsymbol{Y}$, from the spectral sequence of (3.11) we would see that

$$\|\hat{g}_\bullet\| : \|\boldsymbol{V}_\bullet\| \longrightarrow \|c(\boldsymbol{X})_\bullet\| \cong \boldsymbol{X}$$

realizes $\phi : \pi_*\boldsymbol{Y} \to \pi_*\boldsymbol{X}$.

Remark 4.2. We thus have a homotopy-commutative diagram consisting of the strict simplicial space $\boldsymbol{V}_\bullet$ and the space $\boldsymbol{X}$ together with the maps $g_n : \boldsymbol{V}_n \to \boldsymbol{X}$ satisfying (4.1) – in other words, an augmented simplicial object $\boldsymbol{V}_\bullet \to \boldsymbol{X}$ in $ho\mathcal{T}_*$ – and we see that the question of realizing ϕ is reduced to that of *rectifying* this homotopy-commutative diagram: that is, replacing it by a strictly commutative one, or equivalently, by an ∞-homotopy commutative diagram (cf. [BV, Cor. 4.21 & Thm. 4.49] and [DKS, §4]).

As shown in [Bl5, §7], there is a sequence of higher homotopy operations which serve as obstructions to such a rectification, which may be described in simplicial terms, as follows:

Definition 4.3. For each pair of integers k, n $(0 < k \leq n)$, let

$$D(k,n) \stackrel{Def}{=} \{0,1,\dots,k\} \times \{0,1,\dots,k,k+1\} \times \dots \times \{0,1,\dots,n\}$$

where we think of $(i_k,\dots,i_n)$ as corresponding to the composition of face maps

$$d_{i_k} \circ \dots \circ d_{i_n} : \boldsymbol{n} \to \boldsymbol{k-1} \qquad \text{in } \boldsymbol{\Delta}^{op}.$$

We set $D(n+1,n) = \{\emptyset\}$ (where $\emptyset$, the empty sequence, corresponds to $id : \boldsymbol{n} \to \boldsymbol{n}$).

There is an equivalence relation $\sim$ on $D(k,n)$, generated by

$$(i_k,\dots,i_j,i_{j+1},\dots,i_n) \sim (i_k,\dots,i_{j+1}-1,i_j,\dots,i_n) \qquad \text{if} \quad i_j < i_{j+1} \tag{4.4}$$

(that is, $(i_k,\dots,i_n) \sim (j_k,\dots,j_n)$ if the corresponding morphisms in $\boldsymbol{\Delta}^{op}$ are equal: $d_{i_k}d_{i_{k+1}}\dots d_{i_n} = d_{j_k}d_{j_{k+1}}\dots d_{j_n}$ – cf. (3.2).

We call an equivalence class $\gamma \in D(j,n)/\sim$ a *subclass* of $\delta \in D(k,n)/\sim$, written $\gamma \preceq \delta$, if $j \leq k \leq n$ and γ has some representative $(i_j,\dots,i_k,\dots i_n)$ such that $\delta = [(i_k,\dots,i_m)]$ (so in particular $\gamma \preceq \emptyset$ for every γ). This representative is not unique, but the identities (3.2) imply that the correspondence $(i_j,\dots,i_k,\dots i_n) \mapsto (i_j,\dots i_{k-1})$ induces, for each $\delta \in D(k,n)/\sim$ and $j \leq k$ a well-defined function

$$\Phi_j^\delta : \{\gamma \in D(j,n)/\sim \ \mid \gamma \preceq \delta\} \to D(j,k-1)/\sim \tag{4.5}$$

which is readily seen to be a bijection.

Definition 4.6. Let us define an abstract polyhedron $D(n)$ having a k-dimensional facet $D(k)^\gamma$ for each equivalence class $\gamma \in D(k+1,n)/\sim$ $(0 \leq k \leq n)$, where the j-facet $D(j)^\gamma$ corresponding to $\gamma \in D(j,n)/\sim$ belongs to the k-facet $D(k)^\delta$ corresponding to $\delta \in D(k,n)/\sim$ if and only if $\gamma \preceq \delta$.

The bijections Φ_j^δ of (4.5) imply (by induction on $n \geq 0$) that $D(n)$ is just an n-simplex, with a specified labeling of its sub-simplices, and provide canonical identifications

$$\varphi^\delta : D(k)^\delta \xrightarrow{\cong} D(k) \tag{4.7}$$

(In particular, $D(n)$ has $n+1$ vertices, corresponding to the $n+1$ possible composite face maps $d_{i_1} \circ \ldots \circ d_{i_n} : \boldsymbol{n} \to \boldsymbol{0}$ in $\boldsymbol{\Delta}^{op}$.)

We choose a geometric realization $\cong \boldsymbol{\Delta}[n]$ for each $D(n)$, which by abuse of notation we also denote by $D(n)$; its boundary $\partial D(n)$ is homeomorphic to $\boldsymbol{S}^{n-1}$.

Example 4.8. If we represent each vertex of $D(3)$ by a cluster of the six sequences of length 3 representing its equivalence class in $D(1,3)/\sim$, and each side in $D(2,3)/\sim$ by the pair of sequences representing it, we may depict $D(3)$ as in Figure 1. (From this depiction one sees that $D(3)$ can be thought of as a collapsed 3-dimensional permutohedron – compare [Bl5, §4]. See Figure 2 below for a depiction of $D(2)$).

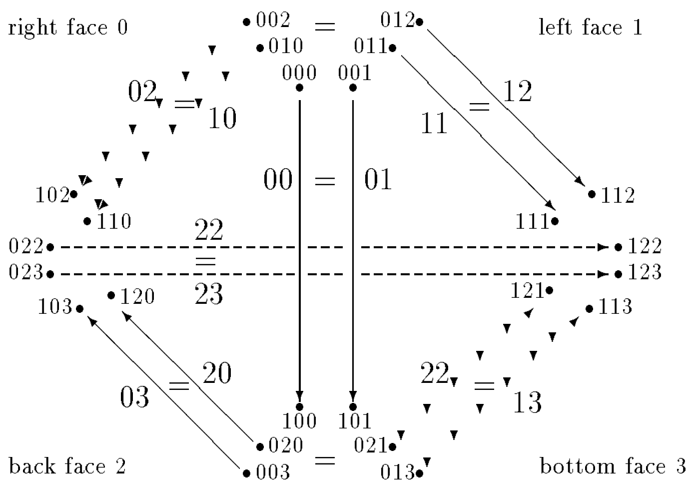

FIGURE 1. Depiction of $D(3)$

Remark 4.9. By comparing the descriptions of $D(n)$ and the "face-map polyhedra" (or permutohedra) $P_n(\delta)$ of [Bl5, §4.1-4.3], we see that the two constructions are dual to each other in an appropriate sense.

Note also that the "n-lattice polyhedra" $L^n(A,B)$ of [Bl5, §7.5], used to define the obstructions to realizing a Π-algebra morphism, are simply barycentric subdivisions of our n-simplex $D(n)$, and the comment in [Bl5, §4.10] applies here, too – so in fact the description of [Bl5, §7] was needlessly complicated.

As with other sequences of convex polyhedra (cf., e.g. [Bl5, §§4,7], [St2, §11]), we can associate to $\{D(n)\}_{n=0}^{\infty}$ a sequence of higher homotopy operations:

Definition 4.10. First recall that the *half-smash* of two spaces $\boldsymbol{X}, \boldsymbol{Y}$ is

$$\boldsymbol{X} \ltimes \boldsymbol{Y} \stackrel{Def}{=} (\boldsymbol{X} \times \boldsymbol{Y})/(\boldsymbol{X} \times \{*\}).$$

If $\boldsymbol{Y}$ is a suspension, there is a homotopy equivalence $\boldsymbol{X} \ltimes \boldsymbol{Y} \simeq \boldsymbol{X} \wedge \boldsymbol{Y} \vee \boldsymbol{Y}$.

Now assume given a simplicial space $\boldsymbol{V}_{\bullet}$ which extends to an augmented simplicial object $[g] : \boldsymbol{V}_{\bullet} \to \boldsymbol{X}$ in $ho\mathcal{T}_*$ (as in §4.2). For each $n \in \mathbb{N}$, we define a $\partial D(n)$-*compatible sequence* (for $\boldsymbol{V}_{\bullet} \to \boldsymbol{X}$) to be a set of maps $h^k : D(k) \ltimes \boldsymbol{V}_k \to \boldsymbol{X}$, one for each $0 \leq k < n$, subject to the following requirements:

(a) $h^0 : D(0) \ltimes \boldsymbol{V}_0 \to \boldsymbol{X}$ is in the prescribed homotopy class of the augmentation $[g] \in [\boldsymbol{V}_0, \boldsymbol{X}]$ – and in fact we may assume without loss of generality that $h_0 = g$, for any representative $g : \boldsymbol{V}_0 \to \boldsymbol{X}$ (cf. [Bl5, §5.8]).

(b) Given $\gamma \in D(j,n)/\sim$ and $\delta \in D(k,n)/\sim$ such that $\gamma \subset \delta$, let $i_\delta^\gamma : D(j) \to D(k)$ denote the composite of

$$D(j) \xrightarrow{(\varphi^\gamma)^{-1}} D(j)^\gamma \xrightarrow{i} D(k)^\delta \xrightarrow{\varphi^\delta} D(k)$$

(where φ^γ is the isomorphism of (4.7) and i is the inclusion).

Then we require that

$$h^k \circ (i_\delta^\gamma \ltimes id_{\boldsymbol{V}_n}) = h^j \circ (id_{D(j)} \ltimes d_{\Phi_\gamma^k}) \ : \ D(j) \ltimes V_k \to \boldsymbol{X}.$$

Note that it suffices that this hold for $k = j+1$ – that is, if $\gamma = [(i_j, i_{j+1}, \dots, i_n)]$ and $\delta = [(i_{j+1}, \dots, i_n)]$, then we require that the following diagram commute (in $\mathcal{T}_*$):

$$\begin{tikzcd}
D(j) \ltimes \boldsymbol{V}_{j+1} \arrow[r, "i_\delta^\gamma \ltimes id"] \arrow[d, "id \ltimes d_{i_{j+1}}"'] & D(j+1) \ltimes \boldsymbol{V}_{j+1} \arrow[d, "h^{j+1}"] \\
D(j) \ltimes \boldsymbol{V}_j \arrow[r, "h^{j+1}"'] & \boldsymbol{X}
\end{tikzcd}$$

A sequence of maps $\{h^k : D(k) \ltimes \boldsymbol{V}_k \to \boldsymbol{X}\}_{k=0}^{\infty}$ satisfying conditions (a) and (b) above for all $k \geq 0$ is called a $\partial D(\infty)$-*compatible sequence* for $\boldsymbol{V}_{\bullet} \to \boldsymbol{X}$.

Given such a $\partial D(n)$-compatible sequence $h^k : D(k) \ltimes \boldsymbol{V}_k \to \boldsymbol{X}$ for $\boldsymbol{V}_{\bullet} \to \boldsymbol{X}$, there is an induced map $\bar{h} : \partial D(n) \ltimes \boldsymbol{V}_n \to \boldsymbol{X}$ defined on the "faces" $D(n-1)^\delta \ltimes \boldsymbol{V}_n$ of $\partial D(n) \ltimes \boldsymbol{V}_n \to \boldsymbol{X}$ by: $\bar{h}|_{D(n-1)^\delta} \ \boldsymbol{V}_n = h^{n-1} \circ (id \ltimes d_i)$ (where $\delta = (i) \in D(n-1,n)/\sim = D(n,n-1)$).

The compatibility condition (b) above guarantees that $\bar{h}$ is well-defined.

Definition 4.11. Given a Δ-simplicial space $\boldsymbol{V}_{\bullet}$ augmented $\boldsymbol{V}_{\bullet} \to \boldsymbol{X}$ in $ho\mathcal{T}_*$ as above, the *n-th order homotopy operation* associated to $\boldsymbol{V}_{\bullet} \to \boldsymbol{X}$ is a subset $\langle\langle n \rangle\rangle$ of the track group $[\Sigma^{n-1}\boldsymbol{V}_n, \boldsymbol{X}]$, defined (for $n \geq 2$) as follows:

Let $T_n \subseteq [\partial D(n) \ltimes \boldsymbol{V}_n, \boldsymbol{X}]$ be the set of homotopy classes of maps $h = \bar{h} : \partial D(n) \ltimes \boldsymbol{V}_n \to \boldsymbol{X}$ induced as above by some compatible collection $\{h^k\}_{k=0}^{n-1}$.

Since each $\boldsymbol{V}_n$ is a suspension (up to homotopy), we have a splitting

$$\partial D(n) \ltimes \boldsymbol{V}_n \cong \boldsymbol{S}^{n-1} \ltimes \boldsymbol{V}_n \simeq \boldsymbol{S}^{n-1} \wedge \boldsymbol{V}_n \vee \boldsymbol{V}_n; \tag{4.12}$$

now let $\langle\langle n\rangle\rangle \subseteq [\Sigma^{n-1}\boldsymbol{V}_n, \boldsymbol{X}]$ be the image under the resulting projection of the subset $T_n \subseteq [\partial D(n) \ltimes \boldsymbol{V}_n, \boldsymbol{X}]$.

Note that the projection of a class $[\bar{h}] \in T_n$ on the other summand $[\boldsymbol{V}_n, \boldsymbol{X}]$ coming from the splitting (4.12) is of no interest, since it is just the homotopy class of the map g_n of 4.1(c).

On the other hand, since by assumption 4.1(a) each $\boldsymbol{V}_n \simeq \bigvee_{k=1}^{\infty} \bigvee_{x\in T_{n,k}} \boldsymbol{S}^k_x$ is homotopy equivalent to a wedge of spheres, so is $\Sigma^{n-1}\boldsymbol{V}_n$, so $\langle\langle n\rangle\rangle$ is in fact a collection of subsets of $\bigoplus_{j=n}^{\infty} \pi_j\boldsymbol{X}$, and as such deserves the name of a higher homotopy operation.

Example 4.13. Let $\boldsymbol{W}_\bullet \to \boldsymbol{X} \times \boldsymbol{X}$ be the augmented simplicial space of Example 3.12; then the part of the secondary operation $\langle\langle 2\rangle\rangle$ in $\pi_5\boldsymbol{X}$ corresponding to $\boldsymbol{S}^4_\theta \subseteq \boldsymbol{V}_2$ is obtained from maps $\bar{h} : \partial D(2) \ltimes \boldsymbol{S}^4_\theta \to \boldsymbol{X}$, defined as follows:

- Define $g = g_0 : \boldsymbol{W}_0 = \boldsymbol{S}^2_{\alpha^1} \vee \boldsymbol{S}^2_{\alpha^2} \vee \boldsymbol{e}^4_{H^1} \vee \boldsymbol{e}^4_G \vee \boldsymbol{e}^5_{K^1} \cup_{\boldsymbol{S}^4_\varphi} \boldsymbol{e}^5_{L^1} \to \boldsymbol{X}$ by letting $g|_{\boldsymbol{S}^2_{\alpha^i}} = \alpha$ for $i = 1, 2$, and $g|_{\boldsymbol{e}^4_{H^1} \vee \boldsymbol{e}^4_G \vee \boldsymbol{e}^5_{K^1} \cup_{\boldsymbol{S}^4_\varphi} \boldsymbol{e}^5_{L^1}} = *$.
- We can thus set $h^1 : g \circ d_0|_{\boldsymbol{W}_1} \sim g \circ d_1|_{\boldsymbol{W}_1}$ (cf. §4.10) equal to
  - the constant homotopy on $D(1) \ltimes s_0\boldsymbol{W}_0$ – since $d_0 = d_1$ on $s_0\boldsymbol{W}_0 \subset \boldsymbol{W}_1$.
  - $h^1|_{D(1)\ \boldsymbol{S}^3_{\beta^1}}$ is the homotopy H between $g\circ d_0|_{\boldsymbol{S}^3_{\beta^1}} = \alpha\circ\eta_2$ and $g\circ d_1|_{\boldsymbol{S}^3_{\beta^1}} = *$.
  - $h^1|_{D(1)\ \boldsymbol{S}^3_\zeta}$ is a homotopy G' between $g \circ d_0|_{\boldsymbol{S}^3_\zeta} = [\alpha, \alpha]$ and $g \circ d_1|_{\boldsymbol{S}^3_\zeta} = *$ (which exists because $\pi_*\boldsymbol{X}$ is abelian).
  - $h^1|_{D(1)\ \partial\boldsymbol{e}^5_K}$ is just the homotopy K between $g \circ d_0|_{\partial\boldsymbol{e}^5_K} = [\iota_{\alpha^2}, \iota_{\alpha^1}] \circ \eta_3 + [\iota_{\alpha^1} \circ \eta_2, \iota_{\alpha^2}] + [[\iota_{\alpha^2}, \iota_{\alpha^1}], \iota_{\alpha^1}]$ and $g \circ d_1|_{\partial\boldsymbol{e}^5_K} = *$
  - Similarly, $h^1|_{D(1)\ \partial\boldsymbol{e}^5_L}$ is the nullhomotopy $L : g \circ d_0|_{\partial\boldsymbol{e}^5_L} = \iota_G \circ \eta_3 + [\iota_{H^1}, \iota_{\alpha^2}] + [\iota_G, \iota_{\alpha^1}] \sim *$.
- Now on $\boldsymbol{S}^4_\theta \subset \boldsymbol{V}_2$ we have
  - $g \circ d_0 \circ d_1 = g \circ d_0 \circ d_0 = [\alpha, \alpha] \circ \eta_3 + [\alpha \circ \eta_2, \alpha] + [[\alpha, \alpha], \alpha]$;
  - $g \circ d_0 \circ d_2 = g \circ d_1 \circ d_0 = * \circ \eta_3 + [*, \alpha] + [*, \alpha]$;
  - $g \circ d_1 \circ d_1 = g \circ d_1 \circ d_2 = *$,

while from the description of h^0 we see that on the three copies of $D(1) \ltimes \boldsymbol{S}^4_\theta$ in $\partial D(2) \ltimes \boldsymbol{S}^4_\theta$:

- $h^1 \circ d_1 : g \circ d_0 \circ d_1 \sim g \circ d_1 \circ d_1$ is K,
- $h^1 \circ d_2 : g \circ d_0 \circ d_2 \sim g \circ d_1 \circ d_2$ is L, and
- $h^1 \circ d_0 : g \circ d_0 \circ d_0 \sim g \circ d_1 \circ d_0$ is $G' \circ \eta_3 + [H, \alpha] + [G', \alpha]$.

Thus we have obtained a map $\bar{h} : \partial D(2) \ltimes \boldsymbol{S}^4_\theta \cong \boldsymbol{S}^5 \to \boldsymbol{X}$, representing the part of $\langle\langle 2\rangle\rangle$ corresponding to $\boldsymbol{S}^4_\theta$, depicted in Figure 2:
(Other choices of H, G', K, L may yield other classes $[\bar{h}] \in \pi_5\boldsymbol{X}$; for a discussion of the indeterminacy see [Bl5, §5.10]).

In more familiar terms, we can say that this "Toda bracket" exists because

$$[\alpha, \alpha] \circ \eta_3 + [\alpha \circ \eta_2, \alpha] + [[\alpha, \alpha], \alpha]$$

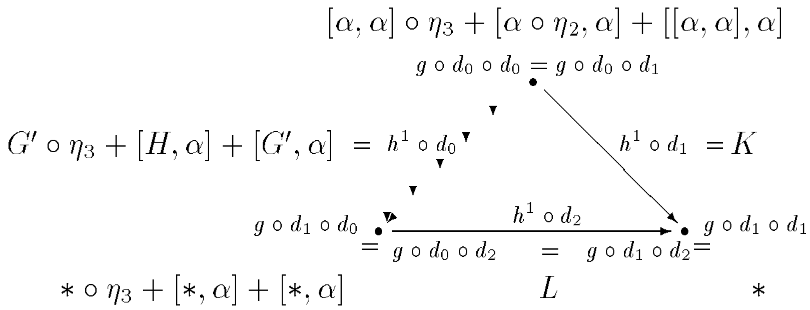

FIGURE 2. The operation $\langle\langle 2\rangle\rangle$ on $\boldsymbol{S}^4_\theta$

vanishes in $\pi_4\boldsymbol{X}$ for two different reasons: the vanishing of $\alpha\circ\eta_2$, as well as all Whitehead products, in $\pi_*\boldsymbol{X}$, combined with the bilinearity of the Whitehead product; and the identity (3.6) (which we have specialized here to the case $\alpha=\beta$ only for simplicity of the presentation).

In general, each secondary operation corresponds to one or more relations in the category Π-*Alg*, with higher order operations corresponding to relations among the relations, etc. (see [Ha2] and [Bl4, §2]).

Remark 4.14. It should be observed that E.C. Zeeman and K.A. Hardie have also considered a secondary homotopy operation associated to a triple Whitehead product, and Hardie has established various properties it has (including vanishing for H-spaces) – see [Ha1, §5]. In [P1], G.J. Porter defined higher Whitehead products of all orders (see also [Ba1, Ba2] for some properties and examples).

As noted in §3.7, there is an analogous rational operation, corresponding to the case $[[\alpha,\alpha],\beta]=-2[[\beta,\alpha],\alpha]$ of the Jacobi identity for $\alpha\in\pi_{2m}\boldsymbol{X}\otimes\mathbb{Q}$ and $\beta\in\pi_q\boldsymbol{X}\otimes\mathbb{Q}$. The rational higher Whitehead products have been studied by various authors (e.g. [Al1, Al2, AA]).

Definition 4.15. It is clearly a necessary and sufficient condition for the subset $\langle\langle n\rangle\rangle$ to be non-empty that all the lower order operations $\langle\langle k\rangle\rangle$ $(2\leq k<n)$ *vanish* – i.e., contain the null class – because that means that some $\bar{h}:\partial D(n)\ltimes \boldsymbol{V}_n\to\boldsymbol{X}$ obtained from a $\partial D(n)$-compatible sequence $\{h^k\}_{k=0}^{n-1}$ extends over all of $D(n)\ltimes\boldsymbol{V}_n$, yielding a $\partial D(n+1)$-compatible sequence $\{h^k\}_{k=0}^{n}$. We say that the higher order operations $\{\langle\langle n\rangle\rangle\}_{n=2}^{\infty}$ *vanish coherently* if there is a $\partial D(\infty)$-*compatible sequence* for $\boldsymbol{V}_\bullet\to\boldsymbol{X}$.

Remark 4.16. If we choose a *functorial* construction of the simplicial resolution by wedges of spheres $\boldsymbol{V}_\bullet\to\boldsymbol{Y}=\boldsymbol{X}\times\boldsymbol{X}$ in §4.1(a) above (as in [Stv1]), and let $\boldsymbol{W}_\bullet\to\boldsymbol{X}$ be the corresponding resolution of $\boldsymbol{X}$, then by functorality $m:\boldsymbol{X}\times\boldsymbol{X}\to\boldsymbol{X}$ induces a map of simplicial spaces $m_\bullet:\boldsymbol{V}_\bullet\to\boldsymbol{W}_\bullet$, which composes with the augmentation $\boldsymbol{W}_0\to\boldsymbol{X}$ to yield a rectification of the homotopy-commutative diagram $\boldsymbol{V}_\bullet\to\boldsymbol{X}$ of §4.2.

Thus the coherent vanishing of the operations $\{\langle\langle n\rangle\rangle\}_{n=2}^{\infty}$, which is equivalent to the rectifiability of this diagram (cf. [DKS, Cor. 4.5]), is not only sufficient but also necessary in order for $m:\boldsymbol{X}\times\boldsymbol{X}\to\boldsymbol{X}$ to exist.

Note further that since $\boldsymbol{X}$ is a retract of $\boldsymbol{X}\times\boldsymbol{X}$ (in two different ways), by functoriality of the resolution there are two disjoint retracts ${\boldsymbol{W}^{(1)}}_{\bullet}$, ${\boldsymbol{W}^{(2)}}_{\bullet}$ (isomorphic to $\boldsymbol{W}_{\bullet}$) in $\boldsymbol{V}_{\bullet}$. If we now restrict the map $g:\boldsymbol{V}_0\to\boldsymbol{X}$ to ${\boldsymbol{W}^{(1)}}_0\vee{\boldsymbol{W}^{(2)}}_0\subseteq\boldsymbol{V}_0$, we get a *strict* augmentation to $\boldsymbol{X}$ (namely, two copies of the augmentation $\varepsilon:\boldsymbol{W}\to\boldsymbol{X}$, followed by the fold map $\boldsymbol{X}^{(1)}\vee\boldsymbol{X}^{(2)}\to\boldsymbol{X}$) – so the corresponding higher homotopy operations must vanish.

Thus in calculating $\langle\!\langle n\rangle\!\rangle\subseteq[\Sigma^{n-1}\boldsymbol{V}_n,\boldsymbol{X}]$ (which may clearly be done on each sphere summand $\boldsymbol{S}^k_x$ in $\boldsymbol{V}_n\simeq\bigvee_{k=1}^{\infty}\bigvee_{x\in T_{n,k}}\boldsymbol{S}^k_x$ separately), it suffices to consider those spheres $\boldsymbol{S}^k_x$ which are in the cross-term $\boldsymbol{V}_n\setminus{\boldsymbol{W}^{(1)}}_n\vee{\boldsymbol{W}^{(2)}}_n$ – in fact, only those which map non-trivially into both sub-simplicial spaces ${\boldsymbol{W}^{(1)}}_{\bullet}$ and ${\boldsymbol{W}^{(2)}}_{\bullet}$. Moreover, we may disregard those sphere wedge summands $\boldsymbol{S}^k_x\hookrightarrow\boldsymbol{V}_n$ which are in the image of some degeneracy map $s_j:\boldsymbol{W}_{n-1}\to\boldsymbol{W}_n$ (see Example 3.12), since the face maps on them, and thus all the homotopies between these, are determined by those of $\boldsymbol{W}_0,\dots,\boldsymbol{W}_{n-1}$.

Note however that it while the higher operations may be calculated separately on each (nondegenerate cross-term) sphere $\boldsymbol{S}^k_x\hookrightarrow\boldsymbol{V}_n$, and their *non-vanishing* on one such $\boldsymbol{S}^k_x$ is an obstruction to realizing μ, their independent vanishing on the different spheres does not suffice for realizing μ, since they must vanish *coherently* – that is, for a consistent choice of the maps h^k on $D(k)\ltimes(\boldsymbol{W}_k)_x$, where $(\boldsymbol{W}_{\bullet})_x$ is that part of the simplicial space $\boldsymbol{V}_{\bullet}$ through which the face maps on $\boldsymbol{S}^k_x$ factor.

One could in fact set up a further obstruction theory for such coherence, as in [Bl5, §5.9], but we shall not do so here.

In fact, it seems reasonable to suppose that one can further restrict the set of spheres $\boldsymbol{S}^k_x\hookrightarrow\boldsymbol{V}_n$ for which we must check the obstructions, as follows:

Conjecture 4.17. *The n-th order higher homotopy operations which must vanish coherently in order for $\mu:\pi_*\boldsymbol{X}\times\pi_*\boldsymbol{X}\to\pi_*\boldsymbol{X}$ to be realizable are in one-to-one correspondence with the elements of the relative homology group $H_n(\pi_*(\boldsymbol{X}\times\boldsymbol{X}),\pi_*\boldsymbol{X}\amalg\pi_*\boldsymbol{X})$ (Def. 3.9).*

This conjecture would follow from Remark 4.16 above, combined with the realizability of arbitrary free simplicial Π-algebra resolutions $A_{\bullet}\to\pi_*(\boldsymbol{X}\times\boldsymbol{X})$ (see §4.1(a) above and [Stv2]), since one could then choose $\boldsymbol{V}_{\bullet}\to\boldsymbol{X}\times\boldsymbol{X}$ so as to realize a *minimal* resolution, in which the non-degenerate cross-term spheres $\boldsymbol{S}^k_x\hookrightarrow\boldsymbol{V}_n$ are (almost) in a bijective correspondence with the elements of $H_n(\pi_*(\boldsymbol{X}\times\boldsymbol{X}),\pi_*\boldsymbol{X}\amalg\pi_*\boldsymbol{X})$ (see also Example 5.10 below).

In light of Remark 4.2, we may summarize the results of this section in the following

Theorem 4.18. *A space $\boldsymbol{X}$ has an H-space structure if and only if*

(a) $\pi_*\boldsymbol{X}$ is an abelian Π-algebra; and

(b) The sequence of higher homotopy operations $\{\langle\!\langle n\rangle\!\rangle\subseteq\pi_*\boldsymbol{X}\}_{n=2}^{\infty}$ associated to the Π-algebra morphism $\mu:\pi_*\boldsymbol{X}\times\pi_*\boldsymbol{X}\to\pi_*\boldsymbol{X}$ (as defined in §4.11 and 4.16) vanish coherently.

Example 4.19. Let $\boldsymbol{X}$ be the 5-th stage in a Postnikov tower for the 4-dimensional complex projective space $\mathbb{C}P^2$, so $\pi_i\boldsymbol{X}\cong\mathbb{Z}$ for $i=2,5$, and $\pi_i\boldsymbol{X}=0$ otherwise. The Π-algebra $\pi_*\boldsymbol{X}$ is abelian (in fact, trivial – all compositions and Whitehead products vanish), but of course $\boldsymbol{X}$ can have no H-space structure since it is rationally equivalent to $\mathbb{C}\mathrm{P}^2$ (this is the example of [BG, §3.10]).

Now we can use the simplicial space $\boldsymbol{W}_\bullet$ of Example 3.12 to define a simplicial resolution by wedges of spheres $\boldsymbol{V}_\bullet \to \boldsymbol{X} \times \boldsymbol{X}$ as follows:

- First, we must add two 5-spheres, say $\boldsymbol{S}^5_{\gamma^i}$ $(i = 1, 2)$, to $\boldsymbol{W}_0$, with $\varepsilon|_{\boldsymbol{S}^5_{\gamma^i}}$ representing the respective generators of $\pi_5(\boldsymbol{X} \times \boldsymbol{X})$.
- Next, we must "symmetrize" $\boldsymbol{W}_\bullet$ by adding spheres corresponding to $S^4_\kappa \subset \bar{A}_2$ in §3.5, (and its "faces"), so that
  - $\boldsymbol{V}_0 = \boldsymbol{W}_0 \vee \boldsymbol{S}^5_{\gamma^1} \vee \boldsymbol{S}^5_{\gamma^2} \vee \boldsymbol{e}^4_{H^2} \vee \boldsymbol{e}^5_{K^2} \cup_{\boldsymbol{S}^4_{\varphi^2}} \boldsymbol{e}^5_{L^2}$, where $\varepsilon|_{\boldsymbol{e}^4_{H^2}} = \iota_{X^2} \circ H$, $\varepsilon|_{\boldsymbol{S}^4_{\varphi^2}} = (\alpha^2 \vee \alpha^1) \circ k$, $\varepsilon|_{\boldsymbol{e}^5_{K^2}} = (\alpha^2 \vee \alpha^1) \circ K$, and $\varepsilon|_{\boldsymbol{e}^5_{L^2}} = (\alpha^2 \vee \alpha^1 \vee H^2 \vee G) \circ L$.
  - Similarly, $\boldsymbol{V}_1 = s_0 \boldsymbol{V}_0 \vee \boldsymbol{S}^3_{\beta^1} \vee \boldsymbol{S}^3_\zeta \vee \boldsymbol{e}^5_{K^1} \vee \boldsymbol{e}^5_{L^1} \vee \boldsymbol{S}^3_{\beta^2} \vee \boldsymbol{e}^5_{K^2} \vee \boldsymbol{e}^5_{L^2}$, where $\boldsymbol{e}^5_{K^1} \vee \boldsymbol{e}^5_{L^1}$ are the $\boldsymbol{e}^5_K \vee \boldsymbol{e}^5_L$ of $\boldsymbol{W}_1$, and $\boldsymbol{e}^5_{K^1} \vee \boldsymbol{e}^5_{L^1}$ are obtained from them by replacing α^1 by α^2, and so on, throughout the definitions, and likewise for the face maps.
  - $\boldsymbol{V}_2 = s\boldsymbol{V}_1 \vee \boldsymbol{S}^4_\theta \vee \boldsymbol{S}^4_\kappa$, with the obvious face maps, and
  - $\boldsymbol{V}_n = s\boldsymbol{V}_{n-1}$ for $n \geq 3$.

Now we can add spheres in dimensions ≥ 6 as necessary to obtain a full resolution by wedges of spheres for $\boldsymbol{X} \times \boldsymbol{X}$ (e.g., using the functorial approach of [Stv1, §2]). However, in fact we need not worry about these higher dimensional spheres, since we see from the spectral sequence of (3.11) that, regardless of what they are, the map $\|\boldsymbol{V}_\bullet\| \to \boldsymbol{X} \times \boldsymbol{X}$ induces an isomorphism in $\pi_i(-)$ for $i \leq 5$, so if we find an obstruction to realizing μ in dimension ≤ 5, it cannot in fact be realized for $\boldsymbol{X}$ itself.

Conversely, we know that $\boldsymbol{X}$ cannot be given an H-space structure, so by Theorem 4.18 we know there is some non-vanishing secondary obstruction, which must necessarily lie in $\pi_5 \boldsymbol{X}$, and thus correspond to a non-degenerate crossterm sphere $\boldsymbol{S}^4_x \hookrightarrow \boldsymbol{V}_2$ (by §4.16). But there are only two such, namely, $\boldsymbol{S}^4_\theta$ and $\boldsymbol{S}^4_\kappa$, and since they both define the same secondary operation, (namely, that of Example 4.13), we can deduce that it does not vanish in $\pi_5 \boldsymbol{X}$ – so that *rationally* it contains a generator of $\pi_5 \boldsymbol{X}$. Of course this holds in $\pi_* \mathbb{C}\mathrm{P}^2$, too, and in fact G.J. Porter has calculated the higher Whitehead products in $\pi_* \mathbb{C}\mathrm{P}^n$ for all n (by other methods – see [P3]).

In fact, the rational secondary operation of §4.14 is the only obstruction to the rational space $\mathbb{C}\mathrm{P}^2$ being an H-space. In [AA, Prop. 6.9], Andrews and Arkowitz have shown that a space $\boldsymbol{X}$ is rationally homotopy equivalent to a product of Eilenberg Mac Lane space (equivalantly: to a rational H-space) if and only if all Whitehead products of all orders vanish in $\pi_* \boldsymbol{X}$. See also [R, §4].

5. The torsion Whitehead product

The obstructions to having an H-space structure described in the previous section are not ordered linearly, so there is no "first obstruction". From the point of view of rational homotopy theory (§3.7), perhaps the secondary operation of §4.14 deserves this name; but from the "periodic" point of view the first case of interest is the following operation:

Definition 5.1. Assume given a space $\boldsymbol{X}$ with elements $\alpha \in \pi_q \boldsymbol{X}$, $\beta \in \pi_r \boldsymbol{X}$ such that $k\alpha = 0 = k\beta$ for some integer k. Then $k[\alpha, \beta] \in \pi_{q+r-1}\boldsymbol{X}$ vanishes "for two different reasons", because

$$* \sim [\alpha, *] \sim [\alpha, k\beta] \sim k[\alpha, \beta] \sim [k\alpha, \beta] \sim [*, \beta] \sim * \tag{5.2}$$

and the choice of two corresponding nullhomotopies $G, H : k[\alpha, \beta] \sim *$ yields an element of $\pi_{q+r}\boldsymbol{X}$ as in [Sp, §3], which we call the *torsion Whitehead product* of α and β, and denote by $[[\alpha, \beta]]$. One may verify that $k[[\alpha, \beta]] = 0$, and that it has indeterminacy

$$\{[\alpha', \beta] \mid \alpha' \in \pi_{q+1}\boldsymbol{X}\} \ + \ \{[\alpha, \beta'] \mid \beta' \in \pi_{r+1}\boldsymbol{X}\} \ \subseteq \ \pi_{p+q}\boldsymbol{X} \tag{5.3}$$

(see §5.6 or 5.8 below).

From the description of the torsion Whitehead product in simplicial terms in §5.8 below it is evident that this will be part of the obstruction $\langle\langle 2 \rangle\rangle$ to realizing μ for an abelian Π-algebra $\pi_*\boldsymbol{X}$ with torsion. We may call it the "first torsion obstruction" because of the following alternative description:

5.4. **$\boldsymbol{M}$-Π-algebras.** Note that at least part of the above discussion could have been carried out in a more general context, with spheres replaced by some other model space: Let $\boldsymbol{M}$ be some space (replacing $\boldsymbol{S}^0$), and write $\pi_t(\boldsymbol{X}; \boldsymbol{M})$ for $[\Sigma^t \boldsymbol{M}, \boldsymbol{X}]$ $(t \geq 1)$.

Definition 5.5. A primary *$\boldsymbol{M}$-homotopy operation* is a natural transformation $\vartheta : \pi_{n_1}(-; \boldsymbol{M}) \times \ldots \times \pi_{n_k}(-; \boldsymbol{M}) \to \pi_r(-; \boldsymbol{M})$, and these are in one to one correspondence with homotopy classes $\alpha_\vartheta \in \pi_r(\Sigma^{n_1}\boldsymbol{M} \vee \ldots \vee \Sigma^{n_k}\boldsymbol{M}; \boldsymbol{M})$, with the universal relations among such operations corresponding to the relations in the composites of maps among wedges of copies of $\boldsymbol{M}^n$.

An $\boldsymbol{M}$-Π-*algebra* is then a graded group $\{X_i\}_{i=1}^\infty$, together with an action of the primary $\boldsymbol{M}$-homotopy operations on them, satisfying the universal relations (cf. [BT, §9].

In [Ar, 2.2], Arkowitz defined a generalized Whitehead product

$$[-,-]' : [\Sigma^p \boldsymbol{M}, \boldsymbol{X}] \times [\Sigma^q \boldsymbol{M}, \boldsymbol{X}] \to [\Sigma^{p+q-1}\boldsymbol{M} \wedge \boldsymbol{M}, \boldsymbol{X}] \qquad (\text{for } \ p, q \geq 1),$$

which satisfies many of the properties of the ordinary Whitehead product (including anti-commutativity and bilinearity). In particular, all such products vanish when $\boldsymbol{X}$ is an H-space (cf. [Ar, Prop. 5.4]). Thus we could generalize the discussion of section 2, and in particular Proposition 2.7, to require that $\pi_*(\boldsymbol{X}; \boldsymbol{M})$ be an abelian $\boldsymbol{M}$-Π-algebra as a necessary condition for $\boldsymbol{X}$ to have an H-space structure, for all possible "coefficients" $\boldsymbol{M}$. However, this is not our apporach here (and in any case this will not be a sufficient condition, as Example 4.19 shows - cf. [BG, §3] and [Ar, pp. 18-19]). One could of course try to develop an obstruction theory for realizing the appropriate map of $\boldsymbol{M}$-Π-algebras, as in section 4; but the simplicial spaces corresponding to the $\boldsymbol{V}_\bullet \to \boldsymbol{X} \times \boldsymbol{X}$ of §4.1(a) do not always exist for arbitrary $\boldsymbol{M}$ – see [Bl6, §4].

Remark 5.6. Note that the Hilton-Milnor Theorem (cf. [W, XI, Thm. 6.7]) allows us to express any $\boldsymbol{M}$-homotopy operation as a sum of iterated generalized Whitehead products, precomposed with a map $\alpha : \Sigma^r \boldsymbol{M} \to \bigwedge_{i=1}^k \Sigma^{n_i}\boldsymbol{M}$. This is not in general as satisfactory as Hilton's original theorem (cf. [Hi2, Thm. A]), because in general $\Sigma^p\boldsymbol{M} \wedge \Sigma^q\boldsymbol{M}$ is not a suspension of $\boldsymbol{M}$.

However, if $\boldsymbol{M}^p(k) = \boldsymbol{S}^{p-1} \cup_k \boldsymbol{e}^p$ is the p-dimensional mod k Moore space, and k is odd or $4|k$, then

$$(5.7) \qquad \boldsymbol{M}^p(k) \wedge \boldsymbol{M}^q(k) \simeq \boldsymbol{M}^{p+q-1}(k) \vee \boldsymbol{M}^{p+q}(k) \qquad\qquad \text{for } p, q \geq 3$$

(cf. [N, Cor. 6.6]). In particular, the generalized Whitehead product map $w_{p,q} : \Sigma\boldsymbol{M}^p(k) \wedge \boldsymbol{M}^q(k) \to \boldsymbol{M}^{p+1}(k) \vee \Sigma\boldsymbol{M}^q(k)$ splits up to homotopy as the sum of $w' : \boldsymbol{M}^{p+q}(k) \to \boldsymbol{M}^{p+1}(k) \vee \boldsymbol{M}^{q+1}(k)$ and $w'' : \boldsymbol{M}^{p+q+1}(k) \to \boldsymbol{M}^{p+1}(k) \vee \boldsymbol{M}^{q+1}(k)$.

If we let $\iota_j : \boldsymbol{S}^j \hookrightarrow \boldsymbol{M}^{j+1}(k)$ denote the inclusion of the bottom cell, it is evident from the description in [N, §6] that $w' \circ \iota_{p+q-1} : \boldsymbol{S}^{p+q-1} \to \boldsymbol{M}^{p+1}(k) \vee \boldsymbol{M}^{q+1}(k)$ represents the ordinary Whitehead product $[\iota_p, \iota_q]$, while $(\phi_p \vee \phi_q) \circ w'' \circ \iota_{p+q-1} : \boldsymbol{S}^{p+q} \to \boldsymbol{M}^p(k) \vee \boldsymbol{M}^q(k)$ represents the torsion Whitehead product $[[\iota_p, \iota_q]]$. (See [Hi1, §6-7] for the justification of this last statement.)

Clearly $k \cdot w'' \sim *$, and the indeterminacy of $[[\alpha, \beta]]$ in (5.3) now follows from [N, Prop. 1.4].

Thus from the point of view of the mod k homotopy groups $\pi_*(-; \boldsymbol{M}(k))$, the vanishing of all torsion Whitehead products is part of the *primary* condition to being an H-space – in fact, the only *new* requirement, in addition to $\pi_*\boldsymbol{X}$ being abelian. Therefore, if we consider Moore spaces to be the simplest spaces after the sphere, this perhaps justifies considering the torsion Whitehead product as the first secondary obstruction.

5.8. **a simplicial description.** Let $\boldsymbol{X}$ is any space with torsion in its homotopy groups, one can define the torsion Whitehead product as a secondary homotopy operation, as in §4.11. For concreteness we exemplify this by a special case, as follows:

Example 5.9. Assume $\pi_2\boldsymbol{X} \cong \mathbb{Z}/2$ (generated by α), and $\pi_i\boldsymbol{X} = 0$ for $i = 1, 3$. In the notation of Example 3.5, a free simplicial Π-algebra resolution $A_\bullet \to \pi_*(\boldsymbol{X} \times \boldsymbol{X})$ is given in degrees ≤ 3 by:

(i) $\bar{A}_0 = S^2_{\alpha^1} \amalg S^2_{\alpha^2}$, with the obvious augmentation onto $\pi_2(\boldsymbol{X} \times \boldsymbol{X})$.

(ii) $\bar{A}_1 = S^2_{\beta^1} \amalg S^2_{\beta^2} \amalg S^3_{\gamma^1} \amalg S^3_{\gamma^2} \amalg S^3_{\delta}$, with $d_0|_{S^2_{\beta^i}} = 2\alpha^i$ and $d_0|_{S^3_{\gamma^i}} = \eta_2^{\#}\alpha^i$ for $i = 1, 2$, and $d_0|_{S^3_\delta} = [\alpha^1, \alpha^2]$.

(iii) $\bar{A}_2 = S^3_{\zeta^1} \amalg S^3_{\zeta^2} \amalg S^3_{\theta^1} \amalg S^3_{\theta^2} \amalg S^3_{\kappa}$, with $d_0|_{S^3_{\zeta^i}} = 4\gamma^i - \eta_2^{\#}\beta^i$, $d_0|_{S^3_{\theta^i}} = \eta_2^{\#}\beta^i - [s_0\alpha^i, \beta^i]$ for $i = 1, 2$, and $d_0|_{S^3_\kappa} = [s_0\alpha^1, \beta^2] - [s_0\alpha^2, \beta^1]$.

(iv) $\bar{A}_3 = S^3_{\lambda^1} \amalg S^3_{\lambda^2} \amalg S^3_{\mu}$, with $d_0|_{S^3_{\lambda^i}} = [s_0\beta^i, s_1\beta^i] - 2\eta_2^{\#} s_0\beta^i$ for $i = 1, 2$ and $d_0|_{S^3_\mu} = 2\kappa + ([s_0\beta^1, s_1\beta^2] - [s_1\beta^1, s_0\beta^2]$.

(v) $\bar{A}_n = 0$ for $n \geq 4$.

As in Example 3.12, one can realize $A_\bullet \to \pi_*(\boldsymbol{X} \times \boldsymbol{X})$ in simplicial dimensions ≤ 3 by an augmented simplicial space $\boldsymbol{V}_\bullet \to \boldsymbol{X} \times \boldsymbol{X}$, once we make choices of:

- a map $\alpha : \boldsymbol{S}^2 \to \boldsymbol{X}$ representing the generator of $\pi_2\boldsymbol{X}$, with $A : 2\alpha \sim *$;
- a nullhomotopy $B : \boldsymbol{e}^4 \to \boldsymbol{S}^2_a \vee \boldsymbol{S}^2_b$ for $[2\iota_b, \iota_a] - [2\iota_a, \iota_b]$; and
- a nullhomotopy $C : \boldsymbol{e}^4 \to \boldsymbol{S}^2_a \vee \boldsymbol{S}^2_b \vee \boldsymbol{e}^3_c \vee \boldsymbol{e}^3_d$ for $[\iota_d, \iota_a] - [\iota_b, \iota_c]$.

(Additional choices are needed to define the simplicial space, but not to describe $[[\alpha, \alpha]]$).

As in Example 4.13 we then define $g : \boldsymbol{V}_0 \to \boldsymbol{X}$ by $g|_{\boldsymbol{S}^2_{\alpha^i}} = \alpha$, and $g = *$ elsewhere, and see that the part of the secondary homotopy operation $\langle\langle 2\rangle\rangle$ associated to $\boldsymbol{S}^3_\kappa \subset \boldsymbol{V}_2$ is defined on $\partial D(2) \ltimes \boldsymbol{S}^3_\kappa$ by $g \circ d_0 \circ d_1|_{\boldsymbol{S}^3_\kappa} = [2\alpha, \alpha] - [\alpha, 2\alpha]$, with $g \circ d_0 \circ d_2|_{\boldsymbol{S}^3_\kappa} = g \circ d_1 \circ d_2|_{\boldsymbol{S}^3_\kappa} = *$, and the two nullhomotopies are $h^1 \circ d_0 = \alpha \circ \nabla \circ B$ and $h^1 \circ d_0 = (A \vee \alpha) \circ \nabla \circ C$. The map $h : \Sigma \boldsymbol{S}^3 \to \boldsymbol{X}$ so obtained represents the torsion Whitehead product $[[\alpha, \alpha]] \in \pi_4\boldsymbol{X}$, by definition.

One may similarly define the torsion Whitehead product $[[\alpha, \beta]]$ in general; it is evident that no special role is played in this description by the prime 2 (aside from the inapplicability of Remark 5.6), or by the fact that we specialized to $\alpha = \beta \in \pi_2\boldsymbol{X}$. That the indeterminacy for this secondary operation is as in (5.3) now follows from [Bl5, Lemma 5.12].

Example 5.10. Now let $\boldsymbol{X}$ be a space with $\pi_2\boldsymbol{X} \cong \mathbb{Z}/2 \cong \pi_4\boldsymbol{X}$ and $\pi_i\boldsymbol{X} = 0$ for $i \neq 2, 4$. By inspecting the possible k-invariants $k_2 : K(\mathbb{Z}/2, 2) \to K(\mathbb{Z}/2, 5)$ we see that there are eight such spaces, all with the same (trivial, and in particular abelian) Π-algebra $\pi_*\boldsymbol{X}$.

Precisely as in the beginning of Example 4.19, one may extend the simplicial space described (implcitly) in Example 5.9 to a resolution of $\boldsymbol{X} \times \boldsymbol{X}$ by wedges of spheres (at least in degrees ≤ 4). This will require adding a non-degenerate crossterm sphere $\boldsymbol{S}^3_\mu \subset \boldsymbol{V}_3$ (corresponding to $S^3_\mu \subset \bar{A}_3$), for which we need additional choices of nullhomotopies – namely:

- a nullhomotopy $D : \boldsymbol{e}^4 \to \boldsymbol{S}^2_a \vee \boldsymbol{S}^2_b \vee \boldsymbol{S}^2_c \vee \boldsymbol{S}^2_d$ for $2([\iota_d, \iota_a] - [\iota_b, \iota_c]) + ([\iota_d, 2\iota_a] - [2\iota_b, \iota_c])$;
- a nullhomotopy $E : \boldsymbol{e}^4 \to \boldsymbol{S}^2_a \vee \boldsymbol{S}^2_b \vee \boldsymbol{e}^4_c$ for $2\iota_c + ([\iota_b, \iota_a] - [\iota_b, \iota_a]$; and
- a nullhomotopy $H : \boldsymbol{e}^4 \to \boldsymbol{S}^2_a \vee \boldsymbol{S}^2_b \vee \boldsymbol{S}^2_c \vee \boldsymbol{S}^2_d \vee \boldsymbol{e}^4_e$ for $2\iota_e + [\iota_d, \iota_a] - [\iota_b, \iota_c]$.

It might appear that we have another (third order) homotopy operation defined in this situation, corresponding to $\boldsymbol{S}^3_\mu$, in $\langle\langle 3\rangle\rangle$. However, this operation is associated to the Π-algebra identity

$$2([2\beta, \alpha] - [2\alpha, \beta]) + ([2\alpha, 2\beta] - [2\alpha, 2\beta]) = 0,$$

and as such can be shown to vanish in $\pi_4\boldsymbol{X}$ if $[[\alpha, \beta]]$ does (essentially, because in addition to (5.2) it involves only the group operation).

Thus we end up with a single obstruction to $\boldsymbol{X}$ being an H-space – namely, the torsion Whitehead product $[[\alpha, \alpha]] \in \pi_4\boldsymbol{X}$. For the four possible spaces X having primitive k-invariant k_2, this must vanish (since they are H-spaces, by [C, Thm. 6]); on the other hand, in the four other cases we can deduce from the fact that $\boldsymbol{X}$ is not an H-space that $\pi_4\boldsymbol{X}$ is generated by $[[\alpha, \alpha]]$.

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