

HIGHER HOMOTOPY OPERATIONS AND COHOMOLOGY

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ABSTRACT. We explain how higher homotopy operations, defined topologically, may be identified under mild assumptions with (the last of) the Dwyer-Kan-Smith cohomological obstructions to rectifying homotopy-commutative diagrams.

INTRODUCTION

The first secondary homotopy operations to be defined were Toda brackets, which appeared (in [T1]) in the early 1950's – at about the same time as the secondary cohomology operations of Adem and Massey (in [Ad] and [MU]). The definition was later extended to higher order homotopy and cohomology operations (see [Sp, Ma, Kl]), which have been used extensively in algebraic topology, starting with Toda's own calculations of the homotopy groups of spheres in [T2].

In [BM], a “topological” definition of higher homotopy operations based on the W -construction of Boardman and Vogt, was given in the form of an obstruction theory for rectifying diagrams. The same definition may be used also for higher cohomology operations. This was recently modified in [BC] to take account of the fact that, in practice, higher order operations, both in homotopy and in cohomology, occur in a *pointed* context, which somewhat simplifies their definition and treatment.

Earlier, in [DKSm2], Dwyer, Kan, and Smith gave an obstruction theory for rectifying a diagram $\tilde{X} : \mathcal{K} \rightarrow \text{ho } \mathcal{T}$ in the homotopy category of topological spaces by making it “infinitely-homotopy commutative”: the precise statement involves the simplicial function complexes $\text{map}(\tilde{X}u, \tilde{X}v)$ for all $u, v \in \mathcal{O} = \text{Obj}(\mathcal{K})$, which constitute an $(\mathcal{S}, \mathcal{O})$ -category \mathcal{C}_X (see §3.11 and Section 4). Their results are thus stated in terms of $(\mathcal{S}, \mathcal{O})$ -categories (simplicially enriched categories with object set \mathcal{O}). In particular, the obstructions take values in the corresponding $(\mathcal{S}, \mathcal{O})$ -cohomology groups (see [DKSm1, §2.1]).

The purpose of the present note is to explain the relation between these two approaches. Because the W -construction, and thus higher operations, are defined in terms of cubical sets, it is convenient to work cubically throughout. In this language, $(\mathcal{S}, \mathcal{O})$ -cohomology is replaced by the (equivalent) (\mathcal{C}, Γ) -cohomology (see §2.25), and our main result (Theorem 4.14 below) may be stated roughly as follows:

Assume given a directed graph Γ without loops (cf. §3.13) of length $n + 2$, having initial node v_{init} and terminal node v_{fin} , and let \mathcal{M} be a cubically enriched pointed model category.

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Theorem A. *For each pointed diagram $\tilde{X} : \Gamma \rightarrow \text{ho } \mathcal{M}$, there is a natural pointed correspondence Φ between the possible values of the final Dwyer-Kan-Smith obstruction to rectifying \tilde{X} , in the (\mathcal{C}, Γ) -cohomology group $H^n(\Gamma, \pi_{n-1}\mathcal{C}_X)$, and the n -th order homotopy operation $\langle\langle \tilde{X} \rangle\rangle$, a subset of $[\Sigma^{n-1}\tilde{X}(v_{\text{init}}), \tilde{X}(v_{\text{fin}})]$.*

0.1. *Remark.* The fact that Φ is *pointed* implies that, not surprisingly, the two different obstructions to rectification vanish simultaneously. Our objective here is to explicitly identify each value of a higher homotopy operation (with its usual indeterminacy) with a (\mathcal{C}, Γ) -cohomology class for Γ .

In [BB], a relationship between $(\mathcal{S}, \mathcal{O})$ -cohomology and the cohomology of a Π -algebra is described. Since the latter is a purely algebraic concept, we hope that together with the present result this will provide a systematic way to apply homological-algebraic methods to interpret and calculate higher homotopy and cohomology operations.

0.2. **Notation.** The category of compactly generated topological spaces is denoted by \mathcal{T} , and that of pointed connected compactly generated spaces by \mathcal{T}_* ; their homotopy categories are denoted by $\text{ho } \mathcal{T}$ and $\text{ho } \mathcal{T}_*$, respectively. The categories of (pointed) simplicial sets will be denoted by \mathcal{S} (resp., \mathcal{S}_*), those of groups, abelian groups, and groupoids by $\mathcal{G}p$, $\mathcal{Ab}\mathcal{G}p$, and $\mathcal{G}pd$, respectively. $\mathcal{C}at$ denotes the category of small categories.

If $\langle \mathcal{V}, \otimes \rangle$ is a monoidal category, we denote by $\mathcal{V}\text{-}\mathcal{C}at$ the collection of all (not necessarily small) categories enriched over \mathcal{V} (see [Bor2, §6.2]). A category \mathcal{K} is called *pointed* if it has a *zero object* 0 – that is, 0 is both initial and final. In such a \mathcal{K} , a map factoring through 0 is called a *null* (or *zero*) map, and since there is a unique such map between any two objects, \mathcal{K} is enriched over pointed sets.

0.3. *Remark.* It will be convenient at times to work with *non-unital* categories – that is, categories which need not have identity maps. These have been studied in the literature under various names, beginning with the *semi-categories* of V.V. Vagner (see [V]). The enriched version appears, e.g., in [BBM].

0.4. **Organization.** Section 1 provides a review of cubical sets and their homotopy theory. Section 2 discusses cubically enriched categories, as a replacement for the $(\mathcal{S}, \mathcal{O})$ -categories of Dwyer and Kan, and describes their model category structure (Theorem 2.21). In Section 3 we give a “topological” definition of *pointed* higher homotopy operations in terms of diagrams indexed by certain finite categories called lattices. Finally, in Section 4 the Dwyer-Kan-Smith obstruction theory is described and the main result (Theorem 4.14 and Corollary 4.15) is proved.

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1. CUBICAL SETS

Even though the obstruction theory of Dwyer, Kan, and Smith was originally defined simplicially, for our purposes it appears more economical to work cubically. This is because cubical sets are the natural setting for the W -construction of Boardman and Vogt, which was used for constructing higher homotopy operations in [BM] and

[BC]. Since our goal is to identify these operations with the cohomological obstructions of Dwyer-Kan-Smith, we simplify the exposition by framing their theory in cubical terms as well. Because cubical homotopy theory is less familiar than the simplicial version, and the relevant information and definitions are scattered throughout the literature, we summarize them here.

1.1. Definition. Let \square denote the *Box category*, whose objects are the abstract cubes $\{\mathcal{I}^n\}_{n=0}^\infty$ (where $\mathcal{I} := \{0, 1\}$ and \mathcal{I}^0 is a single point). The morphisms of \square are generated by the inclusions $d_\varepsilon^i : \mathcal{I}^{n-1} \rightarrow \mathcal{I}^n$ and projections $s^i : \mathcal{I}^n \rightarrow \mathcal{I}^{n-1}$ for $1 \leq i \leq n$ and $\varepsilon \in \{0, 1\}$.

One can identify \square with a category of topological cubes, where \mathcal{I}^n corresponds to $[0, 1]^n$ (an n -fold product of unit intervals), the linear map $d_\varepsilon^i : [0, 1]^{n-1} \rightarrow [0, 1]^n$ is defined $(t_1, \dots, t_{n-1}) \mapsto (t_1, \dots, t_{i-1}, \varepsilon, t_i, \dots, t_{n-1})$, and $s^i : [0, 1]^n \rightarrow [0, 1]^{n-1}$ is defined by omitting the i -th coordinate.

A contravariant functor $K : \square^{\text{op}} \rightarrow \text{Set}$ is called a *cubical set* (or cubical complex), and we write K_n for the set $K(\mathcal{I}^n)$ of n -cubes (or n -cells) of K . The (i, ε) -face map $d_\varepsilon^i : K_n \rightarrow K_{n-1}$ and the i -th degeneracy $s_i : K_{n-1} \rightarrow K_n$ are induced by d_ε^i and s^i , respectively. (See, for example, [BH1, 1.1] for the cubical relations.) A cubical set K is called *finite* if all but finitely many n -cubes of K are degenerate (that is, in the image of some s_i). The category of cubical sets is denoted by \mathcal{C} . See [KP, I, §5], [BH1, §1], or [FRS].

Several obvious constructions carry over from simplicial sets: for example, the n -truncation functor τ_n on cubical sets has a left adjoint, and composing the two yields the cubical n -skeleton functor $\text{sk}_n^c : \mathcal{C} \rightarrow \mathcal{C}$. Thus $\text{sk}_n^c K$ is generated (under the degeneracies) by the k -cubes of K for $k \leq n$.

1.2. Notation. There is a standard embedding of \square in \mathcal{C} , in which $\mathcal{I}^n \in \square$ is taken to the standard n -cube $I^n \in \mathcal{C}$ (with one non-degenerate cell in dimension n , and all its faces). Applying sk_n^c to the standard $(n+1)$ -cube I^{n+1} , we obtain its *boundary* $\partial I^{n+1} := \text{sk}_n^c I^{n+1}$. By omitting the d_i^ε -face from ∂I^{n+1} , we obtain the (i, ε) -square horn $\square_i^{n, \varepsilon}$.

1.3. Remark. There is also a version of cubical sets without degeneracies, sometimes called *semi-cubical sets*, but these are not suitable for homotopy theoretic purposes (cf. [An1]). On the other hand, Brown and Higgins have proposed adding further “adjacent degeneracies”, called *connections* (see [BH1, §1] and [GM]). These have proved useful in various contexts (see, e.g., [An2, BH2]).

1.4. The cubical enrichment of \mathcal{C} . As a functor category, all limits and colimits in \mathcal{C} are defined levelwise. In particular, the k -cubes of a given cubical set $K \in \mathcal{C}$ ($k \geq 0$) form a category \mathcal{C}_K (under inclusions), and $K \cong \text{colim}_{I^k \in \mathcal{C}_K} I^k$.

However, it turns out the products in \mathcal{C} do not behave well with respect to realization (see Remark 1.10 below), so another monoidal operation is needed:

1.5. Definition. If K and L are two cubical sets, their *cubical tensor* $K \otimes L \in \mathcal{C}$ is defined

$$K \otimes L := \text{colim}_{I^j \in \mathcal{C}_K, I^k \in \mathcal{C}_L} I^{j+k}.$$

This defines a symmetric monoidal structure $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ on cubical sets (see [J3, §3]).

More generally, let $\langle \mathcal{V}, \otimes \rangle$ be a monoidal category with (finite) colimits – for example, $\langle \mathcal{T}, \times \rangle$, $\langle \mathcal{S}, \times \rangle$, or $\langle \mathcal{C}, \otimes \rangle$ – and assume we have “standard cubes” in \mathcal{V} , defined by a (faithful) monoidal functor $T : \langle \square, \times \rangle \rightarrow \langle \mathcal{V}, \otimes \rangle$ – that is, a compatible choice of “standard cubes” $T\mathcal{I}^n$ in \mathcal{V} . Given a (finite) cubical set K , for any $X \in \mathcal{V}$ define

$$X \otimes K := \operatorname{colim}_{\mathcal{C}_K} T_X ,$$

where the diagram $T_X : \mathcal{C}_K \rightarrow \mathcal{V}$ is defined by $T_X I^n := X \otimes T\mathcal{I}^n$.

1.6. Definition. For $\langle \mathcal{V}, \otimes \rangle$ as above, the *cubical mapping complex* $\operatorname{map}_{\mathcal{V}}^c(X, Y) \in \mathcal{C}$ is defined for any $X, Y \in \mathcal{V}$ by setting

$$\operatorname{map}_{\mathcal{V}}^c(X, Y)_n := \operatorname{Hom}_{\mathcal{C}}(X \otimes T\mathcal{I}^n, Y) ,$$

with the cubical structure inherited from $\mathcal{I}^n \in \square$ (cf. [K]). We shall generally abbreviate $\operatorname{map}_{\mathcal{V}}^c(X, Y)$ to $\mathcal{V}^c(X, Y)$.

In particular, when \mathcal{V} is \mathcal{C} itself, this makes $\langle \mathcal{C}, \otimes, I^0, \mathcal{C}^c \rangle$ into a symmetric monoidal closed category (see [Bor2, §6.1]).

1.7. Comparison to \mathcal{S} . Cubical sets are related to simplicial sets by a pair of adjoint functors

$$(1.8) \quad \mathcal{C} \underset{\mathcal{S}_{\text{cub}}}{\overset{T}{\rightleftarrows}} \mathcal{S} .$$

The *triangulation* functor T is defined $TK := \operatorname{colim}_{I^n \in \mathcal{C}_K} \Delta[1]^n$ (compare Definition 1.5), where $\Delta[1]^n = \Delta[1] \times \dots \times \Delta[1]$ is the standard simplicial n -cube. The *cubical singular* functor $S_{\text{cub}} : \mathcal{S} \rightarrow \mathcal{C} = \operatorname{Set}^{\square^{\text{op}}}$ is defined adjointly by $(S_{\text{cub}} X)(I^n) := \operatorname{Hom}_{\mathcal{S}}(T\mathcal{I}^n, X)$. This is a singular-realization pair in the sense of [DK4]; composing (1.8) with the usual adjoint pair:

$$(1.9) \quad \mathcal{S} \underset{\mathcal{S}}{\overset{|\cdot|}{\rightleftarrows}} \mathcal{T}$$

yields a similar adjunction to topological spaces.

1.10. Remark. Note that $T : \langle \mathcal{C}, \otimes \rangle \rightarrow \langle \mathcal{S}, \times \rangle$ is strongly monoidal (cf. [Bor2, §6.1]), in that there is a natural isomorphism

$$(1.11) \quad T(K \otimes L) \cong (TK) \times (TL) .$$

On the other hand, $S_{\text{cub}} : \langle \mathcal{S}, \times \rangle \rightarrow \langle \mathcal{C}, \otimes \rangle$ is not strongly monoidal, as we now show: as a right adjoint, S_{cub} commutes with (levelwise) products up to natural isomorphism, so

$$S_{\text{cub}}(X \times Y) \cong S_{\text{cub}}(X) \times S_{\text{cub}}(Y) .$$

Thus, if S_{cub} were strongly monoidal, one would have a levelwise isomorphism

$$S_{\text{cub}}(X) \otimes S_{\text{cub}}(Y) \cong S_{\text{cub}}(X) \times S_{\text{cub}}(Y) .$$

Note this is unlikely, since an n -cube of $K \otimes L$ corresponds to a pair consisting of a j -cube of K (for some $0 \leq j \leq n$) and an $(n-j)$ -cube of L , while an n -cube of $K \times L$ corresponds to a pair consisting of an n -cube of K and an n -cube of L . In fact, $K \times L$ is in general not even homotopy equivalent to $K \otimes L$ for $K, L \in \mathcal{C}$, – for example, $T(I^1 \times I^1) \simeq S^1$ in \mathcal{S} while $T(I^1 \otimes I^1) \cong \Delta[1] \times \Delta[1]$ (see [J1, §1, Remark 8]).

Nevertheless, since I^0 is both terminal in \mathcal{C} and the unit for \otimes , the projections $\pi_K : K \otimes L \rightarrow K \otimes I^0 \cong K$ and $\pi_L : K \otimes L \rightarrow I^0 \otimes L \cong L$ induce a natural map

$$(1.12) \quad \vartheta : K \otimes L \rightarrow K \times L,$$

which is symmetric monoidal in the sense that it commutes with the obvious associativity and switch-map isomorphisms.

1.13. Fact ([J3, §3]). *For any $L \in \mathcal{C}$, the functor $- \otimes L$ preserves monomorphisms in \mathcal{C} .*

1.14. Remark. Note that $- \otimes I^n$ preserves colimits, since it has a right adjoint (defined by constructing the cubical set of maps between two cubical sets as one does in \mathcal{S} – see [J3, §4]). Finally, observe that the cubical mapping complex for \mathcal{S} (Definition 1.6) is simply $\text{map}_{\mathcal{S}}^c(-, -) = S_{\text{cub}} \text{map}_{\mathcal{S}}(-, -)$.

1.15. The model category.

Cubical sets were used quite early on as models for topological spaces – see [Se], [EM], [Mu], [Mc1], [P1, P2], and especially [K1, K2]. However, it was Grothendieck, in [G], who suggested that more generally presheaf categories modeled on certain “test categories” D can serve as models for the homotopy category of topological spaces. Cisinski, in his thesis [C], carried out this program for $D = \square$ (see also the exposition in [J3]). The model category structure is very similar to the analogous one for simplicial sets ($D = \Delta$):

1.16. Definition. A map $f : K \rightarrow L$ in \mathcal{C} is

- a) a *weak equivalence* if $Tf : TK \rightarrow TL$ is a weak equivalence in \mathcal{S} (or equivalently, if $|Tf|$ is a weak equivalence of topological spaces);
- b) a *cofibration* if it is a monomorphism.
- c) a *fibration* if it has the right lifting property (RLP) with respect to all acyclic cofibrations (i.e., those which are also weak equivalences) – that is, if in all commuting squares in \mathcal{C} :

$$(1.17) \quad \begin{array}{ccc} A & \xrightarrow{g} & K \\ i \downarrow & \tilde{h} \nearrow & \downarrow f \\ B & \xrightarrow{h} & L \end{array}$$

where i is an acyclic cofibration, a map $\tilde{h} : B \rightarrow K$ exists making the full diagram commute.

The model category defined here is proper, by [J3, Theorem 8.2].

1.18. Definition. The *cubical spheres* are $S^n := S_{\text{cub}}(\Delta[n]/\partial\Delta[n])$ for $n \geq 1$, with the obvious basepoint. These corepresent the *homotopy groups* $\pi_n(-) := [S^n, -]_*$. Similarly, $S^0 := I^0 \amalg \{*\}$ corepresents π_0 , and a map $f : K \rightarrow L$ in \mathcal{C}_* is a weak equivalence if and only if it induces a π_n -isomorphism for all $n \geq 0$.

Note that we may define the fundamental groupoid $\hat{\pi}_1 K$ of an unpointed cubical set $K \in \mathcal{C}$ as for simplicial sets or topological spaces (cf. [Hig, Chapter 2]).

1.19. *Remark.* In analogy with the case of simplicial sets (see [GJ, Ch. I]) one can show that cofibrations which are weak equivalences are the same as the *anodyne maps* – that is the closure of the set of inclusions of the form

$$(1.20) \quad i : \square_i^{n,\varepsilon} \hookrightarrow I^{n+1}$$

(see §1.2) under cobase change, retracts, coproducts, and countable compositions (see [J3, §4]). Furthermore, the fibrant objects and the fibrations in \mathcal{C} can also be characterized by Kan conditions – having the RLP with respect to maps of the form (1.20) (see [K1] and [J3, Theorem 8.6]).

As noted above (§1.6), \mathcal{C} is a symmetric monoidal closed category (enriched over itself), with cubical mapping complexes $\mathcal{C}^c(-, -)$. As shown in [J1, §3]), it also satisfies the cubical analogue of Quillen’s Axiom SM7 (cf. [Q, II, §2]), so \mathcal{C} deserves to be called a *cubical* model category. In particular, if L is a fibrant (Kan) cubical set, the function complex $\mathcal{C}^c(K, L)$ is fibrant, too, for any (necessarily cofibrant) $K \in \mathcal{C}$.

Finally, the following result shows that \mathcal{C} indeed serves as a model for the usual homotopy category of topological spaces:

1.21. **Proposition** (Cf. [J3, Theorem 8.8]). *The adjoint functors of (1.8) induce equivalences of homotopy categories $\mathrm{ho}\mathcal{C} \cong \mathrm{ho}\mathcal{S}$ (so together with the pair (1.9), we have $\mathrm{ho}\mathcal{C} \cong \mathrm{ho}\mathcal{T}$).*

Note that since I^0 is a final object in \mathcal{C} , the under category $\mathcal{C}_* := I^0/\mathcal{C}$ of pointed cubical sets constitutes a pointed version of \mathcal{C} , and we have:

1.22. **Fact.** *There is a model category structure on \mathcal{C}_* , with the same weak equivalences, fibrations, and cofibrations as \mathcal{C} .*

Proof. See [Ho, Proposition 1.1.8]. □

1.23. Spherical model categories.

Like many other model categories, \mathcal{C}_* enjoys a collection of additional useful properties that were axiomatized in [Bl, §1] under the name of a *spherical* model category. This means that:

- (a) \mathcal{C}_* has a set \mathcal{A} of *spherical objects*: cofibrant homotopy cogroup objects (namely, the cubical spheres $\mathcal{A} = \{S^n\}_{n=1}^\infty$ – Definition 1.18). Furthermore, a map $f : K \rightarrow L$ in \mathcal{C}_* is a weak equivalence if and only if $[A, f]$ is an isomorphism for all $A \in \mathcal{A}$.
- (b) Each $K \in \mathcal{C}_*$ has a functorial *Postnikov tower* of fibrations:

$$(1.24) \quad \dots \rightarrow P_n K \xrightarrow{p^{(n)}} P_{n-1} K \xrightarrow{p^{(n-1)}} \dots \rightarrow P_0 K ,$$

as well as a weak equivalence $r : K \rightarrow P_\infty K := \lim_n P_n K$ and fibrations $r^{(n)} : P_\infty K \rightarrow P_n K$ such that $r^{(n-1)} = p^{(n)} \circ r^{(n)}$ for all n , and $r_\#^{(n)} : \pi_k P_\infty K \rightarrow \pi_k P_n K$ is an isomorphism for $k \leq n$ and zero for $k > n$.

- (c) For every groupoid Λ , there is a functorial *classifying object* $B\Lambda$ with $B\Lambda \simeq P_1 B\Lambda$ and fundamental groupoid $\hat{\pi}_1 B\Lambda \cong \Lambda$, unique up to homotopy.
- (d) Given a groupoid Λ and a Λ -module G (that is, an abelian group object over Λ), for each $n \geq 2$ there is a functorial *extended G-Eilenberg-Mac Lane*

object $E = E^\Lambda(G, n)$ in $\mathcal{C}_*/B\Lambda$, unique up to homotopy, equipped with a section s for $(r^{(1)} \circ r) : E \rightarrow P_1 E \simeq B\Lambda$, such that $\pi_n E \cong G$ as Λ -modules and $\pi_k E = 0$ for $k \neq 0, 1, n$.

- (e) For every $n \geq 1$, there is a functor that assigns to each $K \in \mathcal{C}_*$ a homotopy pull-back square

$$(1.25) \quad \begin{array}{ccc} P_{n+1}K & \xrightarrow{p^{(n+1)}} & P_n K \\ \downarrow \boxed{\text{PB}} & & \downarrow k_n \\ B\Lambda & \longrightarrow & E^\Lambda(M, n+2) \end{array}$$

called an n -th k -invariant square for K , where $\Lambda := \hat{\pi}_1 K$, $M := \pi_{n+1} K$, and $p^{(n+1)} : P_{n+1}K \rightarrow P_n K$ is the given fibration of the Postnikov tower.

The map $k_n : P_n K \rightarrow E^\Lambda(M, n+2)$ is called the n -th (functorial) k -invariant for K .

1.26. Proposition. *The category \mathcal{C}_* is spherical.*

Proof. All the properties for \mathcal{C}_* follow from Proposition 1.21, Fact 1.22, and the analogous results for \mathcal{S}_* or \mathcal{T}_* (see [BJT, Theorem 3.15]). Note that homotopy groups for cubical sets appear in [K1, K2], while (minimal, and thus non-functorial) Postnikov towers for cubical sets were constructed by Postnikov in [P1, P2].

For functorial cubical Postnikov towers, let the n -coskeleton functor $\text{cosk}_n^c : \mathcal{C} \rightarrow \mathcal{C}$ be the right adjoint to sk_n^c , with $r^{(n)} : \text{Id} \rightarrow \text{cosk}_n^c$ the obvious natural transformation, and similarly for \mathcal{C}_* . By construction, $r^{(n)}$ is an isomorphism in dimensions $\leq n$. If $K \in \mathcal{C}_*$ is fibrant, so is $\text{cosk}_n^c K$, and $\pi_i \text{cosk}_n^c K = 0$ for $i > n$, since $\text{sk}_n^c S^i = *$ for $i > n$. Thus if $K \rightarrow K'$ is a functorial fibrant replacement, and we change

$$(1.27) \quad K' \dots \rightarrow \text{cosk}_{n+1}^c K' \rightarrow \text{cosk}_n^c K' \rightarrow \text{cosk}_{n-1}^c K' \dots$$

functorially into a tower of fibrations, we obtain (1.24).

For (strictly) functorial Eilenberg-Mac Lane objects, use [BDG, Prop. 2.2], and apply S_{cub} . For functorial k -invariants in \mathcal{C}_* , use the construction in [BDG, §5-6] (which works in \mathcal{C}_* , too). \square

1.28. Remark. In general, the maps $\text{cosk}_n^c K \rightarrow \text{cosk}_{n-1}^c K$ in (1.27) (adjoint to the inclusion of skeleta) are *not* fibrations (though the original construction of Kan, when applied to a fibrant cubical set K , yields a tower of fibrations with no further modification – see, e.g., [GJ, VI, §2]). However, if we are only interested in a specific Postnikov section P_n , as long as K is fibrant we can use $\text{cosk}_{n+1}^c K$ as a fibrant model for $P_n K$, and need only modify the next section if we want $p^{(n+1)} : P_{n+1}K \rightarrow P_n K$ to be a fibration.

2. CUBICALLY ENRICHED CATEGORIES

In [DK2], Dwyer and Kan showed how any model category (more generally, any small category \mathcal{M} equipped with a class of weak equivalences) can be enriched by simplicial function complexes, so that the resulting simplicially enriched category encodes the homotopy theory of \mathcal{M} (see Remark 3.10 below). Thus the category $s\mathcal{C}at$

of simplicial small categories can be thought of as a “universal model category”, providing a setting for a “homotopy theory of homotopy theories”. Other such universal models were later provided in [DKSm2, §7], [R], and [Be].

An important subcategory of $s\mathcal{C}at$ consists of those simplicial categories with a fixed set of objects. This is a special case of the following:

2.1. Definition. For any set \mathcal{O} , denote by $\mathcal{O}\text{-Cat}$ the category of all small categories \mathcal{D} with $\mathcal{O} = \text{Obj } \mathcal{D}$. More generally, assume $\Gamma \in \mathcal{O}\text{-Cat}$ is a small category, possibly non-unital, and let $\langle \mathcal{V}, \otimes \rangle$ be a monoidal category. A (\mathcal{V}, Γ) -category is a category $\mathcal{D} \in \mathcal{O}\text{-Cat}$ enriched over \mathcal{V} , with mapping objects $\text{map}_{\mathcal{D}}^{\mathcal{V}}(-, -) \in \mathcal{V}$, such that

$$(2.2) \quad \text{Hom}_{\Gamma}(u, v) = \emptyset \Rightarrow \text{map}_{\mathcal{D}}^{\mathcal{V}}(u, v) \text{ is the initial object in } \mathcal{V}.$$

Thus when \mathcal{V} is pointed, we require $\text{map}_{\mathcal{D}}^{\mathcal{V}}(u, v) = *$ whenever $\text{Hom}_{\Gamma}(u, v) = \emptyset$.

The category of all (\mathcal{V}, Γ) -categories will be denoted by $(\mathcal{V}, \Gamma)\text{-Cat}$. The morphisms in $(\mathcal{V}, \Gamma)\text{-Cat}$ are enriched functors which are the identity on \mathcal{O} .

When $\text{Hom}_{\Gamma}(u, v)$ is never empty (so that we may disregard condition (2.2)) we write $(\mathcal{V}, \mathcal{O})\text{-Cat}$ instead of $(\mathcal{V}, \Gamma)\text{-Cat}$. Dwyer and Kan call these \mathcal{O} -diagrams in \mathcal{V} .

2.3. Remark. If Γ is non-unital, $\text{Hom}_{\Gamma}(u, u)$ may be empty, in which case $\text{map}_{\mathcal{D}}^{\mathcal{V}}(u, u)$ will be empty, if $\mathcal{V} = \mathcal{S}et$ or \mathcal{S} . This is allowed in the enriched version of semi-categories (see Remark 0.3). However, the discussion below can be readily carried out in the context of ordinary (enriched) categories, at the cost of paying attention to units. Thus if \mathcal{V} is pointed, $\text{Hom}(u, u)$ has (at least) two maps: the identity and the zero map; these will coincide if u is the zero object.

We shall in fact concentrate on the case where Γ has no self-maps $u \rightarrow u$ – e.g., a non-unital partially ordered set. The main examples of $\langle \mathcal{V}, \otimes \rangle$ to keep in mind are $\langle \mathcal{S}et, \times \rangle$, $\langle \mathcal{G}p, \times \rangle$, $\langle \mathcal{G}pd, \times \rangle$, $\langle \mathcal{S}, \times \rangle$, and $\langle \mathcal{C}, \otimes \rangle$.

2.4. $(\mathcal{S}, \mathcal{O})$ -categories.

Although we shall be mainly concerned with (\mathcal{C}, Γ) -categories, we first recall the more familiar simplicial version:

Note that when $\mathcal{V} = \mathcal{S}$, an (\mathcal{S}, Γ) -category can be thought of as a simplicial object over $\mathcal{O}\text{-Cat}$ (or $(\mathcal{S}et, \Gamma)\text{-Cat}$). Thus each $\mathcal{M}_{\bullet} \in (\mathcal{S}, \mathcal{O})\text{-Cat}$ is a simplicial category with fixed object set \mathcal{O} in each dimension, and all face and degeneracy functors are the identity on objects (cf. [DK1, §1.4]).

2.5. Fact. *The forgetful functor $U : \mathcal{C}at \rightarrow \mathcal{D}i\mathcal{G}$ to the category of directed graphs has a left adjoint $F : \mathcal{D}i\mathcal{G} \rightarrow \mathcal{C}at$, the free category functor (cf. [Ha]).*

2.6. Definition. A simplicial category $\mathcal{E}_{\bullet} \in (\mathcal{S}, \mathcal{O})\text{-Cat}$ is *free* if each category E_n , and each degeneracy functor $s_j : E_n \rightarrow E_{n+1}$, is in the essential image of the functor F .

The pair of adjoint functors of Fact 2.5 defines a comonad $FU : \mathcal{C}at \rightarrow \mathcal{C}at$, and thus for each small category \mathcal{D} , an augmented simplicial category $\mathcal{E}_{\bullet} \rightarrow \mathcal{D}$ with $\mathcal{E}_n := (FU)^{n+1}\mathcal{D}$. If $\mathcal{D} \in (\mathcal{S}et, \Gamma)\text{-Cat}$, then $\mathcal{E}_{\bullet} \in (\mathcal{S}, \Gamma)\text{-Cat}$. We denote this canonical *free simplicial resolution* of \mathcal{D} by $F_s\mathcal{D}$.

2.7. Remark. In [DK1, §1], Dwyer and Kan define a model category structure on $(\mathcal{S}, \mathcal{O})\text{-Cat}$ (also valid for $(\mathcal{S}, \Gamma)\text{-Cat}$), which turns out to be a *resolution model*

category in the sense of [Bou] (see also [J2], [DKSt, §5] and [BJT, §2]). The spherical objects for $(\mathcal{S}, \mathcal{O})\text{-Cat}$ (cf. §1.23(a)) are objects of the form $\mathcal{M}_\bullet := \mathbf{S}_{(u,v)}^n$ for $n \geq 1$ and $\text{Hom}_\Gamma(u, v) \neq \emptyset$, defined by:

$$(2.8) \quad \mathcal{M}(u', v') = \begin{cases} \mathbf{S}^n & \text{for } u' = u \text{ and } v' = v \\ * & \text{otherwise,} \end{cases}$$

One can also show that $(\mathcal{S}, \mathcal{O})\text{-Cat}$ and $(\mathcal{S}, \Gamma)\text{-Cat}$ are spherical – that is, endowed with the additional structure described in §1.23 (of which only the existence of models is guaranteed in a resolution model category).

2.9. The model category $(\mathcal{C}, \Gamma)\text{-Cat}$.

In the case of (\mathcal{C}, Γ) -categories, the situation is somewhat complicated by the fact that they cannot simply be viewed as cubical objects in Cat , because \otimes , and thus the composition maps, are not defined dimensionwise (see Remark 1.10). Berger and Moerdijk have defined a model category structure for algebras over coloured operads in a suitable symmetric monoidal model category, which applies in particular to $(\mathcal{C}, \Gamma)\text{-Cat}$ (see [BM2], and compare [BM1]). However, in this paper we only need to consider (\mathcal{C}, Γ) -categories for a special type of category Γ , for which it is easy to describe an explicit model category structure in which $W\Gamma$ is cofibrant:

2.10. Definition. A small non-unital category Γ will be called a *quasi-lattice* if it has no self-maps; in this case there is a partial ordering on $\mathcal{O} = \text{Obj}(\Gamma)$, with $u \prec v$ if and only if $\text{Hom}_\Gamma(u, v) \neq \emptyset$, and we require in addition that Γ be *locally finite* in the sense that for any $u \prec v$ in \mathcal{O} , the interval $\text{Seg}[u, v] := \{w \in \mathcal{O} \mid u \preceq w \preceq v\}$ is finite.

2.11. Example. The simplest example is a *linear lattice* of length $n + 1$, which we denote by Γ_{n+1} : this consists of a single composable $(n + 1)$ -chain:

$$v_{\text{init}} = (\mathbf{n} + \mathbf{1}) \xrightarrow{\phi_{n+1}} \mathbf{n} \xrightarrow{\phi_n} (\mathbf{n} - \mathbf{1}) \rightarrow \dots \rightarrow \mathbf{2} \xrightarrow{\phi_2} \mathbf{1} \xrightarrow{\phi_1} \mathbf{0} = v_{\text{fin}} .$$

Another example is a commuting square:

$$\begin{array}{ccc} v_{\text{init}} & \xrightarrow{\phi'} & v' \\ \phi'' \downarrow & & \downarrow \psi' \\ v'' & \xrightarrow{\psi''} & v_{\text{fin}} \end{array}$$

Observe that for categories of diagrams indexed on a directed Reedy category (i.e., one for which the “inverse subcategory” is trivial), the Reedy model structure (cf. [Hir, §15.2.2]) agrees with the projective model structure. In this situation, cofibrations of diagrams are those morphisms whose “latching maps” are all cofibrations in the target category, while fibrations and weak equivalences of diagrams are defined objectwise.

Our current context is sufficiently similar to allow an analogous inductive argument, depending on the following analog of Reedy’s latching objects and maps:

2.12. Definition. Given a quasi-lattice Γ , a map $F : \mathcal{A} \rightarrow \mathcal{B}$ in $(\mathcal{C}, \Gamma)\text{-Cat}$, and $u \prec v$ in \mathcal{O} , the *composition category* $(J_{(u,v)}^{\mathcal{A}, \mathcal{B}}, <)$ is a partially ordered set, whose objects are pairs $\langle \omega, \mathcal{X} \rangle$, where ω is a chain $\langle u = w_0 \prec w_1 \prec \dots \prec w_{k-1} \prec w_k = v \rangle$

in $\langle \mathcal{O}, \prec \rangle$, and the index \mathcal{X} is either \mathcal{A} or \mathcal{B} . We omit the copy of the trivial chain $\langle u \prec v \rangle$ indexed by \mathcal{B} .

The partial order is defined by setting $\langle \omega, \mathcal{X} \rangle \leq \langle \omega', \mathcal{X}' \rangle$ whenever ω' is a (not necessarily proper) subchain of ω , and either $\mathcal{X} = \mathcal{X}'$ or $\mathcal{X} = \mathcal{A}$, $\mathcal{X}' = \mathcal{B}$.

The corresponding *composition diagram* $D = D_{(u,v)}^{\mathcal{A},\mathcal{B}} : J_{(u,v)}^{\mathcal{A},\mathcal{B}} \rightarrow \mathcal{C}$ is defined by sending $\langle \omega, \mathcal{X} \rangle$ to $\bigotimes_{j=1}^k \mathcal{X}^c(w_{j-1}, w_j)$. The morphisms are generated by the following two types of maps:

- (i) If ω' is obtained from ω by omitting internal node w_j ($1 < j < k$), the map $D\langle \omega, \mathcal{X} \rangle \rightarrow D\langle \omega', \mathcal{X} \rangle$ is $\text{Id} \otimes \cdots \otimes \text{cmp}_{(w_{j-1}, w_j, w_{j+1})}^{\mathcal{X}} \cdots \otimes \text{Id}$, where

$$\text{cmp}_{(w_{j-1}, w_j, w_{j+1})}^{\mathcal{X}} : \mathcal{X}^c(w_{j-1}, w_j) \otimes \mathcal{X}^c(w_j, w_{j+1}) \rightarrow \mathcal{X}^c(w_{j-1}, w_{j+1})$$

is the cubical composition map in $\mathcal{X} \in \{\mathcal{A}, \mathcal{B}\}$;

- (ii) The map $D\langle \omega, \mathcal{A} \rangle \rightarrow D\langle \omega, \mathcal{B} \rangle$ is $\bigotimes_{i=1}^k F_{(w_{i-1}, w_i)}$.

Note that $F_{(u,v)} : \mathcal{A}^c(u, v) \rightarrow \mathcal{B}^c(u, v)$, together with the composition maps of \mathcal{B} ending in $\mathcal{B}^c(u, v)$, induce a map $\varphi_{(u,v)} : \text{colim} D_{(u,v)}^{\mathcal{A},\mathcal{B}} \rightarrow \mathcal{B}^c(u, v)$. In particular, when $\text{Seg}[u, v] = \{u, v\}$ is minimal, $\text{colim} D_{(u,v)}^{\mathcal{A},\mathcal{B}}$ is simply $\mathcal{A}^c(u, v)$ and $\varphi_{(u,v)}$ is $F_{(u,v)} : \mathcal{A}^c(u, v) \rightarrow \mathcal{B}^c(u, v)$.

We now provide the details of the model category structure on $(\mathcal{C}, \Gamma)\text{-Cat}$ – inter alia, in order to allow the reader to verify that the construction works in the non-unital setting:

2.13. Lemma. *If Γ is a quasi-lattice, the category $(\mathcal{C}, \Gamma)\text{-Cat}$ has all limits and colimits.*

Proof. For any small category Γ , the limits in $(\mathcal{C}, \Gamma)\text{-Cat}$ are constructed by taking the limit at each $(u, v) \in \mathcal{O}^2$, with compositions defined for the product $\prod_{i \in I} \mathcal{A}_i$ by the obvious maps:

$$\left(\prod_{i \in I} \mathcal{A}_i[u, w] \right) \otimes \left(\prod_{i \in I} \mathcal{A}_i[w, v] \right) \rightarrow \prod_{i \in I} (\mathcal{A}_i[u, w] \otimes \mathcal{A}_i[w, v]) \xrightarrow{\text{cmp}_{(u,w,v)}} \prod_{i \in I} \mathcal{A}_i[u, v],$$

and similarly for the other limits.

For the colimits, note that \otimes is defined as a colimit (cf. Definition 1.5), so it commutes with colimits in \mathcal{C} . For (\mathcal{C}, Γ) -categories $\{\mathcal{A}_i\}_{i \in I}$, the coproduct $\mathcal{D} := \coprod_{i \in I} \mathcal{A}_i$ is defined by induction on the cardinality of $\text{Seg}[u, v]$ in $\langle \mathcal{O}, \prec \rangle$. When $\text{Seg}[u, v] = \{u, v\}$ is minimal, we let $\mathcal{D}(u, v) := \coprod_{i \in I} \mathcal{A}_i(u, v)$. In general, set

$$\mathcal{D}(u, v) := \coprod_{i \in I} \mathcal{A}_i(u, v) \amalg \coprod_{u \prec w \prec v} \mathcal{D}(u, w) \otimes \mathcal{D}(w, v),$$

with the obvious (tautological) composition on the right-hand summands.

Now given maps $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{A} \rightarrow \mathcal{E}$ in $(\mathcal{C}, \Gamma)\text{-Cat}$, the pushout \mathcal{PO} is once more defined by induction on the cardinality of $\text{Seg}[u, v]$, as follows:

In the initial case, when $\text{Seg}[u, v]$ is minimal, $\mathcal{PO}(u, v)$ is simply the pushout of $\mathcal{E}(u, v) \leftarrow \mathcal{A}(u, v) \rightarrow \mathcal{B}(u, v)$ in \mathcal{C} .

In the induction step, we let $J = J_{(u,v)}^{\mathcal{PO}}$ denote the union of the composition categories $J_{(u,v)}^{\mathcal{A},\mathcal{B}}$, $J_{(u,v)}^{\mathcal{A},\mathcal{E}}$, and $J_{(u,v)}^{\mathcal{B},\mathcal{PO}}$ (see Definition 2.12). Thus the objects of

J are pairs $\langle \omega, \mathcal{X} \rangle$, where ω is a chain $\langle u = w_0 \prec w_1 \prec \dots w_k = v \rangle$ and $\mathcal{X} \in \{\mathcal{A}, \mathcal{B}, \mathcal{E}, \mathcal{PO}\}$, again omitting $\langle u \prec v, \mathcal{PO} \rangle$. Again J is a partially ordered set, with the order relation defined to be the union of those for $(\mathcal{A}, \mathcal{B})$, $(\mathcal{A}, \mathcal{E})$, and $(\mathcal{B}, \mathcal{PO})$.

The composition diagrams $D_{(u,v)}^{\mathcal{A},\mathcal{B}}$, $D_{(u,v)}^{\mathcal{A},\mathcal{E}}$, $D_{(u,v)}^{\mathcal{B},\mathcal{PO}}$, and $D_{(u,v)}^{\mathcal{E},\mathcal{PO}}$ fit together to form a composition diagram $D_{(u,v)}^{\mathcal{PO}} : J_{(u,v)}^{\mathcal{PO}} \rightarrow \mathcal{C}$. The last two diagrams are well-defined, because we omit the trivial chain $\langle u \prec v, \mathcal{PO} \rangle$, and all other values of $D_{(u,v)}^{\mathcal{B},\mathcal{PO}}$ and $D_{(u,v)}^{\mathcal{E},\mathcal{PO}}$ have already been defined by our induction assumption. We now let $\mathcal{PO}(u, v)$ be the colimit in \mathcal{C} of the diagram $D_{(u,v)}^{\mathcal{PO}} : J_{(u,v)}^{\mathcal{PO}} \rightarrow \mathcal{C}$.

The constructions of the coproducts and pushouts implies that all colimits exist in $(\mathcal{C}, \Gamma)\text{-Cat}$, by the dual of [Bor1, Thm. 2.8.1 & Prop. 2.8.2]. \square

2.14. Definition. Let Γ be a quasi-lattice, and let \mathcal{A} and \mathcal{B} be (\mathcal{C}, Γ) -categories. A map $F : \mathcal{A} \rightarrow \mathcal{B}$ in $(\mathcal{C}, \Gamma)\text{-Cat}$ is

- (a) a *weak equivalence* if $F_{(u,v)} : \mathcal{A}^c(u, v) \rightarrow \mathcal{B}^c(u, v)$ is a weak equivalence in \mathcal{C} (see §1.15) for any $u \prec v$ in \mathcal{O} .
- (b) a *fibration* if $F_{(u,v)} : \mathcal{A}^c(u, v) \rightarrow \mathcal{B}^c(u, v)$ is a (Kan) fibration in \mathcal{C} for all $u \prec v$ in \mathcal{O} .
- (c) a (*acyclic*) *cofibration* if for all $u \prec v$ in \mathcal{O} the maps $F_{(u,v)} : \mathcal{A}^c(u, v) \rightarrow \mathcal{B}^c(u, v)$ and $\varphi_{(u,v)} : \text{colim } D_{(u,v)}^{\mathcal{A},\mathcal{B}} \rightarrow \mathcal{B}^c(u, v)$ are (acyclic) cofibrations in \mathcal{C} .

2.15. Remark. A straightforward induction shows that the acyclic cofibrations so defined are precisely those cofibrations which are weak equivalences.

The following lemmas show that these choices yield a model category structure on $(\mathcal{C}, \Gamma)\text{-Cat}$:

2.16. Lemma. *If Γ is a quasi-lattice, $F : \mathcal{A} \rightarrow \mathcal{B}$ is a cofibration and $P : \mathcal{D} \rightarrow \mathcal{E}$ is an fibration in $(\mathcal{C}, \Gamma)\text{-Cat}$, and either F or P is a weak equivalence, then there is a lifting \tilde{H} in any commutative square*

$$(2.17) \quad \begin{array}{ccc} \mathcal{A} & \xrightarrow{G} & \mathcal{D} \\ F \downarrow & \tilde{H} \nearrow & \downarrow P \\ \mathcal{B} & \xrightarrow{H} & \mathcal{E} \end{array} .$$

Proof. We choose $\tilde{H}_{(u,v)} : \mathcal{B}^c(u, v) \rightarrow \mathcal{D}^c(u, v)$ by induction on the cardinality of the interval $\text{Seg}[u, v]$ in $\langle \mathcal{O}, \prec \rangle$:

When $\text{Seg}[u, v] = \{u, v\}$ is minimal, we simply choose a lift $\tilde{H}_{(u,v)}$ in:

$$\begin{array}{ccc} \mathcal{A}_{(u,v)}^c & \xrightarrow{G_{(u,v)}} & \mathcal{D}_{(u,v)}^c \\ F_{(u,v)} \downarrow & \tilde{H}_{(u,v)} \nearrow & \downarrow P_{(u,v)} \\ \mathcal{B}_{(u,v)}^c & \xrightarrow{H_{(u,v)}} & \mathcal{E}_{(u,v)}^c \end{array}$$

using the fact that $F_{(u,v)}$ is a cofibration and $P_{(u,v)}$ an acyclic fibration in \mathcal{C} (see (1.17) above).

In the induction step, assume we have chosen compatible lifts $\tilde{H}_{(u',v')}$ for all proper subintervals $\text{Seg}[u', v'] \subset \text{Seg}[u, v]$. These yield a map \hat{G} making the following solid square commute in \mathcal{C} :

$$\begin{array}{ccc} \text{colim } D_{(u,v)}^{\mathcal{A},\mathcal{B}} & \xrightarrow{\hat{G}} & \mathcal{D}_{(u,v)}^c \\ \varphi_{(u,v)} \downarrow & \nearrow \tilde{H}_{(u,v)} & \downarrow P_{(u,v)} \\ \mathcal{B}^c(u, v) & \xrightarrow{H_{(u,v)}} & \mathcal{E}^c(u, v) \end{array}$$

and since $\varphi_{(u,v)}$ is a cofibration by Definition 2.14, and $P_{(u,v)}$ is an acyclic fibration by assumption, the lifting $\tilde{H}_{(u,v)}$ exists.

The same argument shows that there exists a lifting in (2.17) when $F : \mathcal{A} \rightarrow \mathcal{B}$ is an acyclic cofibration and $P : \mathcal{D} \rightarrow \mathcal{E}$ is a fibration. \square

2.18. Lemma. *If Γ is a quasi-lattice, any map $F : \mathcal{A} \rightarrow \mathcal{B}$ in $(\mathcal{C}, \Gamma)\text{-Cat}$ factors as:*

$$(2.19) \quad \begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ & \searrow I & \nearrow P \\ & \mathcal{D} & \end{array}$$

where I is a cofibration and P is a fibration; and we can require either I or P to be a weak equivalence.

Proof. Again construct \mathcal{D} , I , and P in (2.19) by induction on the cardinality of $\text{Seg}[u, v]$. When $\text{Seg}[u, v] = \{u, v\}$ is minimal, choose any factorization:

$$\begin{array}{ccc} \mathcal{A}_{(u,v)}^c & \xrightarrow{F_{(u,v)}} & \mathcal{B}_{(u,v)}^c \\ & \searrow I_{(u,v)} & \nearrow P_{(u,v)} \\ & \mathcal{D}_{(u,v)}^c & \end{array}$$

where $I_{(u,v)}$ is an acyclic cofibration and $P_{(u,v)}$ is a fibration in \mathcal{C} .

Now assume by induction that we have chosen compatible factorizations

$$(2.20) \quad \begin{array}{ccccc} \mathcal{A}_{(u,w)}^c \otimes \mathcal{A}_{(w,v)}^c & \xrightarrow{\zeta_{(u,v)}^{\mathcal{A}}} & \text{Col}^{(\mathcal{A})}(u, v) & \xrightarrow{\omega^{\mathcal{A}}} & \mathcal{A}_{(u,v)}^c \\ I_{(u,w)} \otimes I_{(w,v)} \downarrow & & \downarrow \phi_{(u,v)} & & \downarrow \eta_{(u,v)} \\ \mathcal{D}_{(u,w)}^c \otimes \mathcal{D}_{(w,v)}^c & \xrightarrow{\zeta_{(u,v)}^{\mathcal{D}}} & \text{Col}^{(\mathcal{D})}(u, v) & \xrightarrow{\theta_{(u,v)}} & \mathcal{PO}_{(u,v)} \\ P_{(u,w)} \otimes P_{(w,v)} \downarrow & & \downarrow \psi_{(u,v)} & & \downarrow \xi_{(u,v)} \\ \mathcal{B}_{(u,w)}^c \otimes \mathcal{D}_{(w,v)}^c & \xrightarrow{\zeta_{(u,v)}^{\mathcal{B}}} & \text{Col}^{(\mathcal{B})}(u, v) & \xrightarrow{\omega^{\mathcal{B}}} & \mathcal{B}_{(u,v)}^c \end{array}$$

where each cubical set $\text{Col}^{(\mathcal{E})}(u, v)$ (for $\mathcal{E} = \mathcal{A}, \mathcal{B}, \mathcal{D}$) is the colimit over all proper subintervals $\text{Seg}[u', v'] \subset \text{Seg}[u, v]$ and $u' \prec w \prec v'$ of the diagram of composition maps

$$\text{cmp}_{(u',w,v')}^{\mathcal{E}} : \mathcal{E}^c(u', w) \otimes \mathcal{E}^c(w, v') \rightarrow \mathcal{E}^c(u', v').$$

in each row, $\zeta^\mathcal{E} : \mathcal{E}_{(u,w)}^c \otimes \mathcal{E}_{(w,v)}^c \rightarrow \text{Col}^{(\mathcal{E})}(u, v)$ is the structure map for the colimit, while $\omega^\mathcal{E} : \text{Col}^{(\mathcal{E})}(u, v) \rightarrow \mathcal{E}_{(u,v)}^c$ is induced by the compositions. The cubical set $\mathcal{PO}_{(u,v)}$ is the pushout of the upper right-hand square, with structure maps $\eta_{(u,v)}$ and $\theta_{(u,v)}$, and $\xi_{(u,v)}$ is induced on the pushout by $F_{(u,v)}$ and the maps $\psi_{(u,v)}$ (from the naturality of the colimit) and $\omega^\mathcal{B}$.

Note that the map $I_{(u,w)} \otimes I_{(w,v)}$ is an acyclic cofibration in \mathcal{C} (see Fact 1.13), so the induced map $\phi_{(u,v)}$ is, too, as is $\eta_{(u,v)}$, by cobase change. The map $P_{(u,w)} \otimes P_{(w,v)}$, as well as the induced map $\psi_{(u,v)}$, comes from the compatible factorizations (2.20).

Finally, choose a factorization

$$\begin{array}{ccc} \mathcal{PO}(u, v) & \xrightarrow{\xi_{(u,v)}} & \mathcal{B}^c(u, v) \\ & \searrow \zeta_{(u,v)} & \nearrow P_{(u,v)} \\ & & \mathcal{D}^c(u, v) \end{array}$$

where $\zeta_{(u,v)}$ is an acyclic cofibration and $P_{(u,v)}$ is a fibration in \mathcal{C} . This defines the cubical set $\mathcal{D}^c(u, v)$, which is equipped with composition maps

$$\text{cmp}_{(u,w,v)} := \zeta_{(u,v)} \circ \theta_{(u,v)} \circ \zeta_{(u,v)}^{\mathcal{D}}.$$

Setting $I_{(u,v)} := \zeta_{(u,v)} \circ \eta_{(u,v)}$ yields the required acyclic cofibration, and since $\xi_{(u,v)}$ is induced by F , we have $P_{(u,v)} \circ I_{(u,v)} = F_{(u,v)}$, as required.

The same construction, *mutatis mutandis*, yields a factorization (2.19) where I is a cofibration and P an acyclic fibration. \square

2.21. Theorem. *If Γ is a quasi-lattice, Definition 2.14 provides a model category structure on $(\mathcal{C}, \Gamma)\text{-Cat}$.*

Proof. The category $(\mathcal{C}, \Gamma)\text{-Cat}$ is complete and cocomplete by Lemma 2.13. The classes of weak equivalences and fibrations are clearly closed under compositions, and include all isomorphisms. The same holds for cofibrations by an induction argument. Also, if two out of the three maps F , G , and $G \circ F$ are weak equivalences, so is the third. The lifting properties for (co)fibrations are in Lemma 2.16, and the factorizations are given by Lemma 2.18. \square

As expected, the two key types of (\mathcal{V}, Γ) -categories are related by suitable functors (compare [Bor2, Prop. 6.4.3]):

2.22. Proposition. *For any quasi-lattice Γ , the functors $T : \mathcal{C} \rightarrow \mathcal{S}$ and $S_{\text{cub}} : \mathcal{S} \rightarrow \mathcal{C}$ of (1.8) extend to functors $(\mathcal{C}, \Gamma)\text{-Cat} \rightleftharpoons (\mathcal{S}, \Gamma)\text{-Cat}$. Furthermore, this is a strong Quillen pair (cf. [Hir, §8.5.1]), and descends to an adjunction at the level of homotopy categories.*

Proof. The functor T extends to $(\mathcal{C}, \Gamma)\text{-Cat}$ by (1.11). For S_{cub} , given $\mathcal{A}^s \in (\mathcal{S}, \Gamma)\text{-Cat}$, with composition $\xi : \mathcal{A}^s(u, w) \times \mathcal{A}^s(w, v) \rightarrow \mathcal{A}^s(u, v)$ we define the composition map $\text{cmp}_{(u,w,v)} : S_{\text{cub}}(\mathcal{A}^s(u, w)) \otimes S_{\text{cub}}(\mathcal{A}^s(w, v)) \rightarrow S_{\text{cub}}(\mathcal{A}^s(u, v))$ for the (\mathcal{C}, Γ) -category $S_{\text{cub}}\mathcal{A}^s$ to be the composite $S_{\text{cub}}\xi \circ \vartheta$ (see (1.12)). As S_{cub} is a strong right Quillen functor, it follows from the definitions that the extension is also strong right Quillen. \square

2.23. Semi-spherical structure on $(\mathcal{C}, \Gamma)\text{-Cat}$.

The discussion above, including the model category structures, is valid when we replace \mathcal{C} or \mathcal{S} by their pointed versions (see [Ho, Proposition 1.1.8]). Moreover, even though we cannot construct entry-wise spheres for (\mathcal{C}, Γ) -categories as in (2.8), the category $(\mathcal{C}, \Gamma)\text{-Cat}$ may be called *semi-spherical*, in the sense of having the rest of the spherical structure described in §1.23, as follows:

2.24. Definition. Given a quasi-lattice Γ and a (\mathcal{C}_*, Γ) -category \mathcal{A} , its *fundamental groupoid* is the (\mathcal{Gpd}, Γ) -category obtained by applying the fundamental groupoid functor $\hat{\pi}_1$ to \mathcal{A} . Note that because $\hat{\pi}_1 : \mathcal{C} \rightarrow \mathcal{Gpd}$ factors through $T : \mathcal{C} \rightarrow \mathcal{S}$, using (1.11) we see that $\hat{\pi}_1 \mathcal{A}$ is indeed a (\mathcal{Gpd}, Γ) -category (cf. [Bor2, Prop. 6.4.3]).

Similarly, for each $n \geq 2$ the functor π_n , applied entrywise to \mathcal{A} , yields a (\mathcal{Gp}, Γ) -category, which is actually a $(\hat{\pi}_1 \mathcal{A}\text{-Mod}, \Gamma)$ -category (see Definition 2.1). Note that, as for topological spaces, $\pi_n \mathcal{A}$ is a module over $\hat{\pi}_1 \mathcal{A}$.

- a) Each (\mathcal{C}_*, Γ) -category \mathcal{A} has a functorial Postnikov tower, obtained by applying the functors P_n of §1.24 to each $\mathcal{A}^c(u, v)$, and using

$$\begin{aligned} P_n(\mathcal{A}(u, v)) \otimes P_n(\mathcal{A}(v, w)) &\rightarrow P_n(P_n(\mathcal{A}(u, v)) \otimes P_n(\mathcal{A}(v, w))) \\ &\cong P_n(\mathcal{A}(u, v) \otimes \mathcal{A}(v, w)) \rightarrow P_n(\mathcal{A}(u, w)) . \end{aligned}$$

- b) For every (\mathcal{Gpd}, Γ) -category Λ , there is a functorial *classifying object* $B\Lambda \in (\mathcal{C}_*, \Gamma)\text{-Cat}$.
- c) Given a (\mathcal{Gpd}, Γ) -category Λ , and a Λ -module G (i.e., an abelian group object in $(\text{Set}, \Gamma)\text{-Cat}/\Lambda$), for each $n \geq 2$ there is a functorial *extended G-Eilenberg-Mac Lane object* $E^\Lambda(G, n)$ in $(\mathcal{C}_*, \Gamma)\text{-Cat}/B\Lambda$.
- d) For $n \geq 1$, there is a functorial k -invariant square for \mathcal{A} as in (1.25).

All these properties are straightforward for $(\mathcal{S}_*, \Gamma)\text{-Cat}$ (by applying the analogous functors for \mathcal{S}_* componentwise), and they may be transferred to $(\mathcal{C}_*, \Gamma)\text{-Cat}$ using Proposition 2.22.

2.25. Definition. Given a (\mathcal{Gpd}, Γ) -category Λ , a Λ -module G , a (\mathcal{C}_*, Γ) -category \mathcal{A} , and a *twisting map* $p : \mathcal{A} \rightarrow B\Lambda$, we define the n -th (\mathcal{C}, Γ) -cohomology group of \mathcal{A} with coefficients in G to be

$$H_\Lambda^n(\mathcal{A}, G) := [\mathcal{A}, E^\Lambda(G, n)]_{(\mathcal{C}, \Gamma)\text{-Cat}/B\Lambda}.$$

2.26. Remark. Typically, we have $\Lambda = \hat{\pi}_1 \mathcal{A}$, with the obvious map p .

More generally, in [DKSm1] Dwyer, Kan, and Smith give a definition of the $(\mathcal{S}, \mathcal{O})$ -cohomology of any $(\mathcal{S}, \mathcal{O})$ -category with coefficients in a Λ -module G ; and there is also a relative version, for a pair $(\mathcal{A}, \mathcal{B})$ (cf. [DKSm1, §2.1]). It is straightforward to verify that the two definitions of cohomology coincide (when they are both defined) under the correspondence of Proposition 2.22.

3. LATTICES AND HIGHER HOMOTOPY OPERATIONS

We can now define higher homotopy operations as obstructions to rectifying a homotopy commutative diagram $X : \mathcal{K} \rightarrow \text{ho } \mathcal{T}$, using the approach of [BM], with the modification in the pointed case given in [BC]. For this purpose, it is convenient to work with a specific cofibrant cubical resolution of the indexing category \mathcal{K} . We need make no special assumptions about \mathcal{K} at this stage.

Boardman and Vogt originally defined their “bar construction” $W\mathcal{K}$ topologically (see [BV, III, §1]). The $(\mathcal{C}, \mathcal{O})$ -version may be described as follows:

3.1. Definition. The W -construction on a small category \mathcal{K} with $\mathcal{O} = \text{Obj } \mathcal{K}$ is the $(\mathcal{C}, \mathcal{O})$ -category $W\mathcal{K}$, with the cubical mapping complex $W\mathcal{K}(a, b)$ for every $a, b \in \text{Obj}(\mathcal{K})$, constructed as follows:

For every composable sequence

$$(3.2) \quad f_{\bullet} = (a = a_{n+1} \xrightarrow{f_{n+1}} a_n \xrightarrow{f_n} a_{n-1} \dots a_1 \xrightarrow{f_1} a_0 = b)$$

of length $n + 1$ in \mathcal{K} , there is an n -cube $I_{f_{\bullet}}^n$ in $W\mathcal{K}(a, b)$, subject to two conditions:

- (a) The i -th 0-face of $I_{f_{\bullet}}^n$ is identified with $I_{f_1 \circ \dots \circ (f_i \cdot f_{i+1}) \circ \dots \circ f_{n+1}}^{n-1}$, that is, we carry out the i -th composition in the sequence f_{\bullet} (in the category \mathcal{K}).
- (b) The cubical composition

$$W\mathcal{K}(a_0, a_i) \otimes W\mathcal{K}(a_i, a_{n+1}) \rightarrow W\mathcal{K}(a_0, a_{n+1}) = W\mathcal{K}(a, b)$$

identifies the “product” $(n - 1)$ -cube $I_{f_0 \circ \dots \circ f_i}^i \otimes I_{f_{i+1} \circ \dots \circ f_{n+1}}^{n-i-1}$ with the i -th 1-face of $I_{f_{\bullet}}^n$.

3.3. Notation. Note the three different kinds of composition that occur in $W\mathcal{K}$:

- (a) The *internal* composition of \mathcal{K} is denoted by $f \cdot g$, or simply fg .
- (b) The *cubical* composition of $W\mathcal{K}$, denoted by $f \otimes g$, which corresponds to the \otimes -product of the associated cubes.
- (c) The *potential* composition of $W\mathcal{K}$, denoted by $f \circ g$, is the heart of the W -construction: it provides another dimension in the cube for the homotopies between $f \otimes g$ and $f \cdot g$.

Thus a composable sequence f_{\bullet} as in (3.2) (indexing a cube in $W\mathcal{K}$) will be denoted in full by $f_1 \circ \dots \circ f_{n+1}$; the composed map $f_1 f_2 \dots f_{n+1} : a \rightarrow b$ in \mathcal{K} is denoted by $\text{comp}(f_{\bullet})$; and the cubical composite $f_1 \otimes f_2 \otimes \dots \otimes f_{n+1}$ will be denoted by $\otimes f_{\bullet}$ (as an index for a suitable cube in $W\mathcal{K}$).

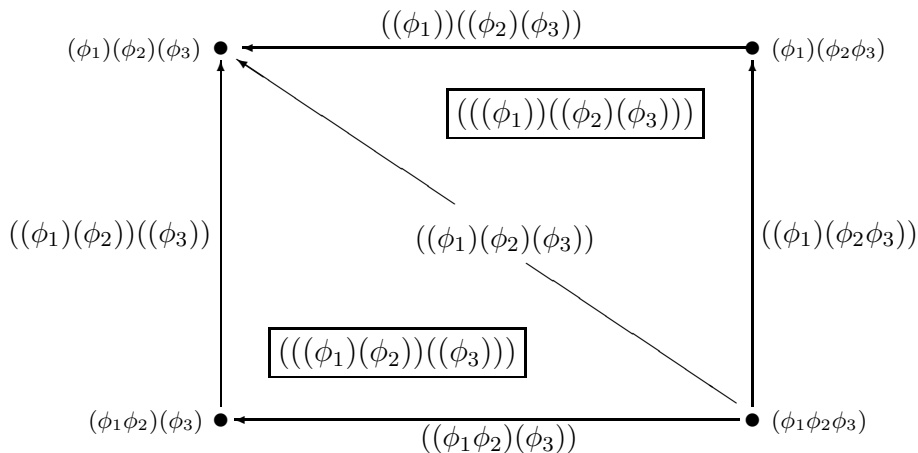
3.4. Definition. The *minimal* vertex of $I_{f_{\bullet}}^n$ is $I_{\text{comp}(f_{\bullet})}^0$, which is in the image of all 0-face maps. The opposite *maximal* vertex, in the image of all 1-face maps, is indexed by $\otimes f_{\bullet}$ according to the convention above, with $I_{f_1}^0 \otimes I_{f_2}^0 \otimes \dots \otimes I_{f_{n+1}}^0$ identified with $I_{\otimes f_{\bullet}}^0$ under the iterated cubical compositions.

If we think of a small category \mathcal{K} as a constant cubical category in $(\mathcal{C}, \mathcal{O})\text{-Cat}$ for $\mathcal{O} = \text{Obj } \mathcal{K}$, there is an obvious map of $(\mathcal{C}, \mathcal{O})$ -categories $\gamma^c : W\mathcal{K} \rightarrow \mathcal{K}$, and following work of [Le] and [Co] we show:

3.5. Lemma. *The map $T\gamma^c : TW\mathcal{K} \rightarrow T\mathcal{K} = \mathcal{K}$ may be identified with $\gamma^s : F_s\mathcal{K} \rightarrow \mathcal{K}$ (see §1.7 ff.).*

Proof. Consider an individual cube $I_{\phi_{\bullet}}^n$ of $W\mathcal{K}$: this is isomorphic to $W\Gamma_{n+1}$, where Γ_{n+1} (Example 2.11) consists of a composable sequence of $n + 1$ maps:

$$(\mathbf{n} + \mathbf{1}) \xrightarrow{\phi_{n+1}} \mathbf{n} \xrightarrow{\phi_n} (\mathbf{n} - \mathbf{1}) \rightarrow \dots \rightarrow \mathbf{2} \xrightarrow{\phi_2} \mathbf{1} \xrightarrow{\phi_1} \mathbf{0}.$$

FIGURE 1. The triangulated 2-cube $F_s \Gamma_3$

The free simplicial resolution of $F_s \Gamma_{n+1}$ is the triangulation of the n -cube $I_{\phi_\bullet}^n$ by $n!$ n -simplices, corresponding to the possible full parenthesizations of ϕ_\bullet (see Figure 1).

This may be identified canonically with the standard triangulation $\Delta[1]^n \in \mathcal{S}$ of $I^n \in \mathcal{C}$ (see [BB, §3]), thus indeed identifying $F_s \mathcal{K}$ with TWK . \square

3.6. Proposition. *If Γ is a quasi-lattice, the map of (\mathcal{C}, Γ) -categories $\gamma^c : W\Gamma \rightarrow \Gamma$ is a cofibrant resolution.*

Proof. The map of $(\mathcal{S}, \mathcal{O})$ -categories $\gamma^s : F_s \mathcal{K} \rightarrow \mathcal{K}$ is a weak equivalence, since F_s is defined by a comonad (see [CP, §1]). Thus $F_s \mathcal{K}$ is indeed a free simplicial resolution of \mathcal{K} (see [DK1, §2.4], [CP, §2], and [BM, §2.21]). Having identified $\gamma^s : F_s \mathcal{K} \rightarrow \mathcal{K}$ with $T\gamma^c : TW\mathcal{K} \rightarrow T\mathcal{K} = \mathcal{K}$, it follows from Proposition 2.22 that γ^c is a weak equivalence.

By construction, each composition map $W\mathcal{K}(a, b) \otimes W\mathcal{K}(b, c) \rightarrow W\mathcal{K}(a, c)$ of $W\mathcal{K}$ is an inclusion of a sub-cubical complex, since on every “product” cube $I_{f_\bullet}^n \otimes I_{g_\bullet}^k \cong I_{f_\bullet \otimes g_\bullet}^{n+k} \subseteq I_{f_\bullet \circ g_\bullet}^{n+k+1}$ it is the inclusion of a 1-face. Thus the map $\varphi_{(u,v)} : \text{colim } D_{(u,v)}^{*, \mathcal{B}} \rightarrow \mathcal{B}^c(u, v)$ of Definition 2.12 is just the inclusion of the sub-cubical complex consisting of all the 1-faces, which is a cofibration (in fact, an anodyne map). This shows that $W\mathcal{K}$ is cofibrant. \square

3.7. Rectifying homotopy commutative diagrams.

We can use the cofibrant resolution $W\mathcal{K} \rightarrow \mathcal{K}$ to study the rectification of a homotopy-commutative diagram $\tilde{X} : \mathcal{K} \rightarrow \text{ho } \mathcal{M}$ in some model category \mathcal{M} (such as \mathcal{T} or \mathcal{T}_*).

Since the 0-skeleton of $W\mathcal{K}$ is isomorphic to $F\mathcal{K}$, choosing an arbitrary representative $X_0(f)$ for each homotopy class $\tilde{X}(f)$ for each morphism f of \mathcal{K} , yields a lifting of \tilde{X} to $X_0 : \text{sk}_0^c W\mathcal{K} \rightarrow \mathcal{M}$.

Note that a choice of a 0-realization $X_0 : F\mathcal{K} \rightarrow \mathcal{M}$ is equivalent to choosing basepoints in each relevant component of each $\mathcal{M}^c(u, v)$, although of course this cannot be done coherently unless \tilde{X} is rectifiable.

3.8. *Remark.* Our goal is to extend X_0 over the skeleta of $W\mathcal{K}$. However, the “naive” cubical skeleton functor $\text{sk}_k^c : \mathcal{C} \rightarrow \mathcal{C}$ (§1.1) is not monoidal with respect to \otimes (unlike the simplicial analogue), so it does not commute with composition maps. Nevertheless, one can define a k -skeleton functor for $(\mathcal{C}, \mathcal{O})$ -categories in general; when Γ is a quasi-lattice (§2.24) and \mathcal{A} is a cofibrant (\mathcal{C}, Γ) -category (such as $W\Gamma$), $\text{sk}_k^c \mathcal{A}$ can be defined by simply including all \otimes -product cubes of i -cubes in \mathcal{A} with $i \leq k$. Of course, if \mathcal{A} is n -dimensional (that is, has no non-degenerate i -cubes for $i > n$), then $\text{sk}_n^c \mathcal{A} = \mathcal{A}$ agrees with the naive n -skeleton.

If \mathcal{M} is cubically enriched (§1.6), extending X_0 to a cubical functor $X_1 : \text{sk}_1^c W\mathcal{K} \rightarrow \mathcal{M}$ is equivalent to choosing homotopies between each $\tilde{X}(f_1 \circ f_2)$ and $\tilde{X}(f_1) \circ \tilde{X}(f_2)$, since the 1-cubes of $W\mathcal{K}$ correspond to all possible (two term) factorizations of maps in \mathcal{K} . Extending X_1 further to $X_2 : \text{sk}_2^c W\mathcal{K} \rightarrow \mathcal{M}$ means choosing homotopies between the homotopies for three-fold compositions, and so on.

This is the idea underlying a fundamental result of Boardman and Vogt:

3.9. **Theorem** ([BV, Cor. 4.21 & Thm. 4.49]). *A diagram $\tilde{X} : \mathcal{K} \rightarrow \text{ho } \mathcal{T}$ lifts to \mathcal{T} if and only if it extends to a simplicial functor $X_\infty : W\mathcal{K} \rightarrow \mathcal{T}$.*

3.10. *Remark.* In fact, for our purposes we do not have to assume that the category \mathcal{M} is cubically enriched, or even has a model category structure: all we need is for \mathcal{M} to have a suitable class of weak equivalences \mathbf{W} , from which we can construct an $(\mathcal{S}, \mathcal{O})$ -category $L(\mathcal{M}, \mathbf{W})$ as in [DK2, §4], and then the corresponding $(\mathcal{C}, \mathcal{O})$ -category $S_{\text{cub}} L(\mathcal{M}, \mathbf{W})$ by Proposition 2.22. Note that when \mathcal{M} and \mathbf{W} are pointed, the construction of Dwyer and Kan is naturally pointed, too. However, to avoid excessive verbiage we shall assume for simplicity that \mathcal{M} is a cubically enriched model category.

We do not actually need the full (usually large) category \mathcal{M} (or $S_{\text{cub}} L(\mathcal{M}, \mathbf{W})$), since we can make use of the following:

3.11. **Definition.** Given a diagram $\tilde{X} : \mathcal{K} \rightarrow \text{ho } \mathcal{M}$ for a model category $\mathcal{M} \in \mathcal{C}\text{-Cat}$, let \mathcal{C}_X be the smallest $(\mathcal{C}, \mathcal{K})$ -category inside \mathcal{M} through which any lift of \tilde{X} to $X : \mathcal{K} \rightarrow \mathcal{M}$ factors. This means that \mathcal{C}_X is the $(\mathcal{C}, \mathcal{K})$ -category having cubical mapping spaces

$$\mathcal{C}_X(Xu, Xv) := \begin{cases} \mathcal{M}^c(Xu, Xv) & \text{if } u \prec v \text{ in } \mathcal{O} := \text{Obj } \mathcal{K} \\ \emptyset & \text{otherwise.} \end{cases}$$

This is a sub-cubical category of \mathcal{M} .

For simplicity, we further reduce the mapping spaces of \mathcal{C}_X so that they consist only of those components of $\mathcal{M}^c(Xu, Xv)$ which are actually hit by \tilde{X} , so that $\pi_0 \mathcal{C}_X = \mathcal{K}$. In particular, if \mathcal{K} is the partially ordered set $\langle \mathcal{O}, \prec \rangle$, we may assume the mapping spaces of \mathcal{C}_X are connected (when they are not empty).

3.12. **Pointed diagrams.** We want to understand the relationship between two possible ways to describe the (final) obstruction to the existence of an extension X_∞ : topologically and cohomologically. Unfortunately, even though these obstructions can be defined for quite general \mathcal{K} , they do not always coincide; this can be seen by comparing the sets in which they take value.

However, we are in fact only interested in the cases where the obstruction can naturally be thought of as the *higher homotopy operation* associated to the data

$\tilde{X} : \mathcal{K} \rightarrow \text{ho } \mathcal{T}$. The usual mantra says that such an operation is defined when “a lower order operation vanishes for two (or more) reasons”. Indeed, the example of the usual Toda bracket shows that the problem cannot be stated simply in terms of rectifying a homotopy-commutative diagram, since any diagram indexed by a linear indexing category Γ_n as above can always be rectified: what we want is to realize certain null-homotopic maps by *zero* maps (see [BM, §3.12]).

This suggests that we restrict attention to *pointed* diagrams, and to the following special type of indexing category:

3.13. Definition. A *lattice* is a finite quasi-lattice Γ (§2.10) equipped with a (*weakly*) *initial* object v_{init} and a (*weakly*) *final* object v_{fin} , satisfying:

- (a) There is a unique $\phi_{\text{max}} : v_{\text{init}} \rightarrow v_{\text{fin}}$.
- (b) For each $v \in \text{Obj } \Gamma$, there is at least one map $v_{\text{init}} \rightarrow v$ and at least one map $v \rightarrow v_{\text{fin}}$.

A composable sequence of n arrows in Γ will be called an *n-chain*. The maximal occurring n (necessarily for a chain from v_{init} to v_{fin} , factorizing ϕ_{max}) is called the *length* of Γ .

3.14. Remark. Note that if the length of Γ is $n + 1$, then $W\Gamma$ is n -dimensional, in the sense that the cubical function complex $W\Gamma(v_{\text{init}}, v_{\text{fin}})$ has dimension n , and $\dim(W\Gamma(u, v)) < n$ for any other pair u, v in Γ .

3.15. Definition. We shall mainly be interested in the case when Γ is pointed (in which case necessarily $\phi_{\text{max}} = 0$). A *null sequence* in Γ is then a composable sequence

$$f_{\bullet} := \left(a_{n+1} \xrightarrow{f_{n+1}} a_n \xrightarrow{f_n} a_{n-1} \dots a_1 \xrightarrow{f_1} a_0 \right)$$

with $\text{comp}(f_{\bullet}) = 0$, but no constituent f_i is zero. It is called *reduced* if all adjacent compositions $f_{i+1} \cdot f_i$ ($i = 1, \dots, n$) are zero. An n -cube $I_{f_{\bullet}}^n$ in $W\Gamma$ indexed by a (reduced) null sequence is called a (reduced) *null cube*.

As noted above, we want to concentrate on the problem of replacing null-homotopic maps with zero maps, given a pointed diagram $\tilde{X} : \Gamma \rightarrow \text{ho } \mathcal{M}$ which commutes up to pointed homotopy. We shall therefore assume from now on that all other (non-zero) triangles in the diagram commute strictly. However, since the non-zero maps in Γ do not form a sub-category, we shall need the following:

3.16. Definition. The *unpointed version* $\mathcal{U}_p(\mathcal{K})$ of a pointed category \mathcal{K} is defined as follows: if $\mathcal{K} \cong F(\mathcal{K})/I$ for some set of relations I in the free category $F(\mathcal{K})$, then the objects of $\mathcal{U}_p(\mathcal{K})$ are those of \mathcal{K} , except for the zero objects, and $\mathcal{U}_p(\mathcal{K}) := F(K')/(I \cap F(K'))$, where K' is obtained from the underlying graph K of \mathcal{K} by omitting all zero objects and maps. The inclusion $K' \hookrightarrow K$ induces a functor $\iota : \mathcal{U}_p(\mathcal{K}) \rightarrow \mathcal{K}$.

Essentially, $\mathcal{U}_p(\mathcal{K})$ is the full subcategory of \mathcal{K} omitting 0 and all maps into or out of the zero object 0. However, if the composite $f \cdot g : a \rightarrow b$ is zero in \mathcal{K} with $f \neq 0 \neq g$, then we add a new (non-zero) map $\varphi : a \rightarrow b$ in $\mathcal{U}_p(\mathcal{K})$ (with $\iota(\varphi) = 0$), to serve as the composite in $\mathcal{U}_p(\mathcal{K})$ of f and g .

3.17. Defining higher operations.

From now on we assume given a pointed lattice Γ and a diagram up-to-homotopy $\tilde{X} : \Gamma \rightarrow \text{ho } \mathcal{M}$ into a pointed cubically enriched model category \mathcal{M} . Setting $\Gamma' := \mathcal{U}_p(\Gamma)$, we also assume that the composite $\tilde{X} \circ \iota$ lifts to a strict diagram $X' : \Gamma' \rightarrow \mathcal{M}$. For simplicity we also denote the factorization of X' through \mathcal{C}_X (§3.11) by $X' : \Gamma' \rightarrow \mathcal{C}_X$.

Our goal is to extend X' to a pointed diagram $X : \Gamma \rightarrow \mathcal{C}_X$. (Note that X' itself cannot be pointed in our sense, but it still takes values in the pointed category \mathcal{M}). Obviously, if X' does extend to such an X , every map $\varphi \in \Gamma'$ which factors through 0 in Γ must be (weakly) null-homotopic in \mathcal{M} . Thus, we additionally include this restriction on the original data as part of our assumptions.

Our approach is to extend X' by induction over the skeleta of $W\Gamma$, where we actually need:

3.18. Definition. Given Γ and X' as above, for each $k \geq 0$ the *relative k -skeleton* for (Γ, Γ') , denoted by $\text{sk}_k^c(\Gamma, \Gamma')$, is the pushout:

$$\begin{array}{ccc} \text{sk}_k^c W\Gamma' & \xrightarrow{\text{sk}_k^c \iota} & \text{sk}_k^c W\Gamma \\ \downarrow \text{sk}_k^c \gamma^c & & \downarrow \\ \Gamma' & \longrightarrow & \text{sk}_k^c(\Gamma, \Gamma') \end{array}$$

in $(\mathcal{C}, \Gamma)\text{-Cat}$ (cf. Lemma 2.13), where $\gamma^c : W\Gamma \rightarrow \Gamma$ is the augmentation of Proposition 3.6.

Note that the natural inclusions $\text{sk}_{k-1}^c \hookrightarrow \text{sk}_k^c$ induce maps $\text{sk}_{k-1}^c(\Gamma, \Gamma') \rightarrow \text{sk}_k^c(\Gamma, \Gamma')$. A map of (\mathcal{C}, Γ) -categories $X'_k : \text{sk}_k^c(\Gamma, \Gamma') \rightarrow \mathcal{C}_X$ extending $X' : \Gamma' \rightarrow \mathcal{C}_X$ is called *k -allowable*.

In particular, if Γ is a lattice of length $n+1$, by Remark 3.14 $W(\Gamma, \Gamma') := \text{sk}_n^c(\Gamma, \Gamma')$ is the pushout

$$\begin{array}{ccc} W\Gamma' & \xrightarrow{\iota} & W\Gamma \\ \downarrow \gamma^c & & \downarrow \\ \Gamma' & \longrightarrow & W(\Gamma, \Gamma') . \end{array}$$

3.19. Remark. X' extends canonically to a pointed map $X_0 : \text{sk}_0^c W\Gamma \rightarrow \mathcal{C}_X$, because $\text{sk}_0^c W\Gamma$ is a free category, and the only new object is 0. Together with γ^c this determines a canonical 0-allowable extension $X'_0 : \text{sk}_0^c(\Gamma, \Gamma') \rightarrow \mathcal{C}_X$.

If Γ is a lattice of length $n+1$, in order to rectify X' we want to extend X'_0 inductively over the relative skeleta $\text{sk}_k^c(\Gamma, \Gamma')$ to an n -allowable map $X'_\infty : W(\Gamma, \Gamma') \rightarrow \mathcal{C}_X$ – equivalently, a map $X_\infty : W\Gamma \rightarrow \mathcal{C}_X$ which agrees with the initial $X' : \Gamma' \rightarrow \mathcal{C}_X$. Recall that because $\dim W\Gamma = \dim W(\Gamma, \Gamma') = n$, X'_n is actually X'_∞ in the sense of Theorem 3.9, so this yields a rectification of X' for suitable \mathcal{M} (such as \mathcal{T}_*).

We assumed in §3.17 that $X' : \Gamma' \rightarrow \mathcal{C}_X$ takes every map $\varphi \in \Gamma'$ which factors through 0 in Γ to one which is null-homotopic in \mathcal{M} . Therefore, by choosing null-homotopies for all such maps we see that X'_0 always extends *non-canonically* to a 1-allowable $X'_1 : \text{sk}_1^c(\Gamma, \Gamma') \rightarrow \mathcal{C}_X$.

However, in general there are obstructions to obtaining k -allowable extensions for $k \geq 2$. These are complicated to define “topologically” (see [BM] and [BC]). Fortunately, in order to define the higher homotopy operation associated to X' , we only need to consider the *last* obstruction.

That is, we assume we have already produced an $(n-1)$ -allowable extension $X'_{n-1} : \text{sk}_{n-1}^c(\Gamma, \Gamma') \rightarrow \mathcal{C}_X$, and want to extend it to X'_n . It may be possible to do so in different ways. In order to define the set $\langle\langle X' \rangle\rangle$ of “last obstructions”, we need the following:

3.20. Lemma. *Assume that $\Gamma = \Gamma_{n+1}$ is a composable $(n+1)$ -chain f_\bullet (§2.11) and that the i -th adjacent composition $f_i \cdot f_{i+1} \neq 0$ in Γ , and let*

$$f'_\bullet := (f_1, \dots, f_{i-1}, f_i \cdot f_{i+1}, f_{i+2}, \dots, f_n).$$

Let $\iota : I_{f'_\bullet}^n \hookrightarrow I_{f_\bullet}^{n+1}$ be the inclusion of the i -th zero face. Let $\tilde{\Gamma}$ be the linear lattice corresponding to f'_\bullet . Then for any $X : \Gamma' \rightarrow \mathcal{C}_X$, the inclusion $\iota : \tilde{\Gamma} \hookrightarrow \Gamma$ induces a one-to-one correspondence between the set of extensions of $X'_0 : \text{sk}_0^c(\Gamma, \Gamma') \rightarrow \mathcal{C}_X$ to $W\Gamma$ and the extensions of $\tilde{X}'_0 : \text{sk}_0^c(\tilde{\Gamma}, \Gamma') \rightarrow \mathcal{C}_X$ to $W\tilde{\Gamma}$.

Proof. The i -th dimension of $I_{f_\bullet}^n$ corresponds to the i -th adjacent composition $f_i \cdot f_{i+1}$ in the $(n+1)$ -chain f_\bullet , and if this composite is not zero, then X'_n , being allowable, is constant along this dimension. Thus the projection $\rho : I_{f_\bullet}^n \rightarrow I_{f'_\bullet}^{n-1}$ induces the inverse to ι^* . \square

3.21. Proposition. *Let Γ be a lattice of length $n+1$ and $X' : \Gamma' \rightarrow \mathcal{C}_X$ a diagram. Let J_Γ be the set of length $n+1$ reduced null sequences of Γ (Definition 3.15). There is a natural correspondence between $(n-1)$ -allowable extensions $X'_{n-1} : \text{sk}_{n-1}^c(\Gamma, \Gamma') \rightarrow \mathcal{C}_X$ of X' and maps $F_{X'_{n-1}} : \bigvee_{f_\bullet \in J_\Gamma} \Sigma^{n-1} X'(v_{\text{init}}) \rightarrow X'(v_{\text{fin}})$, such that $F_{X'_{n-1}}$ is null-homotopic if and only if X'_{n-1} extends to $\text{sk}_n^c(\Gamma, \Gamma')$.*

Proof. In order to extend $X'_{n-1} : \text{sk}_{n-1}^c(\Gamma, \Gamma') \rightarrow \mathcal{C}_X$ to $\text{sk}_n^c(\Gamma, \Gamma')$, we must choose extensions to the n -cubes of $W\Gamma$. These occur only in the full mapping complex $W\Gamma(v_{\text{init}}, v_{\text{fin}})$, and are in one-to-one correspondence with those decompositions

$$f_\bullet = \left(v_{\text{init}} = a_{n+1} \xrightarrow{f_{n+1}} a_n \xrightarrow{f_n} a_{n-1} \dots a_1 \xrightarrow{f_1} a_0 = v_{\text{fin}} \right)$$

of $\phi_{\text{max}} : v_{\text{init}} \rightarrow v_{\text{fin}}$ which are of maximal length $n+1$. Note that the minimal vertex of $I_{f_\bullet}^n$ is indexed by $\phi_{\text{max}} = 0$; the maximal vertex is $I_{\otimes f_\bullet}^0$ (Definition 3.4).

By Lemma 3.20 we need only consider those maximal decompositions f_\bullet for which every adjacent composition $f_i \cdot f_{i+1} = 0$. In this case, we may assume that any facet $I_{f'_\bullet}^{n-1}$ of $I_{f_\bullet}^n$ which touches the vertex labeled by $\phi_{\text{max}} = 0$ has at least one factor of f'_\bullet equal to 0 (in Γ), so $X'_{n-1}|_{I_{f'_\bullet}^{n-1}} = 0$. Thus $X'_{n-1}|_{I_{f_\bullet}^n}$ is given by a map in \mathcal{M} $F'_{(X'_{n-1}, I_{f_\bullet}^n)} : X'(v_{\text{init}}) \otimes \partial I^n \rightarrow X'(v_{\text{fin}})$ which sends $X'(v_{\text{init}}) \otimes I_{\phi_{\text{max}}}^0$ and $*_{X(v_{\text{init}})} \otimes I^n$ to $*_{X(v_{\text{fin}})}$, so it induces $\tilde{F}'_{(X'_{n-1}, I_{f_\bullet}^n)} : X'(v_{\text{init}}) \wedge S^{n-1} \rightarrow X'(v_{\text{fin}})$.

Note further that any two such n -cubes $I_{f_\bullet}^n$ and $I_{g_\bullet}^n$ have distinct maximal vertices $I_{f_\bullet}^0$ and $I_{g_\bullet}^0$, so they can only meet in facets adjacent to the minimal vertex, where \tilde{H} vanishes. Thus altogether X'_{n-1} is described by a map

$$(3.22) \quad F_{X'_{n-1}} : \bigvee_{f_\bullet \in J_\Gamma} \Sigma^{n-1} X'(v_{\text{init}}) \rightarrow X'(v_{\text{fin}}),$$

where J_Γ is the set of length $n + 1$ reduced null sequences of Γ . Clearly, $F_{X'_{n-1}}$ is null-homotopic if and only if X'_{n-1} extends to all of $W\Gamma$, since $W\Gamma(v_{\text{init}}, v_{\text{fin}})$ is $\text{map}(C(\bigvee_{f_\bullet \in J_\Gamma} \Sigma^{n-1} X'(v_{\text{init}})), X'(v_{\text{fin}}))$, up to homotopy, where CK is the cone on K . \square

3.23. Definition. The n -th order *pointed higher homotopy operation* $\langle\langle X' \rangle\rangle$ associated to $X' : \Gamma' \rightarrow \mathcal{C}_X$ as above is defined to be the subset:

$$(3.24) \quad \langle\langle X' \rangle\rangle \subseteq \left[\bigvee_{f_\bullet \in J_\Gamma} \Sigma^{n-1} X'(v_{\text{init}}), X'(v_{\text{fin}}) \right]_{\text{ho } \mathcal{M}}$$

consisting of all maps $F_{X'_{n-1}}$ as above, for all possible choices of $(n - 1)$ -allowable extensions X'_{n-1} , of X' . We say the operation *vanishes* if this set contains the zero class.

4. COHOMOLOGY AND RECTIFICATION

The approach of Dwyer, Kan, and Smith to realizing a homotopy-commutative diagram $\tilde{X} : \Gamma \rightarrow \text{ho } \mathcal{M}$ is also based on Theorem 3.9, which says that \tilde{X} can be rectified if and only if it extends to $W\Gamma$. We do not actually need the full force of their theory, which is why we can work in an arbitrary pointed model category \mathcal{M} , rather than just \mathcal{T}_* (see also Remark 3.10).

Essentially, they define the (possibly empty) moduli space $\text{hc } \tilde{X}$ to be the nerve of the category of all possible rectifications of \tilde{X} (cf. [DKSm2, §2.2]), and $\text{hc}_\infty \tilde{X}$ is the space of all ∞ -homotopy commutative lifts of \tilde{X} in (the simplicial version of) $\text{map}_{\mathcal{C}\text{-cat}}(W\Gamma, \mathcal{M}) = \text{map}_{(\mathcal{C}, \Gamma)\text{-cat}}(W\Gamma, \mathcal{C}_X)$ (§3.11). They then show that $\text{hc } \tilde{X}$ is (weakly) homotopy equivalent to $\text{hc}_\infty \tilde{X}$ (see [DKSm2, Theorem 2.4]). Thus the realization problem is equivalent to finding suitable elements in $\text{map}_{(\mathcal{C}, \Gamma)\text{-cat}}(W\Gamma, \mathcal{C}_X)$. Dwyer, Kan, and Smith also consider a relative version, where \tilde{X} has already been rectified to $Y : \Theta \rightarrow \mathcal{M}$ for some sub-category $\Theta \subseteq \Gamma$ (see [DKSm2, §4]). We shall in fact need only the case $\Theta = \Gamma'$ and $Y = X'$, so we want an element in $\text{map}_{(\mathcal{C}, \Gamma)\text{-cat}}(W(\Gamma, \Gamma'), \mathcal{C}_X)$ (see §3.18).

4.1. The tower. If Γ is a quasi-lattice, $(\mathcal{C}, \Gamma)\text{-Cat}$ has a semi-spherical model category structure (see §2.9 and §2.23). Therefore, the Postnikov tower $\{P^m \mathcal{C}_X\}_{m=0}^\infty$ of the (\mathcal{C}, Γ) -category \mathcal{C}_X allows us to define $\text{hc}_m \tilde{X} := \text{map}_{(\mathcal{C}, \Gamma)\text{-cat}}(W(\Gamma, \Gamma'), P^{m-1} \mathcal{C}_X)$ for $m \geq 1$. Note that $P^0 \mathcal{C}_X$ is homotopically trivial – that is, each component of each mapping space $(P^0 \mathcal{C}_X)(u, v)$ is contractible – so $\text{hc}_1 \tilde{X}$ is, too. Moreover, $\tilde{X} : \Gamma \rightarrow \text{ho } \mathcal{M}$ (or $X' : \Gamma' \rightarrow \mathcal{C}_X$) determines a canonical “tautological” component of $\text{hc}_1 \tilde{X}$ – namely, the component of the map $\tilde{X}_1 : W(\Gamma, \Gamma') \rightarrow P^0 \mathcal{C}_X$, corresponding to the canonical 0-allowable extension $X'_0 : \text{sk}_0^c(\Gamma, \Gamma') \rightarrow \mathcal{C}_X$ of §3.19.

Because \mathcal{C}_X is weakly equivalent to the limit of its Postnikov tower (§1.23(b)), the space $\text{hc}_\infty \tilde{X}$ is the homotopy limit of the tower:

$$(4.2) \quad \text{hc}_\infty \tilde{X} \rightarrow \dots \rightarrow \text{hc}_n \tilde{X} \rightarrow \text{hc}_{n-1} \tilde{X} \dots \rightarrow \text{hc}_1 \tilde{X} .$$

In general, there are \lim^1 problems in determining the components of $\text{hc}_\infty \tilde{X}$ (see [DKSm1, §4.8]), but these will not be relevant to us here, because of the following:

4.3. Lemma. *If Γ has length $n + 1$, the tower (4.2) is constant from $\mathrm{hc}_{n-1} \widetilde{X}$ up.*

Proof. We may assume that \mathcal{C}_X is fibrant (e.g., if $\widetilde{X}v$ is a cubical Kan complex for each $v \in \mathcal{O}$). Then $\mathrm{sk}_n^c(\Gamma, \Gamma') = W\Gamma$ by Remark 3.14, where in this case we are using the naive n -skeleton (see Remark 3.8) which is left adjoint to the n -coskeleton functor. By Remark 1.28, we may use the latter for $P^{n-1}\mathcal{C}_X$. Thus the choices of n -allowable extensions $X'_n : \mathrm{sk}_n^c(\Gamma, \Gamma') = W\Gamma \rightarrow \mathcal{C}_X$ of \widetilde{X} are in natural one-to-one correspondence with lifts $\widetilde{X}_n : W(\Gamma, \Gamma') \rightarrow P^{n-1}\mathcal{C}_X$ of \widetilde{X}_1 . \square

4.4. The obstruction theory.

In view of the above discussion, the realization problem for $\widetilde{X} : \Gamma \rightarrow \mathrm{ho} \mathcal{M}$ – and in particular, the pointed version for $X' : \Gamma' \rightarrow \mathcal{M}$ (see §3.17) – can be solved if one can successively lift the element $\widetilde{X}_1 \in \mathrm{hc}_1 \widetilde{X}$ through the tower (4.2). In fact, we do not really need the (simplicial or cubical) mapping spaces $\mathrm{hc}_m \widetilde{X} := \mathrm{map}_{(\mathcal{C}, \Gamma)\text{-cat}}(W(\Gamma, \Gamma'), P^{m-1}\mathcal{C}_X)$ at all – we simply need to lift the maps $\widetilde{X}_m : W(\Gamma, \Gamma') \rightarrow P^{m-1}\mathcal{C}_X$ in the Postnikov tower for \mathcal{C}_X .

Let $k_{m-1} : \mathcal{C}_X \rightarrow E^G(\pi_m \mathcal{C}_X, m+1)$ be the $(m-1)$ -st k -invariant for \mathcal{C}_X , where $G := \widehat{\pi}_1 \mathcal{C}_X$ (see §2.23 *ff.*). Given a lifting \widetilde{X}_m , composing it with k_{m-1} yields a map $h(X_m) : W(\Gamma, \Gamma') \rightarrow E^G(\pi_m \mathcal{C}_X, m+1)$:

$$\begin{array}{ccc}
 W(\Gamma, \Gamma') & \xrightarrow{\widetilde{X}_m} & P^{m-1}\mathcal{C}_X \\
 \downarrow p & \nearrow \widetilde{X}_{m+1} & \downarrow k_{m-1} \\
 & P^m \mathcal{C}_X \longrightarrow & P^{m-1}\mathcal{C}_X \\
 & \downarrow & \downarrow k_{m-1} \\
 & BG \xrightarrow{s} & E^G(\pi_m \mathcal{C}_X, m+1) \\
 & \longleftarrow \text{proj} &
 \end{array}$$

To identify $h(X_m)$ as an element in the appropriate cohomology group (Definition 2.25), note that in this case the twisting map $p : W(\Gamma, \Gamma') \rightarrow BG$ factors through $\widehat{\pi}_1 X_n : \widehat{\pi}_1 W(\Gamma, \Gamma') \rightarrow \widehat{\pi}_1 P_{m-1}\mathcal{C}_X = \widehat{\pi}_1 \mathcal{C}_X = G$, and by Proposition 3.6, the fundamental groupoid $\widehat{\pi}_1 W(\Gamma, \Gamma') = \Gamma$ is discrete. Thus $[h(X_m)]$ takes value in $H_\Gamma^{m+1}(W(\Gamma, \Gamma'); \pi_m \mathcal{C}_X)$, which we abbreviate to $H^{m+1}(\Gamma; \pi_m \mathcal{C}_X)$.

The lifting property for a fibration sequence (over BG) then yields:

4.5. Proposition ([DKSm2, Prop. 3.6]). *The map \widetilde{X}_m lifts to \widetilde{X}_{m+1} in $\mathrm{hc}_{m+1} \widetilde{X}$ if and only if $[h(X_m)]$ vanishes in $H^{m+1}(\Gamma; \pi_m \mathcal{C}_X)$.*

4.6. Relating the two obstructions.

In order to see how the two obstructions we have described are related, we need some more notation:

For a pointed lattice Γ of length $n+1$, let $\widehat{W}(\Gamma, \Gamma')$ denote the sub- (\mathcal{C}, Γ) -category of $W(\Gamma, \Gamma')$ obtained from $\mathrm{sk}_{n-1}^c(\Gamma, \Gamma')$ by adding all unreduced null n -cubes (Definition 3.15). By Lemma 3.20, any $(n-1)$ -allowable extension $X'_{n-1} : \mathrm{sk}_{n-1}^c(\Gamma, \Gamma') \rightarrow \mathcal{C}_X$ extends canonically to $\widehat{X} : \widehat{W}(\Gamma, \Gamma') \rightarrow \mathcal{C}_X$. If $i_n : \mathrm{sk}_n \widehat{W}(\Gamma, \Gamma') \rightarrow \widehat{W}(\Gamma, \Gamma')$

and $i : \widehat{W(\Gamma, \Gamma')} \rightarrow W(\Gamma, \Gamma')$ are the inclusions, we thus have a commutative diagram in $(\mathcal{C}, \Gamma)\text{-Cat}$:

$$\begin{array}{ccc}
 \text{sk}_{n-1}^c \widehat{W(\Gamma, \Gamma')} & \xrightarrow{i_{n-1}} & \widehat{W(\Gamma, \Gamma')} \\
 \text{sk}_{n-1}^c i = \text{Id} \downarrow & \searrow X'_{n-1} & \downarrow \widehat{x} \\
 \text{sk}_{n-1}^c(\Gamma, \Gamma') & \xrightarrow{X'_{n-1}} & \mathcal{C}_X
 \end{array}$$

Because Γ is a lattice of length $n + 1$, $W\Gamma$ is n -dimensional. Furthermore, if we break up any chain in Γ into disjoint sub-chains of length k and ℓ ($k + \ell = n + 1$), the resulting composite cube has dimension $(k - 1) + (\ell - 1) = n - 1$. Thus the only non-degenerate n -cubes in $W\Gamma$ are indecomposable in $W\Gamma(v_{\text{init}}, v_{\text{fin}})$, which implies that $\text{sk}_{n-1}^c(\Gamma, \Gamma')$ is in fact defined using the naive $(n - 1)$ -skeleton (see Remark 3.8).

Thus by adjointness (using Remark 1.28) we have:

$$(4.7) \quad \begin{array}{ccc}
 \widehat{W(\Gamma, \Gamma')} & \xrightarrow{\widehat{x}} & \mathcal{C}_X \\
 i \downarrow & & \downarrow r \\
 W(\Gamma, \Gamma') & \xrightarrow{\widetilde{X}_n} & \text{cosk}_{n-1}^c \mathcal{C}_X = P_{n-2} \mathcal{C}_X
 \end{array}$$

in which r is the fibration $r^{(n-1)} = p^{(n-1)}$ of §1.23(b).

Now let \mathcal{R}_Γ be the (\mathcal{C}, Γ) -category of all *reduced* null $(n - 1)$ -spheres (that is, boundaries of the reduced null n -cubes) in $W\Gamma$. Thus:

$$(4.8) \quad \mathcal{R}_\Gamma^c(u, v) = \begin{cases} \bigcup_{f_\bullet \in J_\Gamma} \partial I_{f_\bullet}^n & \text{if } (u, v) = (v_{\text{init}}, v_{\text{fin}}) \\ \emptyset & \text{otherwise} \end{cases}$$

(in the notation of (3.22)).

4.9. Fact. *There is a homotopy cofibration sequence of (\mathcal{C}, Γ) -categories*

$$(4.10) \quad \mathcal{R}_\Gamma \xrightarrow{j} \widehat{W(\Gamma, \Gamma')} \xrightarrow{i} W(\Gamma, \Gamma') .$$

Proof. By definition of a pointed lattice, all the n -cubes of $W\Gamma$ (and thus of $W(\Gamma, \Gamma')$) are null cubes. Thus the map $i : \widehat{W(\Gamma, \Gamma')} \rightarrow W(\Gamma, \Gamma')$ is actually an isomorphism in all mapping slots except $(u, v) = (v_{\text{init}}, v_{\text{fin}})$, where the n -cells attached via j provide the missing (necessarily reduced) null n -cubes. \square

4.11. **Definition.** Let Γ be a pointed lattice of length $n+1$, \mathcal{C}_X a (\mathcal{C}, Γ) -category, and define J_Γ as in Proposition 3.21. To each commuting square:

$$(4.12) \quad \begin{array}{ccc} \widehat{W(\Gamma, \Gamma')} & \xrightarrow{\hat{h}} & \mathcal{C}_X \\ i \downarrow & & \downarrow r \\ W(\Gamma, \Gamma') & \xrightarrow{h} & P_{n-2}\mathcal{C}_X \end{array}$$

in $(\mathcal{C}, \Gamma)\text{-Cat}$, we assign the composite $k_{n-2} \cdot h$ in $H^n(\Gamma, \pi_{n-1}\mathcal{C}_X)$. Denote by $K_n(\mathcal{C}_X)$ the subset of $H^n(\Gamma, \pi_{n-1}\mathcal{C}_X)$ consisting of all such elements $k_{n-2} \cdot h$. Finally, define $\Phi_n : K_n(\mathcal{C}_X) \rightarrow \prod_{f_\bullet \in J_\Gamma} \pi_{n-1}\mathcal{C}_X(v_{\text{init}}, v_{\text{fin}})$ by assigning to (4.12) the homotopy class of the composite $\sigma := (\hat{h} \cdot j)(v_{\text{init}}, v_{\text{fin}}) : \mathcal{R}_\Gamma(v_{\text{init}}, v_{\text{fin}}) \rightarrow \mathcal{C}_X(v_{\text{init}}, v_{\text{fin}})$.

4.13. **Lemma.** *The map Φ_n is well-defined.*

Proof. Freudenthal suspension gives an isomorphism

$$[\mathcal{R}_\Gamma(v_{\text{init}}, v_{\text{fin}}), \mathcal{C}_X(v_{\text{init}}, v_{\text{fin}})]_{\text{ho } \mathcal{C}} \xrightarrow{\cong} [\Sigma \mathcal{R}_\Gamma, E^{\hat{\pi}_1 W\Gamma}(\pi_{n-1}\mathcal{C}_X, n)]_{\text{ho } (\mathcal{C}, \Gamma)\text{-Cat}},$$

so Φ_n may be equivalently defined by assigning to the composite $k_{n-2} \cdot h$ the extension $e = \Sigma\sigma$ in the following diagram:

$$\begin{array}{ccc} \widehat{W(\Gamma, \Gamma')} & \xrightarrow{h} & \mathcal{C}_X \\ i \downarrow & & \downarrow p \\ W\Gamma & \xrightarrow{h} & P_{n-2}\mathcal{C}_X \\ \partial \downarrow & \searrow^{k_{n-2} \cdot h} & \downarrow k_{n-2} \\ \Sigma \mathcal{R}_\Gamma & \xrightarrow[e]{} & E^G(\pi_{n-1}\mathcal{C}_X, n) \end{array}$$

where $W\Gamma \xrightarrow{\partial} \Sigma \mathcal{R}_\Gamma \xrightarrow{\Sigma j} \Sigma \widehat{W(\Gamma, \Gamma')}$ is the continuation of the cofibration sequence of (4.10). Here we used the fact that \mathcal{R}_Γ is concentrated in the $(v_{\text{init}}, v_{\text{fin}})$ slot, by (4.8).

Note that the extension e (and thus $\sigma = \Phi_n(k_{n-2} \cdot h)$, the adjoint of e with respect to the (Σ, Ω) adjunction) is uniquely determined up to homotopy, since $[\Sigma \widehat{W(\Gamma, \Gamma')}, E^{\hat{\pi}_1 W\Gamma}(\pi_n \mathcal{C}_X, n+1)] = 0$ for dimension reasons. \square

Our main result, Theorem A of the Introduction, is now a consequence of the following Theorem and Corollary:

4.14. **Theorem.** *Given $X' : \Gamma' \rightarrow \mathcal{M}$ as in §3.17, the map Φ_n is a pointed correspondence between the set of elements of $K_n(\mathcal{C}_X)$ obtained from commuting squares of the form (4.7) and $\langle\langle X' \rangle\rangle$ of (3.24) – that is, $\Phi_n(\alpha) = 0$ if and only if $\alpha = 0$.*

Proof. By Proposition 4.5, the composite $h(X_{n-1}) := k_{n-2} \cdot \widehat{X}_{n-1}$ is the obstruction to extending \widehat{X} to $X_n : W(\Gamma, \Gamma') \rightarrow \mathcal{C}_X$, and since

$$\begin{array}{ccc}
 \mathcal{R}_\Gamma & \xrightarrow{\sigma} & E^G(\pi_{n-1}\mathcal{C}_X, n-1) \\
 \downarrow j & & \downarrow \ell \\
 \widehat{W(\Gamma, \Gamma')} & \xrightarrow{\widehat{X}} & \mathcal{C}_X \\
 \downarrow i & \nearrow X_n & \downarrow p \\
 W\Gamma & \xrightarrow{\widehat{X}_{n-1}} & P_{n-2}\mathcal{C}_X \\
 \downarrow \partial & & \downarrow k_{n-2} \\
 \Sigma\mathcal{R}_\Gamma & \xrightarrow{e} & E^G(\pi_{n-1}\mathcal{C}_X, n)
 \end{array}$$

commutes, with the left vertical column a cofibration and the right vertical column a fibration sequence, the fact that $e = 0 \Leftrightarrow \sigma = 0$ implies that the composite $0 = e \cdot \partial = k_{n-2} \cdot \widehat{X}_{n-1} = h(X_{n-1})$. Conversely, if X_n exists, then $\widehat{X} \cdot j = X_n \cdot i \cdot j = 0$, so $\ell \cdot \sigma = 0$, and since $\pi_{n-1}\ell$ is an isomorphism, $\sigma = 0$. \square

4.15. Corollary. *The Dwyer-Kan-Smith obstruction class $[h(X_{n-1})]$ of Proposition 4.5 is zero in $H_\Gamma^n(W\Gamma; \pi_{n-1}\mathcal{C}_X)$ if and only if the corresponding homotopy class $F_{X'_{n-1}}$ is null. Therefore, $\langle\langle X' \rangle\rangle$ vanishes if and only if $K_n(\mathcal{C}_X)$ contains 0.*

4.16. Remark. Evidently, both the classes $[h(X_{n-1})]$ and the set $\langle\langle X' \rangle\rangle$ serve as obstructions to rectifying $\tilde{X} : \Gamma \rightarrow \text{ho}\mathcal{M}$, given the unpointed rectification $X' : \Gamma' \rightarrow \mathcal{M}$. It is therefore clear that they must “vanish” simultaneously. The point of our analysis is to give an explicit correspondence between the individual elements of $\langle\langle X' \rangle\rangle$ and cohomology classes in $H_\Gamma^n(W\Gamma; \pi_{n-1}\mathcal{C}_X)$. By thus describing higher homotopy operations in cohomological terms, we may hope to use algebraic methods to study and calculate them.

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