Derived functors of graded algebras

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Abstract

A number of spectral sequences arising in homotopy theory have the derived functors of a graded algebraic functor as their E^2-term. We here describe conditions for the vanishing of such derived functors, yielding vanishing lines for the spectral sequences. We also show that under these conditions the n-th derived functor, for large n, depends only on low-dimensional information. The applications we have in mind include certain cases of the Bousfield-Kan spectral sequence of [3], the Quillen homology of a graded algebra (with applications to the "Grothendieck spectral sequence" of [14]), and the wedge, smash, and homology spectral sequences of [St] and [1].

1 Introduction

In [1], we described a spectral sequence converging to the homology of a space (see §1.1.3 below), and show that its E^2-term has a lower vanishing line of slope 1/2. In this paper we show that the methods of [1] apply in a more general context, yielding comparable vanishing lines for other spectral sequences. Furthermore, an extension of these methods makes the E^2-terms just above these vanishing lines more accessible to computation.

1.1 vanishing lines for spectral sequences

In each case, the E^2-term the spectral sequence is isomorphic to the left derived functors of a functor $T : \mathcal{C} \to \mathcal{A}$, where $\mathcal{C}$ is a suitable category of graded algebras with additional structure, and $\mathcal{A}$ is Abelian; the vanishing lines follow from a general fact about the vanishing of such derived functors (Theorem 3.1 below). In particular, we have the following results:

1.1.1 the Bousfield-Kan spectral sequence

Bousfield and Kan defined, for any prime p and suitable pointed spaces $\mathbf{X}$ and $\mathbf{Y}$, a spectral sequence abutting to the p-primary part of the homotopy groups of the pointed mapping space, $\mathbf{map}_\star(\mathbf{X}, \mathbf{Y})$ (see [3, §11,12] and [4, ch. X, §6]).

We show that if $\mathbf{X}$ and $\mathbf{Y}$ have locally finite $\mathbf{Z}/p$-homology, $\mathbf{Y}$ is $(r-1)$-connected, and $H_i(\mathbf{X}; \mathbf{Z}/p) = 0$ for $i \geq s$, then $E^2_{n,k} = 0$ for $n > k+s-r$ (see §3.2.1 below).

1.1.2 Quillen homology of algebras

In [14, §4], H. Miller describes a "Grothendieck" spectral sequence converging to the E^2-term of the Bousfield-Kan spectral sequence. The main ingredient needed to calculate its E^2-term is the (graded) Quillen homology $H^Q_\star X$ (cf. [16, §2]) of a suitable graded $\mathbf{F}_p$-algebra X. We show that if X is $(r-1)$-connected, then $(H^Q_n X)_k = 0$ for $n > (k-r)/r$ (see §3.2.3 below). Of course, the Quillen homology of an algebra also has independent interest.

1.1.3 the homology spectral sequence

In [1, §2], we described, for any pointed connected CW-complex $\mathbf{X}$, a spectral sequence converging to $\tilde{H}_\star(\mathbf{X}; G)$ for any coefficients G (see §3.2.4 below). We showed there that if $\mathbf{X}$ is $(r-1)$-connected $(r \geq 3)$, then $E^2_{n,k} = 0$ for $n > 2(k-r)$ ([1, Thm. 4.1]). This is in fact a special case of Proposition 3.1.2 below.

1.1.4 the wedge and smash spectral sequences

Let $\mathbf{X}$ and $\mathbf{Y}$ be pointed CW-complexes, with $\mathbf{X}$ $(r-1)$-connected and $\mathbf{Y}$ $(s-1)$-connected $(r, s \geq 3)$; in [St, §2], C. Stover describes a spectral sequence converging to $\pi_\star(\mathbf{X} \vee \mathbf{Y})$. We show that $E^2_{n,k} = 0$ for $n > 2(k-(r+s)) + 3$ (see §3.3.4(I) below).

Similarly, in the smash spectral sequence of [St, §7], for $\mathbf{X}$ and $\mathbf{Y}$ as above we have $E^2_{n,k} = 0$ for $n > 2(k-(r+s)) + 1$ (see §3.3.4(II) below).

Both these vanishing lines are best possible, by Propositions 4.5.1 and 4.5.2 respectively.

1.2 dependence results

Using a similar result (Theorem 4.3) for relative derived functors (cf. §4.1), we obtain "bands of dependence" for these spectral sequences, showing that, for large n, $E^2_{n,k}$ depends only on low-dimensional information, in the following sense:

1.2.1 the Bousfield-Kan spectral sequence

For $\mathbf{X}$, $\mathbf{Y}$ and s as in §1.1.1 above, Theorem 4.3 implies that $E^2_{n,k}$ of the Bousfield-Kan spectral sequence depends only on $H_\star(\mathbf{Y};\mathbf{Z}/p)$ in degrees $\leq k+s-n$.

1.2.2 the homology spectral sequence

For any 2-connected space $\mathbf{X}$, let $\mathbf{X}^{[t]}$ denote the t-th stage in a Postnikov tower for $\mathbf{X}$; then $E^2_{n,k}$ of the homology spectral sequence for $\mathbf{X}$ (§1.1.3) depends only on $\mathbf{X}^{[t]}$ for $n \geq 2(k-t)+1$, by the same theorem.

1.2.3 the wedge and smash spectral sequences

In the wedge spectral sequence for $\mathbf{X} \vee \mathbf{Y}$ (§1.1.4), we show that if $\mathbf{X}$ and $\mathbf{Y}$ are as in §1.1.4, then for any $t > r+s$, $E^2_{n,k}$ depends only on $\mathbf{X}^{[t-s]}$ and $\mathbf{Y}^{[t-r]}$ for $n \geq 2(k-t)+4$, by Proposition 4.3.3 below.

Similarly, in the smash spectral sequence for $\mathbf{X} \wedge \mathbf{Y}$, $E^2_{n,k}$ depends only on $\mathbf{X}^{[t-s]}$ and $\mathbf{Y}^{[t-r]}$ for $n \geq 2(k-t)+2$, by the same proposition.

1.3 outline

The paper is organized as follows:

In section 2 we define CRGA's, which are essentially categories of graded algebras, possibly with additional structure; we then define the *degree* of a functor $T : \mathcal{C} \to \mathcal{A}$ from a CRGA into an Abelian category, and recall the definition of derived functors in this context.

This allows us to state the Theorem 3.1 (and its variants) in section 3, and apply them to obtain the vanishing lines of §1.1.

In section 4 we then define relative derived functors $(L_nT)f$ for any morphism $f : X \to Y$ in $\mathcal{C}$, and state the relative vanishing theorem 4.3 for such derived functors; this yields the dependence results of §1.2.

In section 5 we review some facts about free simplical resolutions in CRGA's, and show how a resolution with certain connectivity properties can be constructed for any RGA (Proposition 5.1.4), and how this can be used to prove Theorem 3.1.

In section 6 we refine the construction of section 5 by introducing a certain filtration on the resolution, and show how this can be used to prove Theorem 4.3.

In section 7 we make a sample computation, based on section 6, showing that the vanishing lines of §1.1.4 are best possible

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2 Categories of regular graded algebras

In this section we describe the setting needed to state the basic vanishing theorem 3.1. In §2.1 we define CRGA's – essentially, categories of graded algebras with additional structure – and give some examples in §2.2. In §2.3 we define the *degree* of a functor $T : \mathcal{C} \to \mathcal{A}$ from such a category into an Abelian category. In §2.4 we recall the definition of the derived functors of T.

2.1 CRGA's

We first define the categories to which our vanishing theorems apply:

2.1.1 definition

A category $\mathcal{C}$ is called a *category of regular graded algebras*, or *CRGA*, over a ring R iff the following hold:

(i) $\mathcal{C}$ is a *variety of graded algebras* in the sense of [11, V, §6] (or graded *universal algebras* in the sense of [9]): that is, its objects, which we shall call *RGA's*, are positively graded sets $X = \{X_k\}_{k=1}^{\infty}$, together with an action of a fixed set of operators W, satisfying a set of identities E.

(ii) Each RGA has the structure of a positively graded left R-module.

(iii) The ring R has finite *left global dimension*: $l.g.d.(R\text{-}Mod) = d < \infty$ ([12, VII, §1]). In the applications we have in mind, $R = \mathbf{F}_p$ is a finite field, or else $R \subseteq \mathbf{Q}$ – so that $d = 0$ or 1.

(iv) The operators W of $\mathcal{C}$ may include a graded product $X_i \times X_j \to X_{i+j}$; all others (aside from the R-module operators) are required to be unary and dimension-raising, of the form $\omega : X_j \to X_k$ with $k > j$.

2.1.2 remark on the operators

We could in fact allow more general n-ary operators in §2.1.1(iv) – that is, any operator $\omega : X_{j_1} \times \ldots \times X_{j_n} \to X_k$ is permissible, as long as we have $\sum_{i=1}^{n} j_i \leq k$, and strict inequality if $n = 1$.

2.1.3 notation

For each $k \geq 1$ we have the *k-th degree functor* $G_k : \mathcal{C} \to R\text{-}Mod$, which assigns to $X = \{X_k\}_{k=1}^{\infty}$ the R-module $G_k X = X_k$. (We adopt this somewhat cumbersome notation so that subscripts may later be reserved for simplicial dimensions). For each X, we let $: X :$ denote the least $k \leq \infty$ such that $G_k X \neq 0$, and say that that X is $(k-1)$-*connected*.

2.1.4 free RGA's

The forgetful functor $\mathcal{C} \to grSet$ into the category of positively-graded sets has a left adjoint $F : grSet \to \mathcal{C}$, which assigns to a graded set $T = \{T_i\}_{i=1}^{\infty}$ the *free* RGA FT generated by T under the operators W, subject to the identities E. We shall consider each element $x \in T_i$ to be in $G_i(FT)$.

Let $\mathcal{F}$ denote the full subcategory of free RGA's in $\mathcal{C}$ – that is, the image of the functor F.

2.2 examples of CRGA's

The following are the basic examples of CRGA's to keep in mind:

(I) For any ring R with $l.g.d.(R\text{-}Mod) < \infty$, the category $gr R\text{-}Mod$ of positively graded R-modules is a CRGA.

(II) For the following graded categories $\mathcal{C}$, the full subcategory of 0-*connected* objects constitutes a CRGA:

1. for any prime p, the categories of (stable or unstable) modules, or algebras, over the mod-p Steenrod algebra;
2. for $R \subset \mathbf{Q}$, or R a field, the categories of graded (associative or commutative) algebras over R, or
3. the category of graded Lie algebras over R, (restricted or not).

(III) The category of simply-connected Π-algebras:

Recall (cf. [1, §3] or [St, §4]) that a Π-*algebra* is a graded group $X = \{X_k\}_{k=1}^{\infty}$ together with an action of the primary homotopy operations (cf. [17, ch. X]) which satisfies all the universal relations on such operations.

In our case, in order to have a graded *Abelian* group, we must restrict to the subcategory $\Pi\text{-}Alg_0 \subset \Pi\text{-}Alg$ of *simply-connected* Π-*algebras* – i.e., those objects $X = \{X_k\}_{k=1}^{\infty}$ for which $X_1 = 0$.

Note that the Whitehead product does not quite satisfy the usual graded product rule $: a \cdot b :=: a : + : b :$. Thus, in order for Π-Alg_0 to satisfy condition 2.1.1(iv) of the definition we must re-index, setting $G_k X = X_{k-1}$.

2.3 cross-effects

Let $\mathcal{C}$ be a CRGA (or any pointed category), and $T : \mathcal{C} \to \mathcal{A}$ a functor into an Abelian category. Since in $\mathcal{C}$ the inclusions of X or Y into the coproduct $X \amalg Y$ have retractions, in $\mathcal{A}$ the objects TX and TY are split summands of $T(X \amalg Y)$. The remainder term, denoted $T_2(X, Y)$, is called the *second cross-effect* of T applied to the coproduct $X \amalg Y$. We thus have a canonical decomposition: $T(X \amalg Y) \cong TX \oplus TY \oplus T_2(X, Y)$.

More generally, set $T_1(X) = TX$, and let $X = X_1 \amalg X_2 \amalg \ldots, X_n$, be any n-fold coproduct in $\mathcal{C}$. We recursively define the n-*th cross-effect* of T on this coproduct, denoted $T_n(X_1, X_2, \ldots, X_n)$, to be the remainder summand in $T(\ X_1 \amalg X_2 \amalg \ldots X_n\ )$ after splitting off the q-fold cross-effects $(q < n)$ for each proper sub-coproduct $X_{i_1} \amalg X_{i_2} \amalg \ldots X_{i_q}$ of X (cf. [5, §4.18]).

2.3.1 cross-effect decomposition

Thus, for $T : \mathcal{C} \to \mathcal{A}$ as above and $X = \coprod_{i=1}^{n} X_i$ a finite coproduct in $\mathcal{C}$, we have a direct-sum *cross-effect decomposition* for TX:

$$T(\ \coprod_{i=1}^{n} X_i\ ) \ \cong\ [\ \bigoplus_{i=1}^{n} T_1(X_i)\ ] \oplus [\ \bigoplus_{i \neq j}^{n} T_2(X_i, X_j)\ ] \oplus \ldots \oplus T_n(X_1, \ldots X_n)$$

2.3.2 degree of a functor

If $\mathcal{C}$ is a CRGA, and $\mathcal{A}$ is Abelian, we say that a functor $T : \mathcal{C} \to \mathcal{A}$ has *degree*$\leq t$ iff for any $q \geq 1$, and any set of q objects $X_1, X_2, \ldots, X_q$ in $\mathcal{C}$ such that $\sum_{i=1}^{q} : X_i :> t$, we have $T_q(X_1, X_2, \ldots X_q) = 0$. (In particular, $TX = 0$ if $: X :> t$.)

2.3.3 examples

(I) If $\mathcal{C}$ is a CRGA and $T : \mathcal{C} \to \mathcal{A}$ is a functor which preserves finite coproducts and vanishes on t-connected objects, then T has degree t.

(II) For $\mathcal{C}$ as above, the t-th degree functor (§2.1.3), restricted to the subcategory of free objects (§2.1.4) – that is, the functor $G_t :_{\mathcal{F}}$: $\mathcal{F} \to$*R-Mod* – has degree t (by condition 2.1.1(iv) and §2.1.2).

2.4 non-Abelian derived functors

We now recall Quillen's definition of derived functors in the context of CRGA's – see [15, II, §4] and [16, §2] :

2.4.1 free simplical RGA's

A simplicial RGA $A_\bullet$ is called *free* iff for each $n \geq 0$ there is a graded set $T^n \subseteq A_n$ such that A_n is the free RGA generated by T^n (§2.1.4), and each degeneracy map $s_j : A_n \to A_{n+1}$ takes T^n into T^{n+1}. The sequence $T^0, T^1, \dots$ will be called a *set of generators* for $A_\bullet$.

2.4.2 free simplical resolutions

We define a *free simplical resolution* of an RGA X to be a free simplical RGA $A_\bullet$, together with an augmentation $d_0 : A_0 \to X$, such that for each $k \geq 1$

(a) the homotopy groups of the simplicial R-module $G_k A_\bullet$ vanish in dimensions $n \geq 1$;

(b) the augmentation induces an isomorphism $\pi_0(G_k A_\bullet) \cong G_k X$.

2.4.3 definition of derived functors

Let $\mathcal{C}$ be a CRGA, and $T : \mathcal{C} \to \mathcal{A}$ a functor into an Abelian category. The n-*th left derived functor* of T is the functor $(L_n T) : \mathcal{C} \to \mathcal{A}$, which assigns to the RGA $X \in \mathcal{C}$ the object $(L_n T)X \cong \pi_n(TA_\bullet) \in \mathcal{A}$, where $A_\bullet \to X$ is any free simplical resolution of X. (As usual, different resolutions yield equivalent derived functors.)

2.4.4 remark on the domain and range of T

It is clear from definition 2.4.3 that the functor T need only be defined on $\mathcal{F} \subset \mathcal{C}$ to determine the derived functors $L_n T$ on all of $\mathcal{C}$. Assuming that $\mathcal{A}$ is an exact subcategory of the category *Abgp* of Abelian groups, we shall therefore restrict attention to functors of the form $T : \mathcal{F} \to$ *Abgp*.

3 Vanishing of derived functors

In this section we state our basic vanishing theorem for functors of finite degree defined on a CRGA, and use it to obtain the vanishing lines of §1.1.

Theorem 3.1, and its variants 3.1.2 and 3.1.4, deal with (covariant) functors $T : \mathcal{C} \to$ *Abgp*; applications are given in §3.2. Bifunctors $T : \mathcal{C} \times \mathcal{C} \to$ *Abgp* are treated in §3.3.

3.1 Theorem. *Let $\mathcal{C}$ be a CRGA over a ring R, with l.g.d.*(R-Mod) $= d$, *and* $T : \mathcal{F} \to Abgp$ *a functor of degree k; let* $r > d$ *and* $N = (d+1)(k-r)+d$. *Then for any* $(r-1)$*-connected* $X \in \mathcal{C}$, $(L_nT)X = 0$ *for* $n > N$.

In certain cases, the vanishing point can be improved by one; for this we need the following definition:

3.1.1 k-monomorphisms

A morphism $f : X \to Y$ of free RGA's (§2.1.4) will be called a *k-monomorphism* iff

(a) X is $(k-1)$-connected;

(b) $Y \cong Z \amalg Z'$, where Z and Z' are free RGA's and Z is $(k-1)$-connected, and f factors through a map $g : X \to Z$ – i.e., $f = i \circ g$, where $Z \overset{i}{\hookrightarrow} Y$ is the inclusion;

(c) $G_k g$ is a monomorphism.

3.1.2 Proposition. *If the assumptions of Theorem 3.1 hold, and in addition T takes k-monomorphisms in $\mathcal{C}$ to monomorphisms of Abelian groups, then* $(L_nT)X = 0$ *for* $n \geq N$.

3.1.3 improved vanishing

If a given CRGA has no unary operaors $\omega : X_j \to X_{j+1}$ (§2.1.1(iv)) raising degree by exactly one, these vanishing results can be improved. We exemplify this principle by the category $\mathcal{A}_F$ of *connected graded algebras* over a field F (having no unary operators at all):

3.1.4 Proposition. *Let* $\mathcal{C} = \mathcal{A}_F$, *let* $T : \mathcal{F} \to Abgp$ *be a functor of degree k, and let* $X \in \mathcal{A}_F$ *be* $(r-1)$*-connected. Then* $(L_nT)X = 0$ *for* $n > (k-r)/r$.

3.1.5 connectivity of X

As we shall see in the proof (§5.4.3), one obtains analogous vanishing of derived functors also for $d \geq r \geq 1$. For instance, if $d = r = 1$ and T is of degree k, then $(L_n T)X = 0$ for $n > 2k$ (and $(L_2 T)X = 0$ if $k = 1$). We omit the details.

3.2 applications to spectral sequences

As noted in §1.1, these results about derived functors yield vanishing lines for the E^2-terms of a number of spectral sequences:

3.2.1 the Bousfield-Kan spectral sequence

Recall that the E^2-term of the Bousfield-Kan spectral sequence, converging to the homotopy groups of the mapping space $\mathbf{map}_\star(\mathbf{X},\ \mathbf{Y})$ (§1.1.1), is isomorphic to certain *right* derived functors ([3, §11,12]):

$$E^2_{n,k} \;\cong\; (R_n(Hom_{\mathcal{CA}}(H_\star(S^k\mathbf{X};\mathbf{Z}/p),\ -)))\ H_\star(\mathbf{Y};\mathbf{Z}/p)$$

Here $\mathcal{CA}$ denotes the category of unstable coalgebras over the mod-p Steenrod algebra – which is not a CRGA. Thus Theorem 3.1 is not directly applicable here.

However, if $H_\star(\mathbf{X};\mathbf{Z}/p)$ and $H_\star(\mathbf{Y};\mathbf{Z}/p)$ are locally finite, then (as in [14, §5]) one can take vector space duals to replace the coalgebras by algebras and right derived functors by left derived functors. The theorem may then be applied to the CRGA $\mathcal{K}$ of algebras over the Steenrod algebra (§2.2(II)1.), and the functors $T^k = Hom_{\mathcal{K}}(-,\ H^\star(S^k\mathbf{X};\mathbf{Z}/p)\ )$, with $E^2_{n,k} \cong (L_n T^k)H^\star(\mathbf{Y};\mathbf{Z}/p)$.

In particular, if $H^\star(\mathbf{X};\mathbf{Z}/p) = 0$ vanishes in dimensions $\geq s$, then the functor T^k, which takes finite coproducts to direct sums, has degree $s+k$. Thus if $\mathbf{Y}$ is $(r-1)$-connected, Theorem 3.1 implies that $E^2_{n,k} = 0$ for $n > (s+k) - r$.

3.2.2 right derived functors

Alternatively, it is possible to develop a dual version of section 2, with suitable categories of "graded coalgebras" over R replacing our CRGA's (§2.1.1), a dual notion of degree (§2.3), and the usual "triple-derived" functors (cf.

[2, §7]) replacing the left derived functors of §2.4, using *injective*, rather than *free*, resolutions.

It should be noted that the analogue of Theorem 3.1 exists only when R is a field, and with certain restrictions on the triple used to define the injectives.

3.2.3 Quillen homology

Let $\mathcal{A}$ denote the category of *connected graded algebras* over $\mathbf{F}_p$, and Q the functor which takes an algebra X to the graded $\mathbf{F}_p$-vector space of its *indecomposables*: $QX = \{Q_k X\}_{k=1}^{\infty}$ (cf. [14, §4]). Each functor Q_k clearly has degree k (§2.3.3(I)), so that if X is $(r-1)$-connected, then $(L_n Q_k)X = 0$ for $n > (k-r)/r$, by Proposition 3.1.4.

This gives a vanishing line of slope r for the Quillen homology of X, (cf. [16, §2], where $(H_n^Q X)_k \cong (L_n Q_k)X$. For an application of this fact to a calculation in the Bousfield-Kan spectral sequence, see [10, §6].

3.2.4 Π-algebra indecomposables

For the category Π-Alg_0, one also has an *indecomposables* functor, which takes a Π-algebra X to the graded Abelian group $QX = \{Q_k X\}_{k=1}^{\infty}$, defined to be the quotient of X by the subgroup of elements which are in the image of a "non-trivial" primary homotopy operation (see [1, §2.2.1]). The Hurewicz spectral sequence of [1], which converges to the reduced homology of a pointed CW-complex $\mathbf{X}$, has $E^2_{n,k} \cong (L_n Q_k)(\pi_\star \mathbf{X})$.

Each functor Q_k again has degree k, and also takes k-monomorphisms to monomorphisms (definition 3.1.1). Thus if $: X := r$, by Proposition 3.1.2 we have $(L_n Q_k)X = 0$ for $n > 2(k-r)$. This is the vanishing line for the Hurewicz spectral sequence of [1, Thm. 4.1].

3.3 bifunctors

There are analogous vanishing results for (covariant) bifunctors. If $\mathcal{C}$ is a CRGA and $(X, Y) \in \mathcal{C} \times \mathcal{C}$, we write $: (X, Y) :=: X : + : Y :$. With this notation, §2.3.2 also defines the degree of T for a *bifunctor* $F : \mathcal{C} \times \mathcal{C} \to \mathcal{A}$, and we have:

3.3.1 Proposition. *Let $\mathcal{C}$ be a CRGA over R, with $l.g.d.$*(R-Mod) $= d$*,* *and* $T : \mathcal{F} \times \mathcal{F} \to Abgp$ *a functor of degree* t *such that* $T(X,0) = T(0,X) = 0$ *for all* $X \in \mathcal{C}$. *Then for* $(X,Y) \in \mathcal{C} \times \mathcal{C}$ *such that* $: X := r$ *and* $: Y := s$ $(r, s > d)$, *we have* $(L_nT)(X,Y) = 0$ *for* $n > N = (d+1)(t-(r+s)) + 2d$.

3.3.2 t- bimonomorphisms

If $\mathcal{C}$ is CRGA, we say that a morphism $(f,g) : (X,Y) \to (U,V)$ in $\mathcal{F} \times \mathcal{F}$ is an t-*bimonomorphism* iff f is an p-monomorphism and g is a q-monomorphism (definition 3.1.1) with $p+q=t$.

Then we by analogy with Proposition 3.1.4 we have the following

3.3.3 Proposition. *If the hypotheses of Proposition 3.3.1 hold with* $d \geq 1$, *and in addition* T *is takes* t*-bimonomorphisms in* $\mathcal{F}\times\mathcal{F}$ *to monomorphisms of Abelian groups, then* $(L_nT)(X,Y) = 0$ *for* $n \geq N$.

3.3.4 applications

We illustrate the results for bifunctors in the category of Π-algebras:

(I) Let $pr_i : \mathcal{F}\times\mathcal{F} \to \mathcal{F}$ be the projections on the two factors, $(i = 1, 2)$, and $COP : \mathcal{F}\times\mathcal{F} \to \mathcal{F}$ the *coproduct* functor: $COP(X,Y) = X \amalg Y$. There is then a *cross-term* functor $CRT : \mathcal{F} \times \mathcal{F} \to \Pi\text{-}Alg_0$, and for each $k \geq 1$ there is a natural isomorphism of functors

$$\phi_t : \; G_t \circ COP \xrightarrow{\sim} (G_t \circ pr_1) \oplus (G_t \circ pr_2) \oplus (G_t \circ CRT) \; : \mathcal{F}\times\mathcal{F} \to Abgp$$

(This is just the canonical 2-fold cross-effect decomposition of §2.3).

Let $T = G_t \circ CRT$; then T has degree t by Hilton's theorem [7, Thm. A], and $T(X,0) = T(0,X) = 0$ for every $X \in \mathcal{F}$ by definition of the functor CRT. Also, T takes t-bimonomorphisms in $\mathcal{F} \times \mathcal{F}$ to monomorphisms of Abelian groups. Thus Proposition 3.3.3 applies to T.

Moreover, if we set $\bar{T} = G_t \circ COP$, then ϕ_t induces a natural isomorphism $\mathbb{L}_n\bar{T} \cong L_nT \; : \Pi\text{-}Alg_0 \times \Pi\text{-}Alg_0 \to Abgp$ for $n \geq 1$.

This gives the vanishing line of §1.1.4 for the wedge spectral sequence, since it has $E^2_{n,k} \cong (L_nG_kCOP)(\pi_\star\mathbf{X}, \pi_\star\mathbf{Y})$ (cf. [St, §2]).

(II) Simliarly, the smash functor in the homotopy category of pointed spaces $\wedge : ho\mathcal{T}_\star \times ho\mathcal{T}_\star \to ho\mathcal{T}_\star$ allows us to define a functor of Π-algebras $S : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ (cf. [St, §7] and [1, §7.1.2]). Let $T = G_t \circ S : \mathcal{F} \times \mathcal{F} \to Abgp$; then T has degree $t-1$ (by Hilton's theorem), and Proposition 3.3.3 applies here, too. Again, this yields the vanishing line of §1.1.4 for the smash spectral sequence.

4 Vanishing of relative derived functors

In this section we extend the results of section 3 to the *relative* derived functors $(L_nT)f$ of morphisms $f : X \to Y$, and then use these to deduce the *dependence* results of §1.2.

In §4.1 we define relative derived functors (in the the general situation of [15]), and the *algebraic* k-*skeleton* in §4.2. We then state the relative vanishing theorem 4.3, and the version for bifunctors 4.3.3. In §4.4 we list some applications.

4.1 relative derived functors

Let $\mathcal{C}$ be a category with finite limits and enough projectives, $\mathcal{A}$ an Abelian category, and $T : \mathcal{C} \to \mathcal{A}$ any functor. We let $s\mathcal{C}$ denote the category of simplicial objects over $\mathcal{C}$, giving it the closed model category structure of [15, II, §4]. As in the Abelian case, one has a relative version of the non-Abelian derived functors defined in §2.4:

4.1.1 definition

Given a morphism $f : X \to Y$ in $\mathcal{C}$, the *relative derived functors* $(L_nT)f \in Obj(\mathcal{A})$ are defined as follows:

For any projective resolutions $A_\bullet \to X$, $B_\bullet \to Y$ (e.g., the *free* resolutions of §2.4.2), there is a morphism $\hat{f} : A_\bullet \to B_\bullet$ in $s\mathcal{C}$, unique up to weak equivalence.

Applying the functor T gives $T\hat{f} : TA_\bullet \to TB_\bullet$ in $s\mathcal{A}$. This factors into

$$TA_\bullet \xrightarrow{i} C_\bullet \xrightarrow{p} TB_\bullet ,$$

with i a cofibration and p a weak equivalence fibration in $s\mathcal{A}$; the factorization is unique up to weak equivalence. We define $(L_nT)f$ to be

$\pi_n cok(i)$, where $cok(i)$ is the cokernel of i in $\mathcal{A}$.

4.1.2 the long exact sequence

Since cofibrations in $s\mathcal{A}$ are monomorphisms when $\mathcal{A}$ is Abelian (see [15, ibid.]), the long exact sequence of the pair $TA_\bullet \hookrightarrow C_\bullet$ gives rise to a natural long exact sequence of derived functors:

$$\textbf{(LES)} \qquad \ldots (L_nT)X \xrightarrow{f_\star} (L_nT)Y \longrightarrow (L_nT)f \xrightarrow{\partial_\star} (L_{n-1}T)X \ldots$$

4.1.3 an equivalent definition

Equivalently, one choose a factorization $T\hat{f} = p \circ i$, with i a weak equivalence cofibration and p a fibration in $s\mathcal{A}$, and set $(L_nT)f = \pi_{n-1}ker(p)$. Note that when $\mathcal{A}$ is not Abelian, the two definitions differ!

4.1.4 a special case

Assume now that we can choose $\hat{f} : A_\bullet \to B_\bullet$ so that for each $n \geq 0$, the morphism $\hat{f}_n : A_n \to B_n$ has a retraction $r_n : B_n \to A_n$. Then we know that $T\hat{f}_n : TA_n \to TB_n$ is a (split) monomorphism, so that $\hat{f}$ is a cofibration and thus

$$(L_nT)f \;\cong\; \pi_n(TB_\bullet,\, TA_\bullet) \;=\; \pi_n(\, TB_\bullet/TA_\bullet\,)\ .$$

4.2 the algebraic k-skeleton

For any CRGA $\mathcal{C}$, we have the following

4.2.1 definition

For each $k \geq 1$, we can construct a new CRGA $\mathcal{C}^{(k)}$ out of $\mathcal{C}$ by disregarding degrees $> k$. There is an obvious *truncation* functor $\tau_k : \mathcal{C} \to \mathcal{C}^{(k)}$; if we restrict τ_k to the free subcategory: $\tau_k :_{\mathcal{F}}: \mathcal{F} \to \mathcal{F}^{(k)}$, it has a left adjoint $\lambda_k : \mathcal{F}^{(k)} \to \mathcal{F}$. We then define the k-*skeleton* functor to be the composition: $\rho_k = \lambda_k \circ (\tau_k :_{\mathcal{F}}) : \mathcal{F} \to \mathcal{F}$. This is equipped with a natural transformation $\theta_k : \rho_k \to id_{\mathcal{F}}$, which is an isomorphism in degrees $\leq k$.

4.2.2 $\rho_k X$

Note that if we apply the 0-th derived functor $L_0\rho_k$ to X, the resulting graded R-module actually has the structure of an RGA; in fact, it is just the RGA freely generated by the elements (and satisfying the relations) of X in degrees $\leq k$.

As usual, we denote $(L_0\rho_k)X$ by $\rho_k X$, too, and call it the *algebraic* k-*skeleton* of X.

We can now state the relative version of Theorem 3.1:

4.3 Theorem.

Let $\mathcal{C}$ be a CRGA over a ring R, such that:

(a) *every projective R-module is free;*

(b) *$l.g.d.(R\text{-}Mod) = d \leq 1$.*

Let $T : \mathcal{F} \to Abgp$ be a functor of degree t. Then for any RGA X with $:X:> d$, the morphism $\theta_k : \rho_k X \to X$ induces an isomorphism $(L_nT)(\ \rho_k X\ ) \cong (L_nT)X$ for $n \geq (d+1)(t-k)+d$, and a monomorphism for $n = (d+1)(t-k)+d-1$.

4.3.1 Corollary. *For $\mathcal{C}$ and T as above, let X and Y be d-connected RGA's such that $\tau_k X \cong \tau_k Y$ – that is, X and Y agree in degrees$\leq k$. Then $(L_nT)X \cong (L_nT)Y$ for $n \geq (d+1)(t-k)+d$.*

4.3.2 remark

In fact, the theorem has more content than the corollary, since it implies that, for sufficiently large n, $(L_nT)X$ can be calculated using a certain free simplicial resolution $A[k]_\bullet \to \rho_k X$ defined below, which is generally more accessible than the full resolution $A_\bullet \to X$. Once more one could obtain better results for specific categories, such as $\mathcal{A}_F$ (as in §3.1.3).

We have a similar result for bifunctors:

4.3.3 Proposition. *Let $\mathcal{C}$ be a CRGA over R, with $l.g.d.$(R-Mod) $= d$, and $T : \mathcal{F} \times \mathcal{F} \to Abgp$ a functor of degree t, such that $T(X,0) = T(0,X) = 0$ for any $X \in \mathcal{C}$. Then for $(X,Y) \in \mathcal{C} \times \mathcal{C}$ such that $:X:= r$ and $:Y:= s$, and $k \geq r+s$, $(\theta_{k-s}, \theta_{k-r})$ induces an isomorphism*

$(L_nT)(\ \rho_{k-s}X,\ \rho_{k-r}Y\ )\ \cong\ (L_nT)(X,Y)$ for $n \geq (d+1)(t-k)+2d$, and a monomorphism for $n = (d+1)(t-k)+2d-1$.

4.4 applications of relative vanishing

These results apply respectively to most of the examples of section 3 – that is, to the $Hom_{\mathcal{K}}(-,H^\star)$ functor of §3.2.1, the Π-algebra indecomposables functor of §3.2.4, and the coproduct and smash functors of §3.3.4. The resulting "bands of dependence" for the corresponding spectral sequences are given in §1.2. A suitable modification also applies to the graded Quillen homology of §3.2.3.

In §7 below we show how they can be used to make specific computations of derived functors – showing, in particular, that the vanishing lines given above for the wedge and smash spectral sequences (§1.1.4) are best possible:

4.4.1 Proposition. *If X and Y are Π-algebras such that $: X := r$ and $: Y := s$ $(r,s \geq 3)$, and $G_rX \cong G_sY \cong \mathbf{Z}/2$, then $L_N(G_kCOP)(X,Y) \neq 0$ for $N = 2(k-(r+s))+1 \geq 0$.*

4.4.2 Proposition. *For X and Y as above, and S the smash functor of §3.3.4(II), also $L_N(G_kS)(X,Y) \neq 0$ for $N = 2(k-(r+s))-1 \geq 0$.*

5 Resolutions of RGA's

In this section we prove Theorem 3.1 and its variants, by constructing a suitable free resolution $A_\bullet \to X$ for any $X \in \mathcal{C}$. This is essentially the construction of [1, §4].

In §5.1 we present the two basic propositions needed in the proof of the theorem: Proposition 5.1.3 describes a general property of functors applied to free simplical RGA's, and Proposition 5.1.4 describes the free simplical resolution $A_\bullet \to X$ needed for 3.1. The first is proved in §5.2, and after some remarks on constructing resolutions in §5.3, the proof of the second is given in §5.4.

5.1 functors of free simplical resolutions

In order to calculate the derived functors of $T : \mathcal{C} \to \mathcal{A}$ evaluated on X, *any* free simplical resolution $A_\bullet \to X$ may be used; we wish to describe a particular one, for which we recall the following definitions from [1, §4.2-3]:

5.1.1 basic algebras

Given a free simplical RGA $A_\bullet$ and a set of generators $T^0, T^1, \ldots$ as in §2.4.1, we define the n-*th basic algebra* for $A_\bullet$, denoted $\bar{A}_n$, to be the sub-free RGA of A_n generated by the *non-degenerate* elements in T^n.

A sequence $\bar{A}_0, \bar{A}_1, \ldots, \bar{A}_n, \ldots$ of basic algebras for a free simplical RGA $A_\bullet$ is called a *CW-basis* for $A_\bullet$ (cf. [8, §5.1]) iff for each $n \geq 0$ we have $d_j :_{\bar{A}_n} = 0$ for $1 \leq j \leq n$. We call the morphism $\bar{d}_0 = d_0 :_{\bar{A}_{n+1}} : \bar{A}_{n+1} \to A_n$ the *attaching map* for $\bar{A}_{n+1}$.

5.1.2 normalized chains

Recall also that for a simplicial Abelian group $B_\bullet$, we have the associated normalized chain-complex $\{N_\star B_\bullet, \partial\}$, where for each $n \geq 0$ we let

$$N_n B_\bullet = \bigcap_{1 \leq j \leq n} ker\{d_j : B_n \to B_{n-1}\} \quad \subset B_n, \quad \text{and} \quad \partial_n = d_0 :_{N_n B_\bullet} .$$

The homotopy groups of $B_\bullet$ may then be computed as the homology of this chain-complex: $\pi_n B_\bullet \cong H_n(N_\star B_\bullet, \partial)$ (cf. [13, §17]).

Moreover, if DB_n denotes the subgroup of B_n generated by the degenerate elements, we have $N_n B_\bullet \cap DB_n = 0$ (cf. [13, Cor. 22.2]).

Using these facts, Theorem 3.1 follows immediately from the following two propositions:

5.1.3 Proposition. *Let $\mathcal{C}$ be a CRGA and $T : \mathcal{F} \to Abgp$ a functor of degree k. Given integers $a, b \geq 0$, let $A_\bullet$ be a free simplical RGA with a CW-basis $\bar{A}_0, \bar{A}_1, \ldots$ such that for each $n \geq 0$, $n \leq a : \bar{A}_n : -b$. Then $N_n T A_\bullet \subseteq T\bar{A}_n$ for $n > ak - 2b$.*

5.1.4 Proposition. *Let $\mathcal{C}$ be a CRGA over R, with $l.g.d.$(R-Mod) $= d$; then any $X \in \mathcal{C}$ with $: X := r > d$ has a free simplical resolution $A_\bullet \to X$, with a CW-basis $\bar{A}_0, \bar{A}_1, \ldots,$ such that*

(a) *for each* $n \geq 0$ *we have* $n \leq (d+1)(:\bar{A}_n : -r) + d$;

(b) *for* $n = (d+1)(k-r)+d$, *the attaching map* $\bar{d}_0 : \bar{A}_n \to A_{n-1}$ *is a* k*-monomorphism (definition 3.1.1).*

5.2 proof of Proposition 5.1.3

This proposition generalizes [1, Lemma 4.4.3]. In analogy with [13, p. 95(i)], any free simplicial object has the following

5.2.1 explicit description of A_n

If $A_\bullet$ is a free simplical RGA with CW-basis $\bar{A}_0, \bar{A}_1, \ldots$, then each A_n may be described as a coproduct of the basic algebras:

For each $n \geq 0$ and $0 \leq \lambda \leq n$, let $\mathcal{I}_{\lambda,n}$ denote the set of all sequences I of λ non-negative integers $i_1 < i_2 < \ldots < i_\lambda$ $(i_\lambda < n)$, with $s_I = s_{i_\lambda} \circ \ldots s_{i_2} \circ s_{i_1}$ the corresponding λ-*fold degeneracy*. (We allow $\lambda = 0$, with the corresponding $s_I = id$). Then

$$A_n \;\cong\; \coprod_{0 \leq \lambda \leq n} \; \coprod_{I \in \mathcal{I}_{\lambda,n}} \; \bar{A}_{n-\lambda} \tag{1}$$

where for each $I \in \mathcal{I}_{\lambda,n}$, the copy of $\bar{A}_{n-\lambda}$ indexed by I is *in the image* of the λ-fold degeneracy s_I, in the obvious sense.

Thus, given $A_\bullet$ as above, for each $n \geq 0$ we can write: $A_n \cong \coprod_{\alpha \in K_n} X_\alpha$, where each X_α is in the image of some λ_α-fold degeneracy $(\lambda_\alpha \geq 0)$. In this situation we have the following (cf. [1, Lemma 4.5.2]):

5.2.2 Lemma. *Assume that* $\coprod_{i=1}^{q} X_{\alpha_i}$ *is a sub-coproduct of the above* $A_n \cong \coprod_{\alpha \in K_n} X_\alpha$, *satisfying*

$$\sum_{i=1}^{q} (n - \lambda_{\alpha_i}) < n. \tag{2}$$

Then for some $0 \leq j \leq n-1$, *each* X_{α_i} *(for* $1 \leq i \leq q$*) is in the image of* s_j.

5.2.3 proof of lemma

Let M be the $q \times n$ matrix with entries $m_{ij} = 1$ if X_{α_i} is in the image of s_j, and $m_{ij} = 0$ otherwise. Then each X_{α_i} is in the image of (at least) λ_{α_i} of the n possible degeneracy maps $s_j : A_{n-1} \to A_n$, so there are at most $(n - \lambda_{\alpha_i})$ entries of 0 in the i-th row of M. Condition (2) then implies that there is some column of 1's in M. $\square$

5.2.4 completion of proof of proposition

Using the *cross-effects* decomposition (§2.3.1) of TA_n with respect to the coproduct (1), any $\gamma \in NTA_n \subset TA_n$ can be written as a sum $\gamma = \sum \gamma_k$, with each $0 \neq \gamma_k \in T_q(\bar{A}_{n-\lambda_{\alpha_1}}, \ldots, \bar{A}_{n-\lambda_{\alpha_q}})$ an element of some q-th cross-effect summand of T. Moreover, since T has degree k, we have $\sum_{i=1}^{q} \ : \bar{A}_{n-\lambda_{\alpha_i}} : \ \leq k$ by definition 2.3.2.

Since $n - \lambda_{\alpha_i} \leq \ a : \bar{A}_{n-\lambda_{\alpha_i}} : -b$ by hypothesis, if $q \geq 2$ we have $\sum_{i=1}^{q} (n - \lambda_{\alpha_i}) \leq ak - 2b < n$. Then Lemma 5.2.2 shows that all coproduct summands $\bar{A}_{n-\lambda_{\alpha_i}}$ for such a γ_k must be in the image of some s_i, so γ_k is degenerate. Thus any *nondegenerate* γ_k – for which necessarily $q = 1$ – is in $T\bar{A}_n$, and so is itself an n-chain (by the definition of a CW-basis in §5.1.1). Then the sum of the *degenerate* γ_k's is also an n-chain, and so must vanish by §5.1.2. We conclude that $\gamma \in T\bar{A}_n$, as required. $\square$

5.3 constructing free simplical resolutions

To prove Proposition 5.1.4, we wish to construct a suitable free simplical resolution $A_\bullet \to X$. First, some definitions:

5.3.1 normalized cycles

For any simplicial object $A_\bullet$ over a CRGA $\mathcal{C}$, one can define the normalized *cycles* and *chains* RGA's of $A_\bullet$ in the usual way; in particular, the n-*cycles* algebra of $A_\bullet$ is the sub-RGA of A_n defined

$$Z_n A_\bullet \;=\; \bigcap_{0 \leq j \leq n} ker\{d_j : A_n \to A_{n-1}\}$$

5.3.2 CW-construction

As with CW-complexes, one can construct free simplical RGA's by an inductive process, in which, given a free simplical RGA $A_\bullet$, one obtains a new free simplical RGA $A'_\bullet$ by "attaching" a free RGA $\bar{A}$ in dimension n, using an *attaching map* $\bar{d}_0 : \bar{A} \to Z_n A_\bullet \subset A_n$.

In the cases we shall be interested in, the given $A_\bullet$ will have a CW-basis $\bar{A}_0, \bar{A}_1, \ldots, \bar{A}_n, 0, \ldots$; a CW-basis for the new $A'_\bullet$ is then obtained by adding $\bar{A}_{n+1} = \bar{A}$ to this as the $(n+1)$-st basic algebra, and using definition 5.1.1 and the description of §5.2.1 above.

Note that a free simplical RGA $A_\bullet$, with a CW-basis $\bar{A}_0, \bar{A}_1, \ldots$ and an augmentation $A_0 \to X$, is a free simplical resolution of X (definition 2.4.2) iff for all $n \geq -1$, the attaching map $\bar{d}_0 : \bar{A}_{n+1} \to A_n$ factors through an *epimorphism* $\bar{d}_0 : \bar{A}_{n+1} \twoheadrightarrow Z_n A_\bullet$ (where we set $Z_{-1} A_\bullet = A_{-1} = X$).

5.3.3 N-resolutions

If the attaching map satisfies this condition only for $0 \leq n < N$, we call $A_\bullet \to X$ an N-*resolution*. Equivalently (cf. §2.4.2), for each $k \geq 1$, we have $\pi_j(G_k A_\bullet) = 0$ for $1 \leq j < N$, and the augmentation induces an isomorphism $\pi_0(G_k A_\bullet) \cong G_k X$.

5.3.4 n-skeleta

Given a free simplical RGA $A_\bullet$ with CW-basis $\bar{A}_0, \bar{A}_1, \ldots$, we define the n-*skeleton* of $A_\bullet$, denoted $sk_n A_\bullet$, to be the free simplical RGA with CW-basis $\bar{A}_0, \bar{A}_1, \ldots, \bar{A}_n, 0, 0, \ldots$ (with the same ataching maps as $A_\bullet$ in dimensions $\leq n$). For any $m \leq n$, we identify $(sk_n A_\bullet)_m$ with A_m, and $Z_m(sk_n A_\bullet)$ with $Z_m A_\bullet$. Note that the n-skeleton of a free simplical resolution $A_\bullet \to X$ is an n-resolution.

5.4 proof of Proposition 5.1.4

The construction of the resolution $A_\bullet \to X$ of 5.1.4 is essentially the same as that of [1, §4.4], by induction on skeleta:

5.4.1 an inductive construction

The free simplical Π-algebra $A_\bullet$ is constructed by induction on $n \geq -1$. At the n-th stage we assume we have an augmented free simplical RGA which we shall denote (by a slight abuse of notation) by $sk_n A_\bullet \to X$, together with a given CW-basis $\bar{A}_0, \bar{A}_1, \ldots, \bar{A}_n, 0, \ldots$ such that:

(i) $sk_n A_\bullet \to X$ is an n-resolution of X (cf. §5.3.3);

(ii) for each $m \geq 0$, we have $:\bar{A}_m: \ \geq \ [m/(d+1)] \ + r$, (where $[x]$ denotes the integral part of x).

(iii) for $m = (d+1)(k-r)+d > 0$, the attaching map $\bar{d}_0 : \bar{A}_m \to A_{m-1}$ is a k-monomorphism (definition 3.1.1);

(iv) for each $0 \leq m \leq n$, we have $:Z_m A_\bullet: \ \geq \ [(m+1)/(d+1)] \ + r$.

Conditions (iv) and (ii) are related by the following

5.4.2 Lemma. *For any free simplical RGA $A_\bullet$ satisfying 5.4.1(ii), we have $G_t Z_n A_\bullet \ \subset \ G_t \bar{A}_n$ for $n \geq (d+1)(t-r)$.*

Proof: As in §2.3.3(II), the functor $G_k : \mathcal{F} \to$*R-Mod* has degree k, while 5.4.1(ii) implies that $m \ \leq \ (d+1)(:\bar{A}_m: -r)+d$ for all $m \geq 0$. Thus the free simplical RGA $A_\bullet$ satisfies the basic-algebras hypothesis of Proposition 5.1.3 with $a = d+1$, $b = r(d+1)-d$ (using the fact that $r > d$). $\Box$

5.4.3 the case $r \leq d$

For $1 \leq r \leq d$, the lemma does not hold as stated, but one still obtains a slope of $1/(d+1)$ for the connectivities of the $\bar{A}_n$'s, so that Proposition 5.1.3 may be applied to derived functors (see §3.1.5).

5.4.4 the inductive step

In the $(n+1)$-st step, we choose the basic algebra $\bar{A}_{n+1}$ as follows:

Let $k = [(n+1)/(d+1)]+r$, and denote $Z_n A_\bullet$ by Z, so that $:Z: \geq k$ by hypothesis 5.4.1(iv).

We choose a free R-module P_{n+1} with an epimorphism $P_{n+1} \twoheadrightarrow G_k Z$; if $G_k Z$ is itself free, let $P_{n+1} = G_k Z$. We also make a special choice of P_{n+1} when $n \equiv d-1$ modulo $(d+1)$ – see §5.4.6 below.

We can then choose a $(k-1)$-connected free RGA B_{n+1} with $G_k B_{n+1} \cong P_{n+1}$, and a morphism $B_{n+1} \to Z$ which is surjective in degree k; if $P_{n+1} = 0$, let $B_{n+1} = 0$.

Let Z' denote the k-connected RGA obtained from Z by setting $G_k Z$ equal to 0, and choose a k-connected free RGA C_{n+1} which has an epimorphism $C_{n+1} \twoheadrightarrow Z'$. Setting $\bar{A}_{n+1} = B_{n+1} \amalg C_{n+1}$, we obtain an epimorphism $\bar{d}_0 : \bar{A}_{n+1} \twoheadrightarrow Z$, yielding the required attaching map.

5.4.5 conditions 5.4.1 for $n+1$

By construction we have $:\bar{A}_{n+1}:\geq: Z_n A_\bullet :$, so 5.4.1(ii) for $n+1$ follows from 5.4.1(iv) for n. 5.4.1(i) for $(n+1)$ is also clear, by §5.3.3. It remains to verify that hypotheses 5.4.1(iii) and (iv) hold when $n+1 = (d+1)(k-r)+d$, (since otherwise (iv) follows from (ii) by Lemma 5.4.2, while (iii) is vacuous).

First, we show that $: Z_{n+1} A_\bullet :\geq [(n+2)/(d+1)] + r = k+1$:

By Lemma 5.4.2 and hypothesis 5.4.1(i), we have an exact sequence of R-modules:

$$\begin{aligned} 0 \to G_k Z_n A_\bullet \hookrightarrow G_k \bar{A}_n \xrightarrow{G_k \bar{d}_0} G_k \bar{A}_{n-1} \xrightarrow{G_k \bar{d}_0} \dots \\ \dots \to G_k \bar{A}_{n+1-d} \to G_k Z_{n-d} A_\bullet \to 0 \quad . \end{aligned} \tag{3}$$

However, since $l.g.d.(R\text{-}Mod) = d$, and each of $G_k \bar{A}_{n+1-d}, \dots, G_k \bar{A}_n$ is a projective R-module – so is $P = G_k Z_n A_\bullet$.

5.4.6 remark on projectives

In fact, by revising the last two steps in the construction, we can actually assume that $P = G_k Z_n A_\bullet$ is a *free* R-module. For $d = 0$ this is obvious, while for $d > 0$, we can use the following "Eilenberg trick":

For some R-module Q, $P \oplus Q = F$ is a free R-module, so that the R-module $F' = (Q \oplus P) \oplus (Q \oplus P) \oplus \dots$ is also free, and thus

$$P \oplus F' \cong (P \oplus Q) \oplus (P \oplus Q) \oplus \dots \cong F \oplus F \oplus \dots$$

is free, too.

Therefore, retracing our steps to the n-th stage in the induction, we now replace our previous choice of the free R-module $P_n \twoheadrightarrow G_k Z_{n-1} A_\bullet$ in §5.4.4 above by $P'_n = P_n \oplus F'$, with $d_0 :_{F'} = 0$, and extend to a suitable free RGA B', and so on, to yield the "revised" n-th basic algebra $\bar{A}'_n$ and n-skeleton $sk_n A'_\bullet$.

All requirements of §5.4.1 are still satisfied – but now $G_k \bar{A}'_n \cong P_n \oplus F'$, so that the "revised" n-cycles object $Z_n A'_\bullet$, in degree k, is

$$G_k Z_n A'_\bullet \cong ker\{G_k \bar{d}'_0 : G_k \bar{A}'_n \to G_k A_{n-1}\}$$

$$\cong ker(G_k \bar{d}_0 :_{P_n}) \oplus ker(G_k \bar{d}_0 :_{F'}) \cong G_k Z_n A_\bullet \oplus F' \cong P \oplus F'$$

which is a free R-module. Thus in the $(n+1)$-st inductive step of §5.4.4, we choose the revised $P'_{n+1} = P \oplus F'$, so that $G_k \bar{d}_0 : G_k \bar{A}'_{n+1} \to G_k A'_n$ is a monomorphism.

5.4.7 completion of proof

Assuming that $sk_{n+1} A_\bullet$ has in fact been constructed as in §5.4.6, we now find that $: Z_{n+1} A_\bullet : \geq k+1$, and that $\bar{d}_0 : \bar{A}_{n+1} \to A_n$ is a $(k+1)$-monomorphism, so that 5.4.1(iii) and (iv) hold for $(n+1)$. This completes the proof of Proposition 5.1.4. □

5.4.8 proof of Proposition 3.1.2

Proposition 3.1.2 follows essentially from from Proposition 5.1.4(b), by §5.1.2, since this implies that for $n = (d+1)(k-r)+1$, the relevant part of the attaching map $\bar{d}_0 : \bar{A}_n \to A_{n-1}$, (i.e., its restriction to B, in the notation of §5.4.4), is a k-monomorphism.

5.4.9 remark on the CRGA

In general, one can only improve the connectivity of the $\bar{A}_n$'s by one in each set of $(d+1)$ induction steps. However, some particular CRGA's, we may have no non-trivial operations in a certain range on any sufficiently connected object. In this situation, the choice of B in §5.4.4 does not interfere with an "efficient" choice for C.

For example, the category of graded algebras over a field (§3.1.3) has the property that any $(k-1)$-connected object is just a graded vector space in degrees $< 2k$. Therefore, in §5.4.4, if $: \bar{A}_n := k$, we can choose $\bar{A}_{n+1} = B \amalg$

C so that the attaching map $\bar{d}_0 : \bar{A}_{n+1} \twoheadrightarrow Z_n A_\bullet$ is in fact a monomorphism in degrees $< 2k$ (rather than just in degree k). This is the only modification needed to prove Proposition 3.1.4.

6 a filtered resolution

In this section we prove Theorem 4.3 by showing that the simplicial resolution $A_\bullet \to X$ of section 5 can be filtered by resolutions of the algebraic k-skeleta (cf. §4.2) of X.

In §6.1 we state an analogue of Proposition 5.1.4 describing this resolution; the construction is given in §6.2.

6.1 free inclusions

If $\mathcal{C}$ is a CRGA, a morphism $f : A \to B$ is is called a *free inclusion*, written $A \hookrightarrow B$, iff A and B are free RGA's and there is a morphism $g : C \to B$ which, together with f, induces an isomorphism $A \amalg C \cong B$. We call C a *coproduct complement* of $A \hookrightarrow B$.

If $A_\bullet$ and $B_\bullet$ are free simplical RGA's, with CW-bases $\bar{A}_0, \bar{A}_1, \ldots$ and $\bar{B}_0, \bar{B}_1, \ldots$, respectively, we call a morphism $f : A_\bullet \to B_\bullet$ a *free inclusion* iff f is induced by a sequence of free inclusions $\bar{f}_n : \bar{A}_n \hookrightarrow \bar{B}_n$ for $n \geq 0$.

With this definition, and a slight additional assumption on the ring R, we have the following refinement of Proposition 5.1.4:

6.1.1 Proposition. *Let $\mathcal{C}$ be a CRGA over a ring R such that every projective R-module is free, and l.g.d.(R-Mod) $= d \leq 1$. Then every $X \in \mathcal{C}$ with $: X := r > d$ has a free simplical resolution $A_\bullet \to X$, filtered by a sequence of free inclusions*

$$0 = A[0]_\bullet \hookrightarrow A[1]_\bullet \hookrightarrow A[2]_\bullet \hookrightarrow \ldots A[k]_\bullet \hookrightarrow \ldots A_\bullet$$

together with a compatible sequence of augmentation maps $A[k]_0 \to \rho_k X$, satisfying the following assumptions:

(i) *$A[k]_\bullet \to \rho_k X$ is a free simplicial resolution;*

(ii) *for each $n \geq 0$, we have $: \bar{A}_n :\geq [n/(d+1)] + r$ (as in §5.4.1(ii));*

(iii) *for each* $k \geq r$, $A[k]_\bullet$ *has a CW-basis* $\bar{A}[k]_0, \bar{A}[k]_1, \ldots,$ *such that for each* $n \geq 0$, *the free inclusion* $\bar{A}[k]_n \hookrightarrow \bar{A}_n$ *induces an isomorphism* $G_t\bar{A}[k]_n \cong G_t\bar{A}_n$ *for* $n \geq (d+1)(t-k)+d$.

We also need a relative version of Proposition 5.1.3, as follows:

6.1.2 Proposition. *Let* $\mathcal{C}$ *be a CRGA and* $T : \mathcal{F} \to Abgp$ *a functor of degree* t. *Given integers* $a, b \geq 0$, *let* $A_\bullet \hookrightarrow B_\bullet$ *be a free inclusion of free simplical RGA's with CW-bases* $\bar{A}_0, \bar{A}_1, \ldots$ *and* $\bar{B}_0, \bar{B}_1, \ldots$, *respectively, such that for each* $m \geq 0$ *we have* $\bar{B}_m \cong \bar{A}_m \amalg \bar{C}_m$, $m \leq a : \bar{A}_m :$, *and* $m \leq a : \bar{C}_m : -b$. *Then* $N_n TB_\bullet \subseteq TA_n$ *for* $n > ak - b$.

Proof: As in §5.2.1, write $B_n \cong \coprod_{0 \leq \lambda \leq n} \coprod_{I \in \mathcal{I}_{\lambda,n}} (\bar{A}_{n-\lambda} \amalg \bar{C}_{n-\lambda})$; then use Lemma 5.2.2 and the argument of §5.2.4. □

Theorem 4.3 now follows directly from Propostion 6.1.1, using Proposition 6.1.2 (and §4.1.4) with $a = d+1$ and $b = (d+1)k - d + 1$ to deduce vanishing of the relative derived functors of $\rho_k X \to X$, and then using the **(LES)** of §4.1.2. Similarly for Proposition 4.3.3.

6.2 constructing the filtered resolution

The construction of the filtered resolution is similar to that of §5.4, but here we proceed by double induction:

For each n, we want to construct $sk_n A[k]_\bullet$ for all k's, by induction on $n \geq 0$. This is done in turn by induction on $k \geq 0$, where for each k it suffices to choose the n-th basic algebra $\bar{A}[k]_n$, together with its attaching map $\bar{d}_0 : \bar{A}[k]_n \to A[k]_{n-1}$ (cf. §5.1.1).

6.2.1 n-induction hypotheses

In the n-th stage of the induction, we shall assume that for each $j \geq 0$ we have chosen the CW-basis $\bar{A}[j]_0, \bar{A}[j]_1, \ldots, \bar{A}[j]_n$, with suitable attaching maps, and have thus constructed the n-skeleton $sk_n A[j]_\bullet$ so that the following induction hypotheses are satisfied for all $0 \leq m \leq n$ and $j \geq 1$:

$$P(m,j)\begin{cases} (a) \quad sk_m A[j]_\bullet \to \rho_j X \ \text{ is an } m\text{-resolution;} \\ (b) \quad G_t\bar{A}[j]_m = G_t\bar{A}[j-1]_m \quad \text{if} \ \ t \leq [(m-1)/(d+1)] + j - d \\ (c) \quad G_t Z_m A[j]_\bullet \subseteq G_t\bar{A}[j-1]_m \quad \text{if} \ \ t \leq [m/(d+1)] + j - d \\ (d) \quad : \bar{A}_m :\geq [m/(d+1)] + r, \quad \text{as in §5.4.1(ii).} \end{cases}$$

(using the conventions of §5.3.4).

6.2.2 the case $n \leq 1$

We begin the induction at $n = -1$ with $A_{-1} = X$ as before, and $A[k]_{-1} = \rho_k X$ for all $k \geq 0$. Note that if X is an RGA and $A_\bullet \to X$ is any free resolution, then by definition 4.2.2 we have:

$$\rho_k X = (L_0\rho_k)X \cong \ \rho_k A_0 / im(d_0 :_{N(\rho_k A_1)}) = \rho_k \bar{A}_0 / im(\bar{d}_0 :_{\rho_k \bar{A}_1}).$$

Thus, for $P(1,k)(a)$ to be satisfied it suffices to choose $\bar{A}[k]_n$ $(n = 0, 1)$ so that $\bar{A}[k]_n \cong \rho_k \bar{A}[k]_n$ and the following sequence of graded R-modules is exact in degrees $\leq k$:

$$0 \to ker(\bar{d}_0) \to \bar{A}[k]_1 \xrightarrow{\bar{d}_0} \bar{A}[k]_0 \to X \to 0.$$

Now assume that $d = 0$. In this case, if we let $\eta_k : \rho_k X \to \rho_{k+1} X$ be the obvious natural morphism, we can write

$$G_{k+1}X = G_{k+1}\rho_{k+1}X \ \cong \ G_{k+1} \ im(\eta_k) \ \oplus \ F,$$

where F is necessarily free. We can then ensure that $G_k\bar{A}[k]_1 \subseteq G_k\bar{A}[k-1]_1$ – i.e., that $P(1,k)(b)$ holds.

6.2.3 the n, k stage

In the n-th stage, we choose $\bar{A}[k]_{n+1}$ by induction on $k \geq 0$, (starting with $\bar{A}[0]_{n+1} = 0$):

We assume that we already have $\bar{A}[j]_{n+1}$ satisfying the hypotheses $P(n+1, j)$ for $0 \leq j \leq k$; we now wish to choose $\bar{A}[k+1]_{n+1}$ so that hypotheses $P(n+1, k+1)$ are satisfied, too. First, we need the following

6.2.4 notation for skeleta

Let A be the free RGA generated by the graded set $\{T_i\}_{i=1}^{\infty}$ (cf. §2.1.4); then its algebraic k-skeleton $\rho_k A$ (definition §4.2.1) is isomorphic to the free RGA generated by $\{T_i\}_{i=1}^{k}$. Moreover, there is a free inclusion $\theta_k : \rho_k A \hookrightarrow A$, with $A \cong \rho_k A \amalg A'$ and $A' \cong cok(\theta_k)$ the free RGA generated by $\{T_i\}_{i=k+1}^{\infty}$.

Now, the main ingredients needed in the construction of $\bar{A}[k+1]_{n+1}$ are the following two lemmas:

6.2.5 Lemma. *Given* $t \geq k \geq r$, *let* $n = (d+1)(t-k) - 1$, *and assume* $P(m,j)$ *holds for all* $j \geq 1$ *and* $m \leq n$; *then* $G_t(Z_n A[k+1]_\bullet) \cong G_t(Z_n \rho_{t-1} A[k]_\bullet) \oplus F$, *where* F *is a free* R*-module.*

Proof: Let us denote $sk_n \rho_{t-1} A[k]_\bullet$ by $B_\bullet$, and $sk_n A[k+1]_\bullet$ by $C_\bullet$, with CW-bases $\bar{B}_0, \ldots, \bar{B}_n$ and $\bar{C}_0, \ldots, \bar{C}_n$ respectively. We want $G_t Z_n C_\bullet \cong G_t Z_n B_\bullet \oplus F$. Since $B_\bullet \hookrightarrow C_\bullet$ is an inclusion of simplicial RGA's, $Z_n B_\bullet \to Z_n C_\bullet$ is a monomorphism, and the lemma is thus trivial for $d = 0$ (including the case $n = -1$, by §6.2.2). Assume therefore that $d = 1$ and $n = 2(t-k) - 1$.

For each $0 \leq m \leq n$, let $\bar{B}'_m = cok\{\theta_{t-1} : \bar{B}_m \hookrightarrow \bar{A}[k]_m\}$, so that $\bar{A}[k]_m \cong \bar{B}_m \amalg \bar{B}'_m$ and $: \bar{B}'_m :\geq t$. Similarly, we have free RGA's $\bar{D}_m$ such that $\bar{C}_m \cong \bar{A}[k]_m \amalg \bar{D}_m$.

As in §5.2.1, we have

$$C_n \cong \coprod_{0 \leq \lambda \leq n} \coprod_{I \in \mathcal{I}_{\lambda,n}} (\bar{B}_{n-\lambda} \amalg \bar{B}'_{n-\lambda} \amalg \bar{D}_{n-\lambda}). \tag{4}$$

But $P(n,k)$(b) implies that $: \bar{D}_n :\geq t$, so if we consider the sub-coproduct

$$C' = \bar{D}_n \amalg \coprod_{0 \leq \lambda \leq n} \coprod_{I \in \mathcal{I}_{\lambda,n}} \bar{B}'_{n-\lambda},$$

we see $: C' :\geq t$. Thus the R-module $G_t C_n$ has a free summand $H \cong G_t C'$.

Let $i : G_t Z_n C_\bullet \to G_t C_n$ denote the inclusion, and $\pi : G_t C_n \to H$ the projection. Since H is a free R-module, $F = im(\pi \circ i) \subset H$ is free, too (by the assumptions on R in Proposition 6.1.1). Therefore, if we let $K = ker(\pi \circ i)$, we have a split short exact sequence:

$$0 \to K \to G_t Z_n C_\bullet \xrightarrow{\pi \circ i} F \to 0,$$

and so $G_tZ_nC_\bullet \cong K \oplus F$. Since $G_tB_n \subseteq G_tC_n \cap ker(\pi)$, we have $G_tZ_nB_\bullet \subseteq K = G_tZ_nC_\bullet \cap ker(\pi)$. We now show the converse inclusion also holds:

Take the cross-effect decomposition (§2.3.1) of G_tC_n with respect to the coproduct (4); then we can write any $\gamma \in G_tZ_nC_\bullet$ as a sum $\gamma = \sum \gamma_j$, with each γ_j an element of some cross-effect summand.

If $\gamma \notin G_tB_n$, the same is true of some non-degenerate summand γ_j. Note that by $P(m,k)(d)$ we have $2 : \bar{B}_m :\geq m + 2r - 1$, while $P(m,k)(b)$ implies $2 : \bar{D}_m :\geq m + 2k$, and by construction $: \bar{B}'_m :\geq t$ for all m. A counting argument as in §5.2.4 then shows that either $\gamma_j \in G_t\bar{D}_n$, or else $\gamma_j \in G_t\bar{B}'_{n-\lambda}$ for some λ, so that in any case $\pi(\gamma_j) \neq 0$ and thus $\pi(\gamma) \neq 0$. This shows $K \subseteq G_tZ_nB_\bullet$, which completes the proof of the lemma. □

6.2.6 Corollary. *In this situation we can choose $\bar{D}_{n+1}$ and the attaching map $\bar{d}_0 : \bar{A}[k+1]_{n+1} \to A[k+1]_n$ so that $\bar{d}_0 :_{\bar{D}_{n+1}}$ is a t-monomorphism (definition 3.1.1).* □

Note that hypothesis $P(n,j)(c)$ holding for all $j \geq 0$ allows us to choose all $\bar{A}[j]_{n+1}$ so that hypotheses (a) and (b) are satisfied. Thus the following analogue of Lemma 5.4.2 completes the proof of Proposition 6.1.1:

6.2.7 Lemma. *Assume that the induction hypotheses of §6.2.1 are satisfied for n, and that Corollary 6.2.6 has been applied wherever relevant. If we choose $\bar{A}[k+1]_{n+1}$ so that $P(n+1,k+1)(a)$ and (b) are satisfied, then $P(n+1,k+1)(c)$ holds, too.*

Proof: Similar to the proof of Proposition 6.1.2. □

6.2.8 an additional property

In fact, the proof of Lemma 6.2.7 allows us to construct $A[k]_\bullet \to \rho_k X$ so as to satisfy also the following requirement:

For $t \geq k \geq r$ and $n = (d+1)(t-k) - 1 \geq 0$, $\quad G_tZ_nA[k]_\bullet \subseteq G_t\rho_{t-1}A[k]_n$.

This will be useful in the construction of the following section.

6.2.9 a spectral sequence

The filtration on the resolution gives rise to a natural spectral sequence converging to the derived functors of any functor $T : \mathcal{F} \to Abgp$ into a category of Abelian groups; its E^1-term consists of the relative derived functors of the natural transformations $\eta_k : \rho_k \to \rho_{k+1}$ of §6.2.2. It is not clear if this is of any use, though we have vanishing results for this spectral sequence, too.

7 A calculation

In this section we restrict attention to the category Π-Alg_0 of simply-connected Π-algebras (see §2.2(III)), and apply the relative vanishing results of section 4 to make a calculation related to the derived functors of the coproduct COP (cf. §3.3.4(II)), proving Propositions 4.4.1 and 4.4.2.

7.1 constructing a resolution

We first describe the first filtration on a certain free simplical resolution:

7.1.1 remark on notation

Using the obvious equvalence of categories (cf. [1, §7.1.2]), we may describe free Π-algebras and morphisms between them in terms of (wedges of) spheres and homotopy classes of maps between them:

If $\mathbf{S}^k$ is the k-sphere, we shall write $S[k-1] = \pi_\star(\mathbf{S}^k)$ for the corresponding free Π-algebra. (We call attention once more to the shift in indexing of §2.2(III): thus $G_{k-1}S[k-1] = \pi_k\mathbf{S}^k$.) For $k > 2$ we denote the morphism corresponding to the suspended Hopf map by $\eta_k : S[k] \to S[k-1]$; likewise the corresponding element in $G_kS[k-1] \cong \pi_{k+1}\mathbf{S}^k$. Similarly, $[a, b] \in G_{p+q}X$ will denote the Whitehead product of $a \in G_pX$, $b \in G_qX$.

We now have the following

7.1.2 Lemma. *Let X be a Π-algebra with $: X := r \geq 2$ and $G_rX \cong \mathbf{Z}/2$. Then there is a free simplical resolution $A[r]_\bullet \to \rho_rX$, with CW-basis $\bar{A}[r]_0, \bar{A}[r]_1, \ldots$ such that for $t \geq r$ and $n = 2(t-r)$ we have:*

(a) : $\bar{A}[r]_n := t$, *and there is a single "$(t+1)$-sphere" in* $\bar{A}[r]_n$, *which we denote* $S[t]_n$;

(b) : $\bar{A}[r]_{n+1} := t$, *with a single* $S[t]_{n+1}$ *in* $\bar{A}[r]_{n+1}$;

(c) *the attaching map* $\bar{d}_0 : \bar{A}[r]_{n+1} \to A[r]_n$, *restricted to* $S[t]_{n+1}$, *is a degree* 2 *map* $S[t]_{n+1} \to S[t]_n \hookrightarrow A[r]_n$.

(d) *the attaching map* $\bar{d}_0 : \bar{A}[r]_{n+2} \to A[r]_{n+1}$, *restricted to* $S[t+1]_{n+2}$, *is a suspended Hopf map* $\eta_t : S[t+1]_{n+2} \to S[t]_{n+1} \hookrightarrow A[r]_{n+1}$.

Proof: This is a straightforward calculation based on Proposition 6.1.1, using remark 6.2.8 (compare [1, Prop. 5.2.1]). □

7.2 non-vanishing of $L_n COP$

We now use Lemma 7.1.2 to show that the derived functors of COP are non-zero just above the vanishing line provided by Proposition 3.3.3 (see §1.1.4), by proving Proposition 4.4.1, which we state again:

7.2.1 Proposition. *Let* X *and* Y *be* Π-*algebras with* $: X := r$ *and* $: Y := s$ $(r, s \geq 3)$ *and* $G_r X \cong G_s Y \cong \mathbf{Z}/2$; *let* $k > r + s$. *Then* $(L_\nu G_k COP)(X, Y) \neq 0$ *for* $\nu = 2(k - r - s) + 1$.

Proof: Fix $k \geq r + s$, and let $\nu = 2(k - (r + s)) + 1$ as above.

I. By Proposition 4.3.3, it suffices to show $(L_\nu G_k COP)(\rho_r X, \rho_s Y) \neq 0$. Let $\hat{A}_\bullet = A[r]_\bullet$ and $\hat{B}_\bullet = B[s]_\bullet$ be the resolutions of Lemma 7.1.2 for $\rho_r X$ and $\rho_s Y$ respectively, and $C_\bullet = \hat{A}_\bullet \amalg \hat{B}_\bullet$ the free simplical Π-algebra which is their dimensionwise coproduct. We must show that $\pi_\nu(G_k C_\bullet) \neq 0$.

II. For any simplicial Abelian group $F_\bullet$, let $N_\star F_\bullet$ denote as usual the normalized chain-complex (§5.1.2); then for each $r \leq i \leq k - s$, the *shuffle map* of [6, §5] provides a chain homomorphism

$$N_\star G_i \hat{A}_\bullet \otimes N_\star G_{k-i} \hat{B}_\bullet \xrightarrow{\nabla} N_\star(\, G_i \hat{A}_\bullet \otimes G_{k-i} \hat{B}_\bullet \,)$$

Also, since $\hat{A}_\bullet$ and $\hat{B}_\bullet$ are simplicial Π-algebras, their face maps commute with the operations, and so the Whitehead product maps

$$G_i \hat{A}_n \otimes G_{k-i} \hat{B}_n$$

for each $n \geq 0$, induce a chain map

$$W : N_\star(\ G_i\hat{A}_\bullet \otimes G_{k-i}\hat{B}_\bullet \ ) \ \to \ N_\star G_k C_\bullet$$

III. Now for each $m \geq 0$, let $i = [m/2] + r$ and choose $\alpha_m \in G_i\hat{A}_m$ to be a generator of $G_iS[i]_m \cong \mathbf{Z}$ (in the notation of Lemma 7.1.2), with $d_0\alpha_{2p+1} = 2\alpha_{2p}$ for all p. Similarly, for $n \geq 0$ let $j = [n/2] + s$ and let $\beta_n \in G_j\hat{B}_n$ be a generator of $G_jS[j]_n$, again with $d_0\beta_{2p+1} = 2\beta_{2p}$.

Then in the chain complex $\bigoplus_{i=r+1}^{k-s} \ N_\star G_i\hat{A}_\bullet \ \otimes \ N_\star G_{k-i}\hat{B}_\bullet$, let

$$e \ = \ \sum_{i=r+1}^{k-s} \ (\ \alpha_{2(i-r)} \otimes \beta_{2(k-i-s)+1} \ - \ \alpha_{2(i-r)+1} \otimes \beta_{2(k-i-s)} \ ).$$

This is clearly a normalized chain, so we may define $\gamma = (W \circ \nabla)(e)$ in $N_N G_k C_\bullet$. Moreover, one may verify that $d_0\gamma = 0$, by using Lemma 7.1.2(c)&(d), the fact that $2\eta_j = 0$, and the identity (cf. [17, ch. X, Thm 8.18]):

$$[a \circ \eta_i, \ b] \ = \ [a, \ b \circ \eta_{k-i}] \ = \ [a, b] \circ \eta_k$$

(for $: a := i, \ : b := k - i$ both ≥ 3). Thus γ is an ν-cycle.

IV. We now show that γ does not bound:

Lemma 5.2.2 implies that any non-degenerate element in $N_{\nu+1}(G_kC_\bullet)$ is a sum of Whitehead products of the form $w = [s_I\alpha_{2q+1}, s_J\beta_{\nu-2q}]$, where $I = (i_{\nu-2q}, \ldots, i_0)$ and $J = (j_{2q}, \ldots, j_0)$ are as in §5.2.1. In fact, the argument of §5.2.4 implies that the pair (I, J) forms a $(\nu - 2q, 2q + 1)$-shuffle on $(0, 1, \ldots, \nu)$.

Thus w.l.o.g. we may assume $j_0 = 0$, and so $d_0w = 2[s_{\bar{I}}\alpha_{2q}, s_{\bar{J}}\beta_{\nu-2q}]$ (for suitable multi-indices $(\bar{I}, \bar{J})$). On the other hand, for $l \geq 1$ we see that d_lw, if non-zero, is of the form $[s_{I'}\alpha_{2q+1}, s_{J'}\beta_{\nu-2q}]$, where (I', J') is now a certain $(\nu - 2q - 1, 2q)$-shuffle on $(0, 1, \ldots, \nu - 2)$.

We conclude that all iterated face maps of w are sums of elements of one of two possible forms: either $2[a, b]$, for some a, b; or else $[s_I\alpha_{2p+1}, s_J\beta_{2q+1}]$ – where both α and β have *odd* index. Since γ cannot be expressed as a sum of (degeneracies of) elements of these forms, we have shown that γ does not bound in $N_\star G_kC_\bullet$. □

7.2.2 the smash product

An identical argument works for the smash product of §3.3.4(II), proving Proposition 4.4.2.

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