THE CONFIGURATION SPACE OF A PARALLEL POLYGONAL MECHANISM

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ABSTRACT. We study the configuration space $\mathcal{C}$ of a parallel polygonal mechanism, and give necessary conditions for the existence of singularities; this shows that generically $\mathcal{C}$ is a smooth manifold. In the planar case, we construct an explicit Morse function on $\mathcal{C}$, and show how geometric information about the mechanism can be used to identify the critical points.

1. INTRODUCTION

The mathematical theory of robotics is based on the notion of a mechanism, consisting of links, joints, and rigid parts known as platforms. The *type* of a mechanism is defined by a q-dimensional simplicial (or polyhedral) complex, where the parts of dimension ≥ 2 correspond to the platforms, and the complementary one-dimensional graph corresponds to the links and joints. There may be restrictions as to the kind of motion allowed at the joints. In this paper the lengths of all links are fixed.

A specific embedding of this complex in the ambient Euclidean space $\mathbb{R}^d$ is called a *configuration* of the mechanism. The collection of all such embeddings forms a topological space, called the *configuration space* of the mechanism (see [Hal]). These spaces have been studied intensively, mostly for simple closed or open chains (cf. [FTY, G, HK, JS, KM, MT, OH]; but see [Ho, KTs, SSB]).

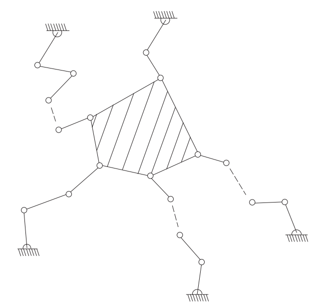

FIGURE 1. A pentagonal planar mechanism

The goal of this note is to study the configuration space of a mechanism comprising a moving planar polygonal platform, having a flexible leg consisting of concatenated links (i.e., rigid rods) attached to each vertex, with the other end fixed in $\mathbb{R}^d$ (see Figure 1). We may think of these fixed ends as forming the *fixed polygon*

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of the mechanism, "parallel" to the *moving polygon* inside. The spatial version of such a mechanism, consisting of a two-dimensional platform free to move in three dimensions, has been studied extensively, but even the planar version, to which we later specialize, has practical applications – for example, in micro-electro-mechanical systems (MEMS). See [BSW].

Our main results are:

(a) The configuration space of a parallel polygonal mechanism is generically a smooth manifold; when it is not, the possible singular points are explicitly described (Theorem 2.2 and Corollary 2.9).
(b) The topology of this manifold can be described for a triangular planar mechanism by means of an explicit Morse function (Theorem 3.1), whose critical points can be identified geometrically (see §3.2).

We start with some terminology and notation:

1.1. **Definition.** A *branch* (of *multiplicity* n) is a sequence $L = (\ell_1, \dots, \ell_n)$ of n positive numbers, which we think of as the lengths of n concatenated links, having $(d-1)$-dimensional spherical joints at the consecutive meeting points (if $d = 2$, the joints are rotational).

1.2. **Definition.** A *configuration* in $\mathbb{R}^d$ for a branch $L = (\ell_1, \dots, \ell_n)$ consists of n vectors $V = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ in $\mathbb{R}^d$ with lengths $\|\mathbf{v}_i\| = \ell_i$ $(i = 1, \dots n)$. A branch configuration $V = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ is *aligned*, with a *direction vector* $\vec{\mathbf{w}} \in \mathbb{R}^d$, if all the vectors $\mathbf{v}_i$ are scalar multiples of $\vec{\mathbf{w}}$.

The *configuration space* $\mathcal{C} = \mathcal{C}(L)$ of a branch L is the product of n $(d-1)$-dimensional spheres S^{d-1} of radii $(R_i = \ell_i)_{i=1}^n$. Up to homeomorphism, $\mathcal{C}$ is independent of the order on L, so we can assume $\ell_1, \dots, \ell_n$ to be in descending order.

1.3. **Definition.** A *polygonal mechanism* $(\mathcal{L}, \mathcal{X}, \mathcal{P})$ in $\mathbb{R}^d$ consists of:

(a) k branches $\mathcal{L} = \{L^{(i)}\}_{i=1}^k$ of multiplicity $\{n^{(i)}\}_{i=1}^k$, respectively;
(b) k distinct *base points* $\mathcal{X} = \{\mathbf{x}^{(i)}\}_{i=1}^k$ in $\mathbb{R}^d$, to which the initial points of the corresponding branches are attached.
(c) An abstract planar k-polygon $\mathcal{P}$ in $\mathbb{R}^d$.

Think of this mechanism as a linkage of k branches, starting at the base points (which form a not necessarily planar polygon in $\mathbb{R}^d$, called the *fixed platform*), and ending at the vertices of a rigid planar polygon congruent to $\mathcal{P}$ (called the *moving platform* of the mechanism). There are spherical joints at either end of each branch, too.

We use parenthesized superscripts to indicate the branch number, and plain subscripts to indicate the link number – e.g., $\ell_j^{(i)}$ denotes the length of the j-th link of the i-th branch.

1.4. *Remark.* Let $\mathcal{P}$ be a convex polyhedron in $\mathbb{R}^d$. It need not be d-dimensional – e.g., we can think of a planar polygon as a degenerate polyhedron in $\mathbb{R}^3$. If $\mathbf{p}^{(1)}, \dots, \mathbf{p}^{(k)}$ $(k > d)$ are its vertices (extremal points), $\mathcal{P}$ is determined up to isometry by the lengths $g^{(i,j)} := \|\mathbf{p}^{(i)} - \mathbf{p}^{(j)}\|$ for $(i,j) \in \mathcal{I}$, where the index set $\mathcal{I}$ consists of the following ordered pairs:

(1) $\mathcal{I} := \{(1,2),\dots,(1,m);(2,3),\dots,(2,m);\dots,(m-1,m);\ (1,m+1),\dots,(d,k)\}$

for $m := \min\{d,k\}$. We first include in $\mathcal{I}$ all ordered pairs (i,j) with $1 \leq i < j \leq m$, which span a "basic" $(d-1)$-simplex Δ (possibly degenerate!) if $d \leq k$. Note that for each $i \geq d+1$, the d edges with second index i, together with Δ, span a d-simplex, and we add to the list $\mathcal{I}$ just enough such d-simplices to rigidly determine $\mathcal{P} \subset \mathbb{R}^d$.

Thus we have:

$$|\mathcal{I}| \;=\; \begin{cases} \binom{k}{2} & \text{if } k \leq d \\ \binom{d}{2} + d(k-d) & \text{if } k \geq d+1 \end{cases} \tag{2}$$

If $\mathcal{P}$ is not convex, the only additional data needed is the discrete information in which half-space the new vertex is to be placed.

1.5. **Definition.** A *configuration* for a polygonal mechanism $(\mathcal{L},\mathcal{X},\mathcal{P})$ in $\mathbb{R}^d$ consists of a set $\mathcal{V} = (V^{(1)},\dots,V^{(k)})$ of k branch configurations for $\mathcal{L}$ (Definition 1.2), satisfying the condition that the endpoints $\mathbf{p}^{(i)} := \mathbf{x}^{(i)} + \sum_{j=1}^{n^{(i)}} \mathbf{v}_j^{(i)}$ $(i = 1,\dots,k)$ of the corresponding branch configurations (attached to the given basepoints) form a planar polygon congruent to $\mathcal{P}$ in $\mathbb{R}^d$. If the branch configuration $V^{(i)}$ is aligned, with direction vector $\vec{\mathbf{w}}_i$ (Definition 1.2), then the line $\mathrm{Line}^{(i)} := \{\mathbf{x}^{(i)} + t\vec{\mathbf{w}}_i \mid t \in \mathbb{R}\}$ is called the *direction line* for $V^{(i)}$ (with $\mathbf{p}^{(i)} \in \mathrm{Line}^{(i)}$).

The set of all configurations for the given mechanism $\Gamma = (\mathcal{L},\mathcal{X},\mathcal{P})$ (as a subspace of the product of the appropriate branch configuration spaces) is its *configuration space* $\mathcal{C} = \mathcal{C}(\Gamma)$.

1.6. **Definition.** Note that the moving platform $\mathcal{P}$ can be translated and rotated in $\mathbb{R}^d$ (subject to the constraints imposed by the branches and the locations of the fixed vertices). The space of all allowable positions for $\mathcal{P}$, denoted by $\mathcal{W} = \mathcal{W}(\Gamma)$, is called the *work space* for $\Gamma = (\mathcal{L},\mathcal{X},\mathcal{P})$. The *work map* $\Phi : \mathcal{C} \to \mathcal{W}$ assigns to each configuration $\mathcal{V}$ the resulting position of $\mathcal{P}$.

1.7. **Organization.** In Section 2 we identify the potential singular points of the configuration spaces $\mathcal{C}$ we consider here, in any ambient dimension, and show that, generically, $\mathcal{C}$ is a manifold. In Section 3 we describe a Morse function for the configuration space of a generic planar mechanism, analyze its critical points geometrically, and give a simple example showing how this analysis may be used to recover $\mathcal{C}$.

1.8. *Acknowledgements.* We would like to thank the referee for his or her comments.

2. Singularities for polygonal mechanisms

We now show that, generically, the configuration space of a polygonal mechanism is a manifold, and give necessary geometric conditions for a configuration to be singular. We note that such "topological" singularities can in fact occur (cf. [FS] and [KTe]), and their analysis is of interest in relation to the kinematic singularities.

2.1. **Definition.** A configuration $\mathcal{V} = (V^{(1)},\dots,V^{(k)})$ of a polygonal mechanism $\Gamma = (\mathcal{L},\mathcal{X},\mathcal{P})$ is called *hyper-aligned* if:

(a) Two of its branch configurations $V^{(i_1)}$ and $V^{(i_2)}$ are aligned, with coinciding direction lines: $\mathrm{Line}^{(i_1)} = \mathrm{Line}^{(i_2)}$ (see Figure 2).

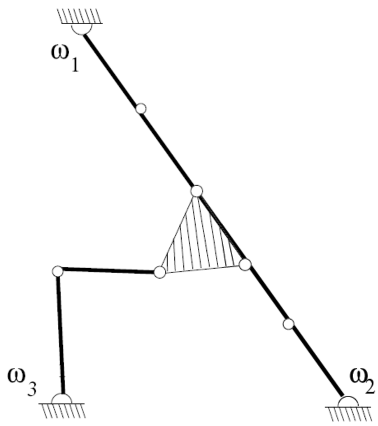

FIGURE 2. Singular configuration of type (a)

(b) Three of its branch configurations are aligned, with direction lines in the same plane meeting in a single point (see Figure 3).

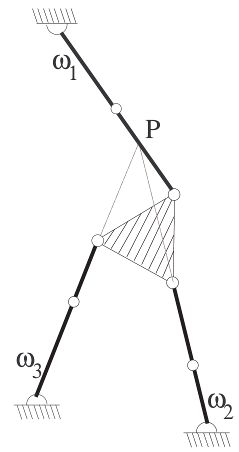

FIGURE 3. Singular configuration of type (b)

(c) Four of its branch configurations are aligned, with direction vectors all lying in the same plane, if $d > 2$.

Compare [Hau].

2.2. **Theorem.** *The configuration space* $\mathcal{C} = \mathcal{C}(\Gamma)$ *of a polygonal mechanism in* $\mathbb{R}^d$ *is smooth at each configuration* $\mathcal{V}$*, unless it is hyper-aligned.*

2.3. *Remark.* Note that we do not claim that any hyper-aligned configuration is necessarily a singular point of $\mathcal{C}$. It is in fact not easy to directly determine all singularities of a configuration space, except for the simple case of a closed chain (i.e., a polygonal linkage), where all hyper-aligned configurations (necessarily of type (a)) are in fact singular (cone points). See [KM, Theorem 2.6]. For a more general analysis, with an explicit description of the form of the singularities occurring, see [BS] .

Proof of Theorem. For any n, define a map $f_n : (\mathbb{R}^d)^n \to \mathbb{R}^n$ by:

$$f_n(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_n) \ := \ (|\vec{\mathbf{v}}_1|^2, \dots, |\vec{\mathbf{v}}_n|^2) \ . \tag{3}$$

Let $N := \sum_{i=1}^{k} n^{(i)}$ (the total number of links in the mechanism), and consider the *constraint map* $F : \mathbb{R}^{dN} \to \mathbb{R}^{N+|\mathcal{I}|}$, defined:

$$F(\mathcal{V}) \ := \ (f_{n^{(1)}}(V^{(1)}), \dots, f_{n^{(k)}}(V^{(k)}), \ \|\mathbf{a}(1,2)\|^2, \dots, \|\mathbf{a}(k-1,k)\|^2) \ . \tag{4}$$

Here $\mathcal{V} = (V^{(1)}, \dots, V^{(i)})$ is the ordered set of branch configurations, which potentially constitute a configuration of our mechanism, $\mathbf{p}^{(i)} := \mathbf{x}^{(i)} + \sum_{t=1}^{n^{(i)}} \mathbf{v}_t^{(i)}$ is the endpoint of the i-th branch for the configuration $V^{(i)} = (\mathbf{v}_1^{(i)}, \dots, \mathbf{v}_{n^{(i)}}^{(i)})$ attached to the basepoint $\mathbf{x}^{(i)} \in \mathbb{R}^d$, and $\mathbf{a}(i,j) := \mathbf{p}^{(i)} - \mathbf{p}^{(j)}$ is the (i,j)-diagonal of the polygon spanned by these endpoints. Recall that the polygonal platform $\mathcal{P}$ determines (and is determined by) the set of diagonals $\mathcal{G} = (g^{(i,j)})_{(i,j)\in\mathcal{I}}$ (see §1.4).

Let

$$Z_{(\mathcal{L},\mathcal{G})} \ := \ ((\ell_1^{(1)})^2, \dots, (\ell_{n^{(1)}}^{(1)})^2, \dots ((\ell_1^{(k)})^2, \dots, (\ell_{n^{(k)}}^{(k)})^2, \ (g^{(1,2)})^2, \dots, (g^{(k-1,k)})^2)$$

be the pre-determined value of F at any configuration belonging to the mechanism. That is, the configuration space $\mathcal{C} = \mathcal{C}(\Gamma)$ is precisely the pre-image of $Z_{(\mathcal{L},\mathcal{G})} \in \mathbb{R}^{N+|\mathcal{I}|}$ under the constraint map F.

Note that this pre-image generally has various connected components, corresponding to different "assembly modes" of the mechanism, in those cases where geometric constraints prevent a continuous motion between certain configurations. A simple example is a (scalene) triangle and its reflection in the plane.

Recall that a sufficient condition for a subset C of a Euclidean space $\mathbb{R}^N$ to be a smooth submanifold is for C to be the preimage of a *regular value* Z of a smooth function $F : \mathbb{R}^N \to \mathbb{R}^m$ $(N \geq m)$ – that is, if the Jacobian matrix $\mathrm{dF}_{\mathcal{V}}$ has maximal rank (i.e., m) at each point $\mathcal{V} \in F^{-1}(Z)$ (see [Hi, I, Theorem 3.2]). We calculate the Jacobian matrix explicitly:

$$\mathrm{dF}_{\mathcal{V}} \ = \ 2 \begin{pmatrix} A^{(1)} & 0 & 0 & 0 & \dots & 0 \\ 0 & A^{(2)} & 0 & 0 & \dots & 0 \\ 0 & 0 & A^{(3)} & & \dots & 0 \\ 0 & 0 & 0 & A^{(4)} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & A^{(k)} \\ & & & & & \\ \vec{\mathbf{b}}^{(1,2)} & \vec{\mathbf{b}}^{(2,1)} & 0 & 0 & \dots & 0 \\ \vec{\mathbf{b}}^{(1,3)} & 0 & \vec{\mathbf{b}}^{(3,1)} & 0 & \dots & 0 \\ 0 & \vec{\mathbf{b}}^{(2,3)} & \vec{\mathbf{b}}^{(3,2)} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \vec{\mathbf{b}}^{(k-1,k)} & \vec{\mathbf{b}}^{(k,k-1)} \end{pmatrix} , \tag{5}$$

Here $A^{(i)}$ is the $n^{(i)} \times dn^{(i)}$ matrix:

$$\begin{pmatrix} \mathbf{v}_1^{(i)} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{v}_{n^{(i)}}^{(i)} \end{pmatrix} . \tag{6}$$

The last $|\mathcal{I}|$ rows of D are:

$$\vec{\mathbf{z}}^{(i,j)} := (\underbrace{0,\dots,0}_{d(n^{(1)}+\dots+n^{(i-1)})}, \vec{\mathbf{b}}^{(i,j)}, \underbrace{0,\dots,0}_{d(n^{(i+1)}+\dots+n^{(j-1)})}, \vec{\mathbf{b}}^{(j,i)}, \underbrace{0,\dots,0}_{d(n^{(j+1)}+\dots+n^{(k)})}) \tag{7}$$

(for $(i,j) \in \mathcal{I}$ – see §1.4), where each edge $\mathbf{a}(i,j) \in \mathbb{R}^d$ appears $n^{(i)}$ times in the sub-vector:

$$\vec{\mathbf{b}}^{(i,j)} := \underbrace{(\mathbf{a}(i,j), \mathbf{a}(i,j), \dots, \mathbf{a}(i,j))}_{n^{(i)}} \in \mathbb{R}^{dn^{(i)}} . \tag{8}$$

Thus we may write the last $|\mathcal{I}|$ rows of D in terms of k matrices $B_1, \dots, B_k$ of size $|\mathcal{I}| \times d$, where the matrix B_i is repeated $n^{(i)}$ times:

$$(\underbrace{B_1, B_1, \dots, B_1}_{n^{(1)}}, \underbrace{B_2, B_2, \dots, B_2}_{n^{(2)}}, \dots, \underbrace{B_k, B_k, \dots, B_k}_{n^{(k)}}) . \tag{9}$$

Since $\mathbf{a}(i,j) = -\mathbf{a}(j,i)$ for any $i \neq j$, we see from (7) that:

$$\sum_{i=1}^{k} B_i = 0 , \tag{10}$$

(the $|\mathcal{I}| \times d$ zero matrix).

In our case, $\mathcal{V}$ will be a smooth point of $\mathcal{C}$ if $D := \frac{1}{2}\mathrm{dF}_{\mathcal{V}}$ is of rank $N + |\mathcal{I}|$ whenever $F(\mathcal{V}) = Z_{(\mathcal{L},\mathcal{G})}$. Let us assume by way of contradiction that D is *not* of maximal rank. This means that there is some non-trivial vanishing linear combination of the rows of D:

$$\sum_{i=1}^{k} \left(\sum_{j=1}^{n^{(i)}} \lambda_j^{(i)} \vec{\mathbf{u}}_j^{(i)} \right) + \sum_{(i,j)\in\mathcal{I}} \gamma^{(i,j)} \vec{\mathbf{z}}^{(i,j)} = \mathbf{0} , \tag{11}$$

where $\vec{\mathbf{u}}_1^{(i)}, \dots, \vec{\mathbf{u}}_{n^{(i)}}^{(i)}$ denote the rows of $A^{(i)}$.

For each $1 \leq i \leq k$, let:

$$\vec{\mathbf{w}}_i := \sum_{\substack{(s,i)\in\mathcal{I} \\ s<i}} \gamma^{(s,i)} \cdot \mathbf{a}(s,i) - \sum_{\substack{(i,t)\in\mathcal{I} \\ i<t}} \gamma^{(i,t)} \cdot \mathbf{a}(i,t) \in \mathbb{R}^d .$$

Note that $\vec{\mathbf{w}}_i \neq \mathbf{0}$ if and only if $\lambda_{j'}^{(i)} \neq 0$ for *some* $1 \leq j' \leq n^{(i)}$, since $\vec{\mathbf{w}}_i$ consists of the total contribution of the last $|\mathcal{I}|$ rows to the sum (11) in each of the columns corresponding to the submatrix $A^{(i)}$. But because of the repeated blocks $B_i, \dots, B_i$ in (9), we see that $\lambda_j^{(i)} = \lambda_{j'}^{(i)}$ for all $1 \leq j, j' \leq n^{(i)}$. If we denote this common value by $\lambda^{(i)}$, we deduce that:

$$\mathbf{v}_j^{(i)} = \frac{1}{\lambda^{(i)}} \cdot \vec{\mathbf{w}}_i ,$$

so that the i-th branch is aligned, with direction vector $\vec{\mathbf{w}}_i$.

By (10), $\sum_{i=1}^{k} \vec{\mathbf{w}}_i = \mathbf{0}$, so if $\vec{\mathbf{w}}_i = \mathbf{0}$ for $i \neq i_0, i_1$, then $\vec{\mathbf{w}}_{i_0} + \vec{\mathbf{w}}_{i_1} = \mathbf{0}$, and thus branches i_0 and i_1 are co-aligned with direction vector $\mathbf{a}(i_0, i_1)$. In other words, $\mathcal{V}$ is hyper-aligned (see §2.1(a)). To complete the proof of the Theorem we need the following:

2.4. **Proposition.** *Any singular configuration $\mathcal{V}_0 \in \mathcal{C}$ having at least three aligned branches is hyper-aligned.*

Proof. The proof is by induction on k, the number of branches in Γ. The initial step of the induction, $k = 3$, will be dealt with below.

Without loss of generality we may assume that the three branches 1, 2 and k are aligned, with direction vectors $\vec{\mathbf{w}}_1$, $\vec{\mathbf{w}}_2$, and $\vec{\mathbf{w}}_k$, respectively. We assume no two of the branches are hyper-aligned (§2.1(a)). Branch 3 may also be aligned, with direction vector $\vec{\mathbf{w}}_3$, but in this case we may assume that $\vec{\mathbf{w}}_1$, $\vec{\mathbf{w}}_2$, $\vec{\mathbf{w}}_3$ and $\vec{\mathbf{w}}_k$ are not coplanar (§2.1(c)).

Step I: Let $\hat{\mathcal{C}}$ denote the configuration space of the "reduced" mechanism $\hat{\Gamma}$, obtained from $\Gamma = (\mathcal{L}, \mathcal{X}, \mathcal{P})$ by omitting the last branch, and $\mathcal{C}(k)$ the configuration space for this branch $\Gamma(k)$ (attached to $\mathbf{x}^{(k)} \in \mathbb{R}^d$, with free end $\mathbf{p}^{(k)}$).

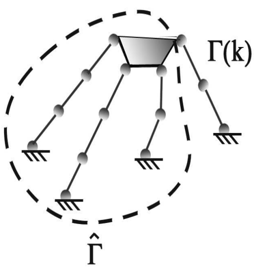

FIGURE 4. The "reduced" mechanism

The *work space* of both mechanisms (i.e., the set of possible locations for the k-th vertex $\mathbf{p}^{(k)}$ of $\mathcal{P}$) is contained in $\mathbb{R}^d$, and we have *work maps* $\psi : \hat{\mathcal{C}} \to \mathbb{R}^d$ and $\phi : \mathcal{C}(k) \to \mathbb{R}^d$ which associate to each configuration the location of this vertex $\mathbf{p}^{(k)}$.

The main idea of the proof is that the configuration space $\mathcal{C} = \mathcal{C}(\Gamma)$ is the *pullback* of the two maps ϕ and ψ – that is, a configuration for the full mechanism $\Gamma = (\mathcal{L}, \mathcal{X}, \mathcal{P})$ is (uniquely determined by) a pair $(\hat{\mathcal{V}}, V^{(k)})$ such that $\psi(\hat{\mathcal{V}}) = \mathbf{p}^{(k)} = \phi(V^{(k)})$, where $\hat{\mathcal{V}} \in \hat{\mathcal{C}}$ and $V^{(k)} \in \mathcal{C}(k)$:

$$
\begin{tikzcd}
\mathcal{V} = (\hat{\mathcal{V}}, V^{(k)}) \in \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C}(k) \ni V^{(k)} \arrow[d, "\phi"] \\
\hat{\mathcal{V}} \in \hat{\mathcal{C}} \arrow[r, "\psi"] & \mathbb{R}^d \ni \mathbf{p}^{(k)}
\end{tikzcd}
\tag{12}
$$

In other words, these are compatible configurations for the reduced mechanism $\hat{\Gamma}$ and the last branch, in the sense that the locations of their two end vertices (which were the same in the original mechanism) coincide. In particular, the specific aligned configuration $\mathcal{V}_0$ whose singularity is in question is determined by the pair $(\hat{\mathcal{V}}_0, V_0^{(k)})$.

2.5. *Remark.* It is convenient to have one of the two configuration spaces $\hat{\mathcal{C}}$ and $\mathcal{C}(k)$ – say, $\hat{\mathcal{C}}$ – be a submanifold of a large ambient manifold Y, in such a way that ψ turns into an embedding. In this case we can guarantee that the pullback $\mathcal{C}$ is a manifold if we can show that the other map ϕ is transversal to $\hat{\mathcal{C}}$.

Note that the manifold $\mathcal{C}(k)$ (an n-torus for $n = n^{(k)}$) is the preimage of $Z_{\ell(k)}$ under the map $f = f_n : \mathbb{R}^{nd} \to \mathbb{R}^n$ of (3), and we write $i : \mathcal{C}(k) \to \mathbb{R}^{nd}$ for the inclusion. Similarly, $\hat{\mathcal{C}}$ is determined by a smooth constraint map:

$$\hat{F} : \widetilde{\mathbb{R}^M} \to \mathbb{R}^{N-n+|\mathcal{I}|}$$

for $M = (N-n)d$, where $\widetilde{\mathbb{R}^M}$ is a fiber bundle over $\mathbb{R}^M$ with fiber $\mathcal{S}$, where $\mathcal{S}$ is a point if $k \geq 4$, and $\mathcal{S} = S^{d-2}$ if $k = 3$. The index set $\mathcal{I}$ is defined as in (1).

The constraint map is defined as in (4) by:

$$\begin{aligned} \hat{F}(\hat{\mathcal{V}}) &= \hat{F}(V^{(1)}, \ldots, V^{(k-1)}, \vec{\omega}) \\ &:= (f_{n^{(1)}}(V^{(1)}), \ldots, f_{n^{(k-1)}}(V^{(k-1)}),\ \|\mathbf{a}(1,2)\|^2,\ \ldots,\ \|\mathbf{a}(i,j)\|^2) \end{aligned} \tag{13}$$

(with the functions $\|\mathbf{a}(i,j)\|^2$ indexed by $(i,j) \in \mathcal{I}$). There is no constraint on $\vec{\omega}$.

2.6. *Remark.* The vector $\vec{\omega} \in \mathcal{S} = S^{d-2}$ is needed if $k = 3$ (so the moving platform $\mathcal{P}$ is a triangle) and $d > 2$, since in this case the location of the two vertices $\mathbf{p}^{(1)}$ and $\mathbf{p}^{(2)}$ of the triangle determines the position of $\mathcal{P}$ only up to rotation around the given edge $\mathbf{a}(1,2)$. The rotation is about $\mathbf{a}(1,2)$ in the hyperplane $\mathbf{a}(1,2)^{\perp}$, and is thus uniquely determined by a rotation vector $\vec{\omega}$ in the unit sphere S^{d-2} in $\mathbf{a}(1,2)^{\perp}$.

In a neighborhood U of any given configuration $\hat{\mathcal{V}}_0$, for which $\mathbf{a}(1,2)$ has the value $\mathbf{a}_0(1,2)$, say, we may choose a fixed copy $\mathcal{S}$ of S^{d-2} in $\mathbf{a}_0(1,2)^{\perp} \cong \mathbb{R}^{d-1}$. If we move from $\hat{\mathcal{V}}_0$ to a neighboring configuration $\hat{\mathcal{V}} \in U$, the associated vector $\mathbf{a}(1,2)$ will still be linearly independent of any $\vec{\omega} \in \mathcal{S}$, so by the Gram-Schmidt process $(\vec{\omega}, \mathbf{a}(1,2))$ determines a unique orthogonal pair $(\hat{\omega}, \mathbf{a}(1,2))$ (still spanning the plane of the moving platform $\mathcal{P}$). This is the reason we have a locally trivial fiber bundle $\mathcal{S} \hookrightarrow \widetilde{\mathbb{R}^M} \to \mathbb{R}^M$, rather than a (global) product $\widetilde{\mathbb{R}^M} \cong \mathbb{R}^M \times \mathcal{S}$.

Again, we may identify $\hat{\mathcal{C}}$ with $\hat{F}^{-1}(Z_{(\hat{\mathcal{L}},\hat{\mathcal{G}})})$ for the obvious value $(\hat{\mathcal{L}}, \hat{\mathcal{G}})$. Moreover, we may assume by our induction hypothesis on k that $(\hat{\mathcal{L}}, \hat{\mathcal{G}})$ is a regular value of $\hat{F}$, since the branches of $\hat{\Gamma}$ are not hyper-aligned at $\hat{\mathcal{V}}_0$. Thus $\hat{\mathcal{C}}$ is smooth, at least in a neighborhood of $\hat{\mathcal{V}}_0$. We denote the inclusion by $\hat{\imath} : \hat{\mathcal{C}} \to \mathbb{R}^M$.

Let $X := \mathcal{C}(k) \times \widetilde{\mathbb{R}^M}$ and $Y := \mathbb{R}^d \times \widetilde{\mathbb{R}^M}$. We define $h : X \to Y$ to be the product map $\phi \times \mathrm{Id}_{\widetilde{\mathbb{R}^M}}$ and $g : \hat{\mathcal{C}} \to Y$ to be $(\psi, \hat{\imath})$, so that g is an embedding of $\hat{\mathcal{C}}$ as a submanifold in Y. Since $\mathcal{C} = \mathcal{C}(\Gamma)$ is simply the pullback (12), it may be identified with the preimage of the submanifold $\hat{\mathcal{C}} \subseteq Y$ under h.

Now let $\hat{\mathcal{V}}_0 \in \hat{\mathcal{C}}$ be a configuration where the first two branches are aligned (but not hyper-aligned), and $V^{(k)} \in \mathcal{C}(k)$ an aligned configuration with direction vector $\mathbf{v}^{(k)}$, such that $\psi(\hat{\mathcal{V}}_0) = \phi(V^{(k)})$. Let $\mathbf{x} \in X$ denote the pair $(V^{(k)}, \hat{\imath}(\hat{\mathcal{V}}_0))$ in the pullback (so that $h(\mathbf{x}) = g(\hat{\mathcal{V}}_0)$).

$$
\begin{tikzcd}
\mathbf{x} \in X \arrow[d, "h"'] & = & \mathcal{C}(k) \ni V^{(k)} \arrow[d, "\phi"'] & \times & \widetilde{\mathbb{R}^M} \ni \hat{\imath}(\hat{\mathcal{V}}) \arrow[d, "\mathrm{Id}"'] \\
h(\mathbf{x}) \in Y & = & \mathbb{R}^d \ni \phi(V^{(k)}) & \times & \widetilde{\mathbb{R}^M} \ni \hat{\imath}(\hat{\mathcal{V}}) \\
\hat{\mathcal{V}} \in \hat{\mathcal{C}} \arrow[u, "g"] \arrow[urr, "\psi"] \arrow[urrrr, "\hat{\imath}"'] & & & &
\end{tikzcd}
$$

Step II: We must show that if the corresponding configuration $\mathcal{V}_0 = (V^{(k)}, \hat{\mathcal{V}}_0)$ is not smooth in $\mathcal{C}$, then it is hyper-aligned. By [Hi, I, Theorem 3.3]), $\mathcal{V}_0$ is smooth if $h \pitchfork \hat{\mathcal{C}}$ there – that is, h is locally transverse to $\hat{\mathcal{C}}$ at the points $\mathbf{x} \in X$ and $\hat{\mathcal{V}}_0 \in \hat{\mathcal{C}}$, which means that:

$$
\operatorname{Im} \mathrm{dh}_{\mathbf{x}} + T_{\hat{\mathcal{V}}_0}(\hat{\mathcal{C}}) \;=\; T_{\hat{\mathcal{V}}_0}(Y) \;=\; \mathbb{R}^d \times \mathbb{R}^{M+\Delta}\,, \tag{14}
$$

where $\Delta = \dim(\mathcal{S})$ is 0 unless $k = 3$, in which case $\Delta = d - 2$.

First note that since $\mathcal{C}(k) = f^{-1}(Z_{\ell(k)}) \subseteq \mathbb{R}^{nd}$, the tangent space $T_{V^{(k)}}(\mathcal{C}(k))$ may be identified with the kernel of $\mathrm{df}_{V^{(k)}} : \mathbb{R}^{nd} \to \mathbb{R}^n$, which is the null space of the matrix $A^{(k)}$ of (6). Since $V^{(k)}$ is aligned, by assumption, with direction vector $\vec{\mathbf{w}}_k$, we see that:

$$
\begin{aligned}
T_{V^{(k)}}(\mathcal{C}(k)) &\cong \{(\mathbf{y}_1^{(k)}, \dots, \mathbf{y}_n^{(k)}) \in \mathbb{R}^{nd} \mid \mathbf{y}_1^{(k)} \cdot \vec{\mathbf{w}}_k = 0,\ \dots, \mathbf{y}_n^{(k)} \cdot \vec{\mathbf{w}}_k = 0\} \\
&= \underbrace{\vec{\mathbf{w}}_k^{\perp} \times \ldots \times \vec{\mathbf{w}}_k^{\perp}}_{n}
\end{aligned}
$$

Furthermore, $\phi : \mathcal{C}(k) \to \mathbb{R}^d$ extends to $\hat{\phi} : \mathbb{R}^{nd} \to \mathbb{R}^d$ (so that $\hat{\phi} \circ i = \phi$), with

$$
\hat{\phi}(\mathbf{v}_1, \dots, \mathbf{v}_n) \;=\; \mathbf{x}^{(k)} + \mathbf{v}_1 + \cdots + \mathbf{v}_n\,.
$$

Since $\hat{\phi}$ is linear, its differential $\mathrm{d}\hat{\phi}$ is represented by the $d \times nd$ matrix $(I_d, I_d, \dots, I_d)$ (n blocks). Thus:

$$
\operatorname{Im} \mathrm{dh}_{\mathbf{x}} \;=\; \vec{\mathbf{w}}_k^{\perp} \times \mathbb{R}^{M+\Delta}\,.
$$

We may disregard the factor $\mathbb{R}^{M+\Delta}$, which was only needed in order to change ψ into the embedding g, as explained above. Essentially, we are using the fact that for an aligned open chain, as for a single rigid link, an infinitesimal movement in the work space is orthogonal to its direction vector $\vec{\mathbf{w}}_k$.

Therefore, in order for (14) to hold it is necessary and sufficient to have:

$$
\vec{\mathbf{w}}_k \text{ is not orthogonal to } T_{\hat{\mathcal{V}}_0}(\hat{\mathcal{C}})\,. \tag{15}
$$

Step III: The tangent space $T_{\hat{\mathcal{V}}_0}(\hat{\mathcal{C}})$ may be identified with the kernel of:

$$
\mathrm{d}\hat{F}_{\hat{\mathcal{V}}_0} : \mathbb{R}^{M+\Delta} \to \mathbb{R}^{N-n+|\mathcal{I}|}\,,
$$

for $\hat{F}$ defined in (13). Thus $\mathrm{d}\hat{F}_{\hat{\mathcal{V}}_0}$ is described as in (5) by the $(N-n+|\mathcal{I}|)\times(M+\Delta)$ matrix:

$$\mathrm{d}\hat{F}_{\hat{\mathcal{V}}_0} \;=\; 2\begin{pmatrix} A^{(1)} & 0 & \dots & 0 & 0\dots0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & A^{(k)} & 0\dots0 \\ & & & & \\ \vec{\mathbf{b}}^{(1,2)} & \vec{\mathbf{b}}^{(2,1)} & \dots & 0 & 0\dots0 \\ \vdots & \vdots & \ddots & \vdots & 0\dots0 \\ 0 & 0 & \dots & \vec{\mathbf{b}}^{(i,j)} & 0\dots0 \end{pmatrix} \tag{16}$$

where the zero columns on the right show the lack of constraint on $\vec{\omega}\in\mathcal{S}$.

Since the first two branches are aligned with direction vectors $\vec{\mathbf{w}}_1$, $\vec{\mathbf{w}}_2$, $T_{\hat{\mathcal{V}}_0}(\hat{\mathcal{C}})$ may be identified with the set of $N-n$ d-dimensional vectors

$$\mathbf{y}^{(1)}_1,\dots,\mathbf{y}^{(1)}_{n^{(1)}};\mathbf{y}^{(2)}_1,\dots,\mathbf{y}^{(2)}_{n^{(2)}};\dots;\mathbf{y}^{(k-1)}_1,\dots,\mathbf{y}^{(k-1)}_{n^{(k-1)}}\;, \tag{17}$$

together with $\mathbf{z}\in\mathbb{R}^{\Delta}$, where the first $n^{(1)}$ vectors are all in $\vec{\mathbf{w}}_1^{\perp}$, the next $n^{(2)}$ are all in $\vec{\mathbf{w}}_2^{\perp}$, and the remainder are in individual orthogonal complements:

$$\mathbf{y}^{(3)}_1\perp\mathbf{v}^{(3)}_1,\ \dots,\mathbf{y}^{(3)}_{n^{(3)}}\perp\mathbf{v}^{(3)}_{n^{(3)}},\ \dots,\ \mathbf{y}^{(k-1)}_1\perp\mathbf{v}^{(k-1)}_1,\ \dots,\mathbf{y}^{(k-1)}_{n^{(k-1)}}\perp\mathbf{v}^{(k-1)}_{n^{(k-1)}}\;. \tag{18}$$

Furthermore, the last $|\mathcal{I}|$ rows of $\mathrm{d}\,\hat{F}_{\hat{\mathcal{V}}}$, as described in (7) and (8), impose additional conditions on (17) – namely, if we let $\vec{\mathbf{y}}^{(i)}:=\sum_{t=1}^{n^{(i)}}\mathbf{y}^{(i)}_t$ $(i=1,\dots,k-1)$, we must have:

$$\mathbf{a}(i,j)\cdot(\vec{\mathbf{y}}^{(i)}-\vec{\mathbf{y}}^{(j)})\;=\;0 \tag{19}$$

for each $(i,j)\in\mathcal{I}$, as well as

$$\vec{\mathbf{w}}_1\cdot\vec{\mathbf{y}}^{(1)}\;=\;0\ \text{ and }\ \vec{\mathbf{w}}_2\cdot\vec{\mathbf{y}}^{(2)}\;=\;0\;. \tag{20}$$

Of course, if branch 3 is aligned, too, we have likewise $\vec{\mathbf{w}}_3\cdot\vec{\mathbf{y}}^{(3)}\;=\;0$, and so on if there are additional aligned branches.

Thus in practice we can simply replace each aligned branch i $(i=1,2$, and possibly $3)$ by a "virtual" branch with a single link (i.e., $n^{(i)}=1$), with corresponding directions $\vec{\mathbf{w}}_i$ and tangent vectors $\vec{\mathbf{y}}^{(i)}\in\vec{\mathbf{w}}_i^{\perp}$.

Step IV: We must now distinguish several cases:

Case 1: $k>3$ and $d>2$:

Because the moving platform $\mathcal{P}$ is planar, and we assumed all vertices are extremal, there are fixed (non-zero) scalars α and β such that $\mathbf{a}(1,k)\;=\;\alpha\mathbf{a}(1,2)+\beta\mathbf{a}(1,3)$, and the position vector for the k-th vertex is

$$\mathbf{p}^{(k)}\;=\;\mathbf{p}^{(1)}\;+\;\alpha\,\mathbf{a}(1,2)+\beta\,\mathbf{a}(1,3)\;.$$

Thus the work function $\psi:\hat{\mathcal{C}}\to\mathbb{R}^d$ for the vertex $\mathbf{p}^{(k)}$ in the reduced mechanism $\hat{\Gamma}=(\hat{\mathcal{L}},\hat{\mathcal{X}},\mathcal{P})$ extends to a smooth function $\hat{\psi}:\mathbb{R}^M\to\mathbb{R}^d$ of the form:

$$\hat{\psi}(V^{(1)},\dots,V^{(k-1)})\;=\;\mathbf{x}^{(1)}+\mathbf{v}^{(1)}_1+\dots+\mathbf{v}^{(1)}_{n^{(1)}}\;+\;\alpha\,\mathbf{a}(1,2)+\beta\,\mathbf{a}(1,3)\;,$$

where

$$\mathbf{a}(1,j) := \mathbf{p}^{(j)} - \mathbf{p}^{(1)} \;=\; (\mathbf{x}^{(j)} + \mathbf{v}_1^{(j)} + \cdots + \mathbf{v}_{n^{(j)}}^{(j)}) \;-\; (\mathbf{x}^{(1)} + \mathbf{v}_1^{(1)} + \cdots + \mathbf{v}_{n^{(1)}}^{(1)})$$

for $j = 2, 3, k$. We see that $\mathrm{d}\hat{\psi}$ is represented by the $d \times M$ matrix:

$$(\underbrace{(1-\alpha-\beta)I_d, \ldots, (1-\alpha-\beta)I_d}_{n^{(1)}d},\ \underbrace{\alpha I_d, \ldots, \alpha I_d}_{n^{(2)}d},\ \underbrace{\beta I_d, \ldots, \beta I_d}_{n^{(3)}d},\ \mathbf{0}, \ldots, \mathbf{0})\,.$$

However, the composite of the inclusion of $T_{\hat{\mathcal{V}}}(\hat{\mathcal{C}})$ into $\mathbb{R}^M \times \mathbb{R}^d$ with the projection onto $\mathbb{R}^d$ is just $\mathrm{d}\psi = \mathrm{d}\hat{\psi}|_{T_{\hat{\mathcal{V}}}(\hat{\mathcal{C}})}$. Combining this with the description of $T_{\hat{\mathcal{V}}}(\hat{\mathcal{C}})$ in (18), (19), and (20), we see that the image of $\mathrm{d}\psi$ in $\mathbb{R}^d$ consists of all vectors $\mathbf{v}$ of the form

$$\vec{\mathbf{z}} \;=\; \vec{\mathbf{y}}^{(1)} \;+\; \alpha\,\vec{\mathbf{u}} \;+\; \beta\,\vec{\mathbf{v}} \tag{21}$$

for $\vec{\mathbf{y}}^{(1)}$, $\vec{\mathbf{u}} := \vec{\mathbf{y}}^{(2)} - \vec{\mathbf{y}}^{(1)}$, and $\vec{\mathbf{v}} := \vec{\mathbf{y}}^{(3)} - \vec{\mathbf{y}}^{(1)}$ satisfying:

$$\vec{\mathbf{w}}_1 \cdot \vec{\mathbf{y}}^{(1)} \;=\; 0, \;\; \mathbf{a}(1,2) \cdot \vec{\mathbf{u}} \;=\; 0, \;\; \mathbf{a}(1,3) \cdot \vec{\mathbf{v}} \;=\; 0, \;\text{ and }\; \vec{\mathbf{w}}_2 \cdot (\vec{\mathbf{u}} + \vec{\mathbf{y}}^{(1)}) \;=\; 0\,. \tag{22}$$

If branch 3 is aligned, too, we have $\vec{\mathbf{w}}_3 \cdot (\vec{\mathbf{v}} + \vec{\mathbf{y}}^{(1)}) = 0$ (otherwise (18) imposes no constraint on $\vec{\mathbf{y}}^{(3)} := \vec{\mathbf{v}} + \vec{\mathbf{y}}^{(1)}$; and in any case additional aligned branches play no role in $\hat{\psi}$).

Assume that $\vec{\mathbf{w}}_k$ is orthogonal to any $\vec{\mathbf{z}}$ as in (21). If $\mathcal{E} := \mathrm{Span}(\mathbf{a}(1,2), \vec{\mathbf{w}}_2)$ in $\mathbb{R}^d$, by setting $\vec{\mathbf{y}}^{(1)} = \vec{\mathbf{v}} = \mathbf{0}$ we see that for any $\vec{\mathbf{u}} \perp \mathcal{E}$ we must have $\vec{\mathbf{w}}_k \perp \vec{\mathbf{u}}$, so $\vec{\mathbf{w}}_k \in \mathcal{E}$. Proceeding in this way, we deduce that a necessary condition for $\mathrm{Im}(\mathrm{d}\psi)$ to be orthogonal to $\vec{\mathbf{w}}_k$ is that $\vec{\mathbf{w}}_1$, $\vec{\mathbf{w}}_2$, $\vec{\mathbf{w}}_k$, $\mathbf{a}(1,2)$, $\mathbf{a}(1,3)$ all lie in the plane of $\mathcal{P}$. This is equivalent to a reduction to Case 3 below ($d = 2$).

Case 2: $k = 3$ and $d > 2$:

In this case $\mathcal{S} = S^{d-2}$, $\widetilde{\mathbb{R}^M}$ is $(M + d - 2)$-dimensional, and the work function $\psi : \hat{\mathcal{C}} \to \mathbb{R}^d$ for the vertex $\mathbf{p}^{(k)}$ in $\hat{\Gamma}$ extends to a smooth function $\tilde{\psi} : U \to \mathbb{R}^d$ of the form:

$$\tilde{\psi}(\mathbf{v}_1^{(1)}, \ldots, \mathbf{v}_{n^{(1)}}^{(1)}, \mathbf{v}_1^{(2)}, \ldots, \mathbf{v}_{n^{(2)}}^{(2)}, \vec{\omega}) \;=\; \mathbf{x}^{(1)} + \mathbf{v}_1^{(1)} + \cdots + \mathbf{v}_{n^{(1)}}^{(1)} + \alpha\,\mathbf{a}(1,2) + \beta\, G(\vec{\omega}, \mathbf{a}(1,2))$$

in some open neighborhood U of the given aligned configuration $\hat{\mathcal{V}}_0 \in \hat{\mathcal{C}}$ in $\widetilde{\mathbb{R}^M}$, where $\mathbf{a}(1,3) = \alpha\,\mathbf{a}(1,2) + \beta\,\mathbf{v}$ for fixed scalars α and β, and $\mathbf{v}$ is the unit normal to $\mathbf{a}(1,2)$ in the plane of $\mathcal{P}$. As before,

$$\mathbf{a}(1,2) \;:=\; \mathbf{p}^{(2)} - \mathbf{p}^{(1)} \;=\; \mathbf{x}^{(2)} - \mathbf{x}^{(1)} + \sum_{i=1}^{n^{(2)}} \mathbf{v}_i^{(2)} \;-\; \sum_{j=1}^{n^{(1)}} \mathbf{v}_j^{(1)}\,,$$

Here $G(\vec{\omega}, \mathbf{a})$ is result of applying the Gram-Schmidt process to the pair $(\vec{\omega}, \mathbf{a})$, so:

$$G(\vec{\omega}, \mathbf{a}) \;=\; \frac{\ell^2 \vec{\omega} - (\vec{\omega} \cdot \mathbf{a})\mathbf{a}}{\ell \sqrt{\ell^2 - (\vec{\omega} \cdot \mathbf{a})^2}}$$

where $\ell := \|\mathbf{a}\|$ is the constant $g^{(1,2)}$ and $\vec{\omega} \in \mathcal{S} \subseteq \mathbb{R}^d$ is the rotation vector for the plane of $\mathcal{P}$ about the edge $\mathbf{a} = \mathbf{a}(1,2)$ as in Remark 2.6 above.

Thus $\mathrm{d}\tilde{\psi}_{\hat{\mathcal{V}}}$ is represented for $\hat{\mathcal{V}} = (V^{(1)}, V^{(2)}, \mathbf{w})$ by the $d \times d(n^{(1)} + n^{(2)} + 1)$ matrix:

$$\begin{aligned}[t] &[\,(1-\alpha)\,I_d \;+\; \beta\,\frac{\partial G(\vec{\omega},\mathbf{a})}{\partial \mathbf{v}^{(1)}_1}\;,\;\dots\;,(1-\alpha)\,I_d + \beta\,\frac{\partial G(\vec{\omega},\mathbf{a})}{\partial \mathbf{v}^{(1)}_{n^{(1)}}}\;, \\ &\alpha\,I_d \;+\; \beta\,\frac{\partial G(\vec{\omega},\mathbf{a})}{\partial \mathbf{v}^{(2)}_1}\;,\dots,\alpha\,I_d \;+\; \beta\,\frac{\partial G(\vec{\omega},\mathbf{a})}{\partial \mathbf{v}^{(2)}_{n^{(2)}}}\;,\;\beta\,\frac{\partial G(\vec{\omega},\mathbf{a})}{\partial \vec{\omega}}\,] \end{aligned} \tag{23}$$

Now calculate:

$$\frac{\partial G(\vec{\omega},\mathbf{a})_i}{\partial w_j} \;=\; \ell\cdot\frac{\frac{\partial w_i}{\partial w_j}\,(\ell^2-(\vec{\omega}\cdot\mathbf{a})^2)\;+\;(w_i(\vec{\omega}\cdot\mathbf{a})-a_j)\,a_j}{[\ell^2-(\vec{\omega}\cdot\mathbf{a})^2]^{3/2}}\;.$$

Since we assumed $\vec{\omega}\cdot\mathbf{a}=0$ at $\hat{\mathcal{V}}_0$ for $\mathbf{a}:=\mathbf{a}_0(1,2)$ and any $\vec{\omega}\in\mathcal{S}$, we have:

$$Q\;:=\;\left(\frac{\partial G(\vec{\omega},\mathbf{a})}{\partial\vec{\omega}}\right)_{\hat{\mathcal{V}}_0}\;=\;I_d-\frac{\mathbf{a}^T\,\mathbf{a}}{\|\mathbf{a}\|^2}\;,$$

where $\mathbf{a}^T\,\mathbf{a}$ is the $d\times d$ matrix with (p,q)-th entry $a_p\cdot a_q$. - that is, Q is simply the projection onto $\mathbf{a}^{\perp}=\mathbf{a}_0(1,2)^{\perp}$. Since $T_{\vec{\omega}}(\mathcal{S})=\vec{\omega}^{\perp}\cap\mathbf{a}_0(1,2)^{\perp}\subseteq\mathbb{R}^d$, we deduce that the image of Q applied to $\mathbf{x}\in T_{\hat{\mathbf{w}}}(\mathcal{S})$ is the orthogonal complement to the plane of $\mathcal{P}$ in $\mathbb{R}^d$.

Similarly, we have:

$$\left(\frac{\partial G(\vec{\omega},\mathbf{a})}{\partial V^{(1)}}\right)_{\hat{\mathcal{V}}_0}\;=\;-\,\frac{\vec{\omega}^T\,\mathbf{a}}{\ell}\quad\text{and}\quad\left(\frac{\partial G(\vec{\omega},\mathbf{a})}{\partial V^{(2)}}\right)_{\hat{\mathcal{V}}_0}\;=\;\frac{\vec{\omega}^T\,\mathbf{a}}{\ell}\;,$$

(both projections onto $\mathbf{a}=\mathbf{a}_0(1,2)$), so when applied to the vector (17), subject to condition (19), we see that $\frac{\partial G(\vec{\omega},\mathbf{a})}{\partial V^{(i)}}$ contributes 0 to $\mathrm{Im}(\mathrm{d}\tilde{\psi}_{\hat{\mathcal{V}}})$.

In summary, $\mathrm{Im}(\mathrm{d}\psi)$ consist of all vectors of the form

$$\vec{\mathbf{z}}\;=\;\vec{\mathbf{y}}^{(1)}\;+\;\alpha\,\vec{\mathbf{u}}\;+\;\vec{\mathbf{v}}$$

for $\vec{\mathbf{y}}^{(1)}\perp\vec{\mathbf{w}}_1$, $\vec{\mathbf{u}}\perp\mathbf{a}(1,2)$ (since $\vec{\mathbf{u}}:=\vec{\mathbf{y}}^{(2)}-\vec{\mathbf{y}}^{(1)}$), and $\vec{\mathbf{v}}\in\mathcal{P}^{\perp}$ with $\vec{\mathbf{w}}_2\cdot(\vec{\mathbf{u}}+\vec{\mathbf{y}}^{(1)})=0$

We see again that $\vec{\mathbf{w}}_3$ is in the image of $\mathrm{d}\psi$, obtained by applying (23) to $T_{\hat{\mathcal{V}}}(\hat{\mathcal{C}})\subseteq\mathbb{R}^{M+d-2}$, except possibly if $\vec{\omega}$, $\vec{\mathbf{w}}_1$, $\vec{\mathbf{w}}_2$, $\vec{\mathbf{w}}_3$, and $\mathbf{a}(1,2)$ (and thus also $\mathbf{a}(1,3)$) all to lie in the plane of $\mathcal{P}$ – again reducing to the following:

Case 3: $d=2$:

Writing $A=\frac{g^{(1,k)}}{g^{(1,2)}}\left(\begin{smallmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{smallmatrix}\right)$ for the rotation-dilitation matrix taking $\mathbf{a}(1,2)$ to $\mathbf{a}(1,k)$, we see that $\hat{\psi}:\mathbb{R}^M\to\mathbb{R}^2$, for $M:=n^{(1)}+n^{(2)}$, is now given by:

$$\hat{\psi}(V^{(1)},V^{(2)})\;=\;\mathbf{x}^{(1)}+A(\mathbf{x}^{(2)}-\mathbf{x}^{(1)})\;+\;\sum_{i=1}^{n^{(1)}}(\mathbf{v}^{(1)}_i-A\mathbf{v}^{(1)}_i)\;+\;\sum_{j=1}^{n^{(2)}}A\mathbf{v}^{(2)}_j\;,$$

and thus $\mathrm{d}\psi_{\hat{\mathcal{V}}}$ is represented by the $2\times 2M$ matrix

$$\left(\underbrace{I_2-A,\dots,I_2-A}_{2n^{(1)}},\;\underbrace{A,\dots,A}_{2n^{(2)}}\right)\;.$$

Applying $\mathrm{d}\psi_{\hat{\mathcal{V}}}$ to a vector of the form (17), and noting that from (20) we have $\vec{\mathbf{y}}^{(1)} = s\,\vec{\mathbf{w}}_1^{\perp}$ and $\vec{\mathbf{y}}^{(2)} = t\,\vec{\mathbf{w}}_2^{\perp}$ for some $s, t \in \mathbb{R}$, we see that $\mathrm{Im}(\mathrm{d}\psi_{\hat{\mathcal{V}}})$ consists of sums of the form:

$$\mathbf{v} \;=\; s\,\vec{\mathbf{w}}_1^{\perp} \;-\; s\,A\vec{\mathbf{w}}_1^{\perp} \;+\; t\,A\vec{\mathbf{w}}_2^{\perp} \;=\; s\,\vec{\mathbf{w}}_1^{\perp} \;+\; A(t\,\vec{\mathbf{w}}_2^{\perp} \;-\; s\,\vec{\mathbf{w}}_1^{\perp})\,.$$

Moreover, by (19) we know that $\vec{\mathbf{y}}^{(2)} - \vec{\mathbf{y}}^{(1)} = t\,\vec{\mathbf{w}}_2^{\perp} - s\,\vec{\mathbf{w}}_1^{\perp}$ is orthogonal to $\mathbf{a}(1,2)$ – so that $t\,\vec{\mathbf{w}}_2^{\perp} - s\,\vec{\mathbf{w}}_1^{\perp} = u\,\mathbf{a}(1,2)^{\perp}$ for some $u \in \mathbb{R}$, and thus:

$$\mathbf{v} \;=\; u\,A\mathbf{a}(1,2)^{\perp} \;+\; s\,\vec{\mathbf{w}}_1^{\perp}\,, \tag{24}$$

subject to the condition that

$$u\,\mathbf{a}(1,2)^{\perp} + s\,\vec{\mathbf{w}}_1^{\perp} \quad \text{is a multiple of} \quad \vec{\mathbf{w}}_2^{\perp}\,. \tag{25}$$

Now for any three vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ in $\mathbb{R}^2$ we have:

$$(\mathbf{x}\cdot\mathbf{y}^{\perp})\,\mathbf{z} \;+\; (\mathbf{y}\cdot\mathbf{z}^{\perp})\,\mathbf{x} \;+\; (\mathbf{z}\cdot\mathbf{x}^{\perp})\,\mathbf{y} \;=\; \mathbf{0}\,. \tag{26}$$

Therefore, setting $\mathbf{x} := \mathbf{a}(1,2)^{\perp}$, $\mathbf{z} := \vec{\mathbf{w}}_1^{\perp}$, and $\mathbf{y} = \vec{\mathbf{w}}_2^{\perp}$, we see that $\vec{\mathbf{w}}_2^{\perp}$ is a multiple of $(\mathbf{a}(1,2)^{\perp}\cdot(-\vec{\mathbf{w}}_2))\,\vec{\mathbf{w}}_1^{\perp} \;+\; (\vec{\mathbf{w}}_2^{\perp}\cdot(-\vec{\mathbf{w}}_1))\,\mathbf{a}(1,2)^{\perp}$, and so (25) holds for $s = -\mathbf{a}(1,2)^{\perp}\cdot\vec{\mathbf{w}}_2 = \mathbf{a}(1,2)\cdot\vec{\mathbf{w}}_2^{\perp}$ and $u = -\vec{\mathbf{w}}_2^{\perp}\cdot\vec{\mathbf{w}}_1 = \vec{\mathbf{w}}_2\cdot\vec{\mathbf{w}}_1^{\perp}$. Therefore, by (24), $\mathrm{Im}(\mathrm{d}\psi_{\hat{\mathcal{V}}})$ is spanned by:

$$\mathbf{v}_0 \;:=\; (\vec{\mathbf{w}}_2\cdot\vec{\mathbf{w}}_1^{\perp})\,A\mathbf{a}(1,2)^{\perp} \;+\; (\mathbf{a}(1,2)\cdot\vec{\mathbf{w}}_2^{\perp})\,\vec{\mathbf{w}}_1^{\perp}$$

where $A\mathbf{a}(1,2) = \mathbf{a}(1,k)$, so $A\mathbf{a}(1,2)^{\perp} = \mathbf{a}(1,k)^{\perp}$.

Thus (15) fails only if $\vec{\mathbf{w}}_k$ is perpendicular to $\mathbf{v}_0$ – in other words, if $\vec{\mathbf{w}}_k$ is proportional to:

$$(\vec{\mathbf{w}}_2\cdot\vec{\mathbf{w}}_1^{\perp})\,\mathbf{a}(1,k) \;+\; (\mathbf{a}(1,2)\cdot\vec{\mathbf{w}}_2^{\perp})\,\vec{\mathbf{w}}_1\,,$$

or equivalently, if $\vec{\mathbf{w}}_k$ is proportional to:

$$\mathbf{e} \;:=\; \frac{\mathbf{a}(1,2)^{\perp}\cdot\vec{\mathbf{w}}_2}{\vec{\mathbf{w}}_1^{\perp}\cdot\vec{\mathbf{w}}_2}\,\vec{\mathbf{w}}_1 \;-\; \mathbf{a}(1,k)\,,$$

which by (26) is precisely the vector connecting the meeting point

$$P \;:=\; \mathbf{p}^{(1)} \;+\; \frac{\mathbf{a}(1,2)^{\perp}\cdot\vec{\mathbf{w}}_2}{\vec{\mathbf{w}}_1^{\perp}\cdot\vec{\mathbf{w}}_2}\,\vec{\mathbf{w}}_1$$

of $\mathrm{Line}^{(1)}$ and $\mathrm{Line}^{(2)}$ with the end point $\mathbf{p}^{(k)} = \mathbf{p}^{(1)} + \mathbf{a}(1,k)$ of $V^{(k)}$, so that $\vec{\mathbf{w}}_k$ is proportional to $\mathbf{e}$ if and only if the direction line $\mathrm{Line}^{(k)}$ passes through P – i.e., the configuration $\mathcal{V}$ is hyper-aligned (see Figure 3 and Definition 2.1(b)).

This completes the proof of Proposition 2.4, and thus of Theorem 2.2. □

2.7. **Definition.** A mechanism $\Gamma = (\mathcal{L}, \mathcal{X}, \mathcal{P})$ is called *generic* if none of its configurations are hyper-aligned (cf. Definition 2.1).

2.8. *Remark.* The moduli space $\mathcal{M}$ of all possible mechanisms of a given combinatorial type – i.e., feasible choices of the parameters $\Gamma = (\mathcal{L}, \mathcal{X}, \mathcal{P})$ – is a semi-algebraic subspace of $\mathbb{R}^{\sum_{i=1}^{k} n^{(i)} + dk + |\mathcal{I}|}$ determined by a set of linear inequalities. The "generic" mechanisms will indeed be generic in the sense of (real) algebraic geometry, since mechanisms which can have singular configurations form a subspace of $\mathcal{M}$ of positive codimension.

2.9. **Corollary.** *The configuration space* $\mathcal{C} = \mathcal{C}(\Gamma)$ *of a generic polygonal mechanism* $\Gamma = (\mathcal{L}, \mathcal{X}, \mathcal{P})$ *in* $\mathbb{R}^d$ *is a smooth closed orientable manifold of dimension* $N(d-1) - |\mathcal{I}|$, *where* k *is the number of vertices of* $\mathcal{P}$, $N := \sum_{i=1}^{k} n^{(i)}$, *and* $|\mathcal{I}|$ *is given by* (2).

3. Morse functions for planar mechanisms

From now on we shall concentrate on the simplest type of polygonal mechanism – namely, planar mechanisms $(d = 2)$ having triangular platforms $(k = 3)$ and exactly two links per branch $(n^{(1)} = n^{(2)} = n^{(3)} = 2)$. These mechanism are known in the robotics literature as 3-*RRR* (rotational) mechanisms.

Recall that a smooth real-valued function on a manifold is called a *Morse function* if all its critical points are non-degenerate (cf. [M, I, §2]). Such functions may be used to deduce the cellular structure of the manifold, and thus recover its homotopy type (see [M, I, §3]). Our goal is to describe a Morse function for the configuration space of a 3-RRR mechanism.

3.1. **Theorem.** *The function* $f(\mathcal{V}) := \sum_{j=1}^{3} \|\mathbf{v}^{(j)}\|^2$ *is generically a Morse function on* $C(\Gamma)$, *where* $\mathbf{v}^{(j)} := \mathbf{v}_1^{(j)} + \mathbf{v}_2^{(j)} = \mathbf{p}^{(j)} - \mathbf{x}^{(j)}$.

Proof. In order to show that the critical points of f are non-degenerate, we must choose a local coordinate system near each such point.

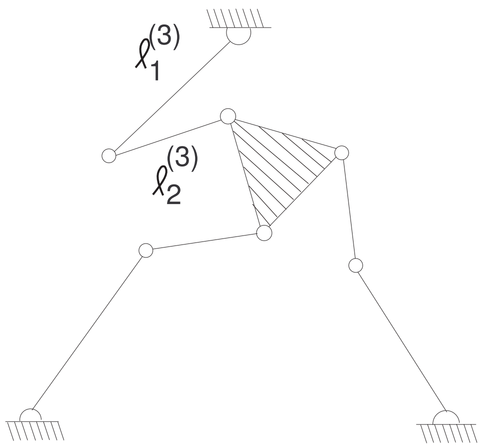

FIGURE 5. Local coordinates

Unfortunately, there is no uniform choice of such a system, so we must distinguish three cases:

Case I: Let $\Phi := (\phi_1, \phi_2, \phi_3)$, where ϕ_j denotes the angle between the vectors $-\mathbf{v}_1^{(j)}$ and $\mathbf{v}_2^{(j)}$ for $j = 1, 2, 3$. Then:

$$h^{(j)}(\phi_j) := \|\mathbf{v}^{(j)}\| = \|\mathbf{v}_1^{(j)} + \mathbf{v}_2^{(j)}\| = \sqrt{(\ell_1^{(j)})^2 + (\ell_2^{(j)})^2 - 2\ell_1^{(j)}\,\ell_2^{(j)} \cos\phi_1} \tag{27}$$

and thus $f(\Phi) = \sum_{j=1}^{3} h^{(j)}(\phi_j)^2$, so that:

$$\begin{aligned} \nabla f = \nabla_\Phi f &= 2\left(\ell_1^{(1)}\ell_2^{(1)}\,\sin(\phi_1),\ \ell_1^{(2)}\ell_2^{(2)}\,\sin(\phi_2),\ \ell_1^{(3)}\ell_2^{(3)}\,\sin(\phi_3)\right) \\ &= 2\,((\mathbf{v}_1^{(1)})^\perp \cdot \mathbf{v}_2^{(1)},\ (\mathbf{v}_1^{(2)})^\perp \cdot \mathbf{v}_2^{(2)},\ (\mathbf{v}_1^{(3)})^\perp \cdot \mathbf{v}_2^{(3)}) \end{aligned} \tag{28}$$

where $\vec{\mathbf{w}}^{\perp} := (b, -a)$ for $\vec{\mathbf{w}} = (a, b)$.

Thus Φ is a critical point if and only if:

$$\Phi \;=\; \frac{\pi}{2}\,(1+\sigma_1,\;1+\sigma_2,\;1+\sigma_3) \qquad \text{for } \sigma_1,\sigma_2,\sigma_3 \in \{\pm 1\}\;. \tag{29}$$

Computing the Hessian at a critical point Φ yields:

$$H_\Phi \;=\; \begin{pmatrix} \sigma_1\,\ell_1^{(1)}\ell_2^{(1)} & 0 & 0 \\ 0 & \sigma_2\,\ell_1^{(2)}\ell_2^{(2)} & 0 \\ 0 & 0 & \sigma_3\,\ell_1^{(3)}\ell_2^{(3)} \end{pmatrix}\;,$$

which is non-degenerate, with index Ind_Φ equal to the number of negative values in $\{\sigma_1,\,\sigma_2,\,\sigma_3\}$. Such critical points will be refered to as *Type I*.

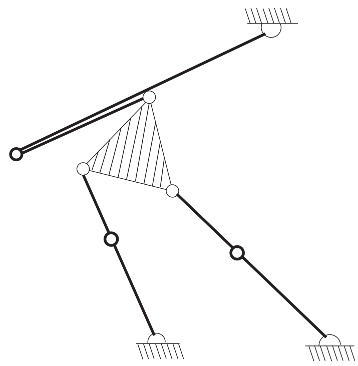

FIGURE 6. Type I critical point

Case II: As we saw, critical points of f appear when all three branches are aligned. However, for some mechanisms this will never happen, because one or two branches can never fully stretch or fold – that is, ϕ_3 (say) takes values in a proper subset $[a_1, a_2] \cup [-a_2, -a_1]$ of $[-\pi, \pi]$ (see Example 3.3). Clearly, ϕ_3 cannot then serve as a local coordinate at a point $(\phi_1, \phi_2, \pm a_k)$.

However, if the first two branches can both be aligned, then in the vicinity of doubly aligned configurations we take $\hat{\Phi} := (\phi_1, \phi_2, \theta_1)$, where ϕ_j $(j = 1, 2)$ as in Case I, and θ_j is the angle between $\mathbf{v}^{(j)} := \mathbf{v}_1^{(j)} + \mathbf{v}_2^{(j)}$ and the vector $\mathbf{x}^{(2)}$ (we assume for simplicity that $\mathbf{x}^{(1)}$ is at the origin).

Since

$$\mathbf{v}^{(j)} = h^{(j)}(\cos\theta_j, \sin\theta_j)\;, \tag{30}$$

using (27) we have:

$$(\partial_{\phi_j}\mathbf{v}^{(j)},\;\partial_{\phi_{j'}}\mathbf{v}^{(j)},\;\partial_{\theta_j}\mathbf{v}^{(j)}) \;=\; (\frac{(\mathbf{v}_1^{(j)})^{\perp}\cdot\mathbf{v}_2^{(j)}}{h^{(j)}(\phi_j)^2}\,\mathbf{v}^{(j)},\;0,\;-(\mathbf{v}^{(j)})^{\perp}))\;, \tag{31}$$

for $\{j, j'\} = \{1, 2\}$.

However, since θ_2 is a dependent variable, we may differentiate the norms in:

$$\mathbf{v}^{(1)} + \mathbf{a}(1,2) - \mathbf{v}^{(2)} \;=\; \mathbf{x}^{(2)} \tag{32}$$

implicitly and deduce that:

$$\partial_{\theta_1} \mathbf{v}^{(2)} \;=\; -\partial_{\theta_1} \theta_2 \, (\mathbf{v}^{(2)})^{\perp} \;=\; -\frac{\mathbf{a}(1,2)\cdot(\mathbf{v}^{(1)})^{\perp}}{\mathbf{a}(1,2)\cdot(\mathbf{v}^{(2)})^{\perp}}\,(\mathbf{v}^{(2)})^{\perp}\,. \tag{33}$$

Differentiating (32) itself and using (31), (33), and (26) yields:

$$\nabla_{\hat{\Phi}}\mathbf{a}(1,2) \;=\; \Big(-\frac{(\mathbf{v}_1^{(1)})^{\perp}\cdot\mathbf{v}_2^{(1)}}{h^{(1)}(\phi_1)^2}\,\mathbf{v}^{(1)},\; -\frac{(\mathbf{v}_1^{(2)})^{\perp}\cdot\mathbf{v}_2^{(2)}}{h^{(2)}(\phi_2)^2}\,\mathbf{v}^{(2)},\; \frac{\mathbf{v}^{(1)}\cdot(\mathbf{v}^{(2)})^{\perp}}{\mathbf{a}(1,2)\cdot(\mathbf{v}^{(2)})^{\perp}}\,(\mathbf{a}(1,2))^{\perp}\Big) \tag{34}$$

Since $\mathbf{x}^{(1)}=\mathbf{0}$, we see $\mathbf{v}^{(3)} \;=\; \mathbf{v}^{(1)}+\mathbf{a}(1,3)-\mathbf{x}^{(3)}$, so:

$$\begin{cases} \partial_{\phi_1} f \;=\; 2\,(\mathbf{v}_1^{(1)})^{\perp}\cdot\mathbf{v}_2^{(1)}\,\frac{2h^{(1)}(\phi_1)^2+\mathbf{v}^{(1)}\cdot(\mathbf{a}(1,3)-\mathbf{x}^{(3)})+(\mathbf{x}^{(3)}-\mathbf{v}^{(1)})\cdot(B_\alpha\mathbf{v}^{(1)})}{h^{(1)}(\phi_1)^2} \\ \partial_{\phi_2} f \;=\; 2\,(\mathbf{v}_1^{(2)})^{\perp}\cdot\mathbf{v}_2^{(2)}\,\frac{h^{(2)}(\phi_2)^2\;+(\mathbf{x}^{(3)}-\mathbf{v}^{(1)})\cdot(B_\alpha\mathbf{v}^{(2)})}{h^{(2)}(\phi_2)^2} \\ \partial_{\theta_1} f \;=\; 2\,(\mathbf{x}^{(3)}-\mathbf{a}(1,3))\cdot(\mathbf{v}^{(1)})^{\perp} \;+\; 2\,\frac{[\mathbf{v}^{(1)}\cdot(\mathbf{v}^{(2)})^{\perp}]\,[(\mathbf{v}^{(1)}-\mathbf{x}^{(3)})\cdot(\mathbf{a}(1,3))^{\perp}]}{\mathbf{a}(1,2)\cdot(\mathbf{v}^{(2)})^{\perp}} \end{cases} \tag{35}$$

where B_α is the rotation-and-dilitation matrix taking $\mathbf{a}(1,2)$ to $\mathbf{a}(1,3)$.

Note that we use the coordinates $\hat{\Phi}$ only at points where the first two branches are aligned, so that $\mathbf{v}_2^{(j)}\cdot(\mathbf{v}_1^{(j)})^{\perp}=0$ for $j=1,2$, and thus the first two entries of $\nabla_{\hat{\Phi}}(f)$ vanish at these points. The vanishing of $\frac{\partial f}{\partial\theta_1}$ is equivalent to the condition:

$$\begin{aligned} &[\mathbf{x}^{(3)}\cdot(\mathbf{v}^{(1)})^{\perp}]\,[\mathbf{a}(1,2)\cdot(\mathbf{v}^{(2)})^{\perp}] \;-\; [\mathbf{a}(1,3))\cdot(\mathbf{v}^{(1)})^{\perp}]\,[\mathbf{a}(1,2)\cdot(\mathbf{v}^{(2)})^{\perp}] \\ &+[\mathbf{v}^{(1)}\cdot(\mathbf{v}^{(2)})^{\perp}]\,[\mathbf{v}^{(1)}\cdot(\mathbf{a}(1,3))^{\perp}] \;-\; [\mathbf{v}^{(1)}\cdot(\mathbf{v}^{(2)})^{\perp}]\,[\mathbf{x}^{(3)}\cdot(\mathbf{a}(1,3))^{\perp}] \;=\; 0 \end{aligned} \tag{36}$$

Note that by (26) again, the intersection of $\mathrm{Line}^{(1)}$ with $\mathrm{Line}^{(2)}$ is at the point:

$$P := \mathbf{x}^{(3)}+\mathbf{v}^{(1)}+\frac{\mathbf{a}(1,2)\cdot(\mathbf{v}^{(2)})^{\perp}}{\mathbf{v}^{(1)}\cdot(\mathbf{v}^{(2)})^{\perp}}\,\mathbf{v}^{(1)}\,,$$

and (36) is equivalent to the colinearity of $\mathbf{x}^{(3)}$, P, and $\mathbf{v}^{(1)}+\mathbf{a}(1,3)$. Such critical points will be refered to as *Type II*.

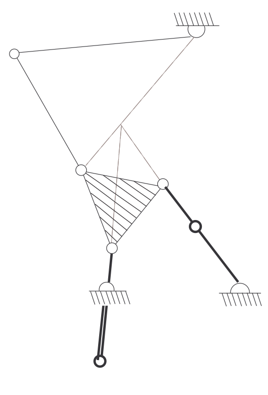

FIGURE 7. Type II critical point

Calculating the Hessian matrix H_f of f at a critical point, we find that it is diagonal, with:

$$\partial_{\phi_1\phi_1} f \;=\; -\,2\,\mathbf{v}_1^{(1)}\cdot\mathbf{v}_2^{(1)}\;\frac{2h^{(1)}(\phi_1)^2+\mathbf{v}^{(1)}\cdot(\mathbf{a}(1,3)-\mathbf{x}^{(3)})+(\mathbf{x}^{(3)}-\mathbf{v}^{(1)})\cdot(B_\alpha\mathbf{v}^{(1)})}{h^{(1)}(\phi_1)^2}$$

$$\partial_{\phi_2\phi_2} f \;=\; -\,2\,\mathbf{v}_1^{(2)}\cdot\mathbf{v}_2^{(2)}\;\frac{h^{(2)}(\phi_2)^2\;+(\mathbf{x}^{(3)}-\mathbf{v}^{(1)})\cdot(B_\alpha\mathbf{v}^{(2)})}{h^{(2)}(\phi_2)^2}$$

$$\begin{aligned}\partial_{\theta_1\theta_1} f \;=\; &\frac{2}{[\mathbf{a}(1,2)\cdot(\mathbf{v}^{(2)})^\perp]^2}\;\big(-\;2\,[\mathbf{v}^{(1)}\cdot(\mathbf{v}^{(2)})^\perp]\,[\mathbf{a}(1,3)\cdot\mathbf{v}^{(1)}]\,[\mathbf{a}(1,2)\cdot(\mathbf{v}^{(2)})^\perp]\\ &+\;[(\mathbf{x}^{(3)}-\mathbf{a}(1,3))\cdot\mathbf{v}^{(1)}]\,[\mathbf{a}(1,2)\cdot(\mathbf{v}^{(2)})^\perp]^2\\ &+\;[\mathbf{v}^{(1)}\cdot\mathbf{v}^{(2)}]\,[\mathbf{a}(1,2)\cdot(\mathbf{v}^{(1)}-\mathbf{v}^{(2)})^\perp]\,[(\mathbf{v}^{(1)}-\mathbf{x}^{(3)})\cdot(\mathbf{a}(1,3))^\perp]\\ &+\;[\mathbf{v}^{(1)}\cdot(\mathbf{v}^{(2)})^\perp]^2\,[(\mathbf{v}^{(1)}-\mathbf{x}^{(3)})\cdot(\mathbf{a}(1,3))]\\ &+\;[\mathbf{a}(1,2)\cdot\mathbf{v}^{(2)}]\,[(\mathbf{v}^{(2)}-\mathbf{a}(1,2))\cdot(\mathbf{v}^{(1)})^\perp]\,[(\mathbf{x}^{(3)}-\mathbf{a}(1,3))\cdot(\mathbf{v}^{(1)})^\perp]\;\big)\end{aligned}$$

If we solve (35) to find explicitly the critical points of f in the coordinates $\hat{\Phi}$, and then substitute into the expression we have found for H_f at these points, we obtain a polynomial expression of degree 6 in the parameters $\Gamma=(\mathcal{L},\mathcal{X},\mathcal{G})$ for the mechanism. Thus the critical point we identified is degenerate only when this polynomial vanishes, so generically f is indeed a Morse function.

Case III: Note that the work space $\mathcal{W}$ for each vertex of $\mathcal{P}$ is the intersection of three annuli (so it is compact), and thus the boundary of $\mathcal{W}$ must intersect at least one of the bounding circles of the annuli. Therefore, at least one of the three branches (say, the first) *can* be aligned.

Thus, at critical points of f where neither Φ nor $\hat{\Phi}$ can be used as local coordinates, the first branch is aligned, and we take $\Psi:=(\theta_1,\phi_1,\psi)$ as our local coordinates, where θ_1 and ϕ_1 are as in Case II above, and ψ denotes the angle between $\mathbf{a}(1,2)$ and $\mathbf{x}^{(2)}$ (see Figure 5). Note that this will not work when the second branch is also aligned, since these coordinates only determine the length of $\mathbf{v}^{(2)}$, and not "elbow up/down" near $\phi_2=\frac{\pi}{2}(1+\sigma_2)$.

Here:

$$f(\Psi)\;=\;\|\mathbf{v}_1^{(1)}+\mathbf{v}_2^{(1)}\|^2\;+\;\|\mathbf{v}^{(1)}+\mathbf{a}(1,2)-\mathbf{x}^{(2)}\|^2\;+\;\|\mathbf{v}^{(1)}+\mathbf{a}(1,3)-\mathbf{x}^{(3)}\|^2\,.$$

and since $\mathbf{a}(1,2)=g^{(1,2)}(\cos\psi,\sin\psi)$, we have $\nabla_\Psi(\mathbf{v}^{(1)})=(-(\mathbf{v}^{(1)})^\perp,\;\frac{(\mathbf{v}_1^{(1)})^\perp\cdot\mathbf{v}_2^{(1)}}{h^{(1)}(\phi_1)^2}\,\mathbf{v}^{(1)},\;0)$ and $\nabla_\Psi(\mathbf{a}(1,j))=(0,\;0,\;-\mathbf{a}(1,j)^\perp$ for $j=2,3,$ so:

$$\partial_{\theta_1} f\;=2(\mathbf{v}^{(1)})^\perp\cdot((\mathbf{x}^{(2)}+\mathbf{x}^{(3)})-(\mathbf{a}(1,2)+\mathbf{a}(1,3)))$$

$$\partial_{\phi_1} f\;=\;-\,(\mathbf{v}_1^{(1)})^\perp\cdot\mathbf{v}_2^{(1)}\;\frac{\mathbf{v}^{(1)}\cdot(4\,\mathbf{v}^{(1)}+\mathbf{a}(1,2)+\mathbf{a}(1,3)-\mathbf{x}^{(2)}-\mathbf{x}^{(3)})}{\|\mathbf{v}^{(1)}\|^2} \tag{37}$$

$$\partial_\psi f\;=\mathbf{a}(1,2)^\perp\cdot(\mathbf{x}^{(2)}-\mathbf{v}^{(1)})\;+\;\mathbf{a}(1,3)^\perp\cdot(\mathbf{x}^{(3)}-\mathbf{v}^{(1)})$$

We are using the coordinates Ψ because the first leg is aligned, so indeed $\partial_{\phi_1} f=0$. In order for this to be a critical point, we have two additional geometric conditions: the vanishing of $\partial_{\theta_1} f$ implies that the vector connecting the midpoints of sides of the fixed and moving platforms opposite the first vertex – that is, $A:=(\mathbf{x}^{(2)}+\mathbf{x}^{(3)})/2$ and $B:=(\mathbf{a}(2)+\mathbf{a}(3))/2$ – is aligned with $\mathbf{v}^{(1)}$ (see Figure

8). On the other hand, the vanishing (in addition) of $\partial_\psi f$ is equivalent to:

$$(\mathbf{v}^{(2)})^\perp \cdot \mathbf{x}^{(2)} \;+\; (\mathbf{v}^{(3)})^\perp \cdot \mathbf{x}^{(3)} \;=\; 0\,, \tag{38}$$

which means that the areas of the triangles spanned by $\mathbf{v}^{(j)}$ and $\mathbf{x}^{(j)}$ $(j = 2, 3)$ are equal. Such critical points will be refered to as *Type III*.

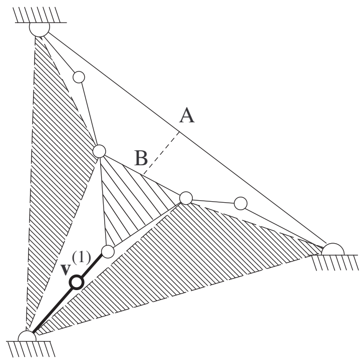

FIGURE 8. Type III critical point

Now, calculating the Hessian of f at the critical points we have:

$$\begin{aligned}
\partial_{\theta_1\theta_1} f \;&=\; -\;2\mathbf{v}^{(1)} \cdot ((\mathbf{x}^{(2)} + \mathbf{x}^{(3)}) - (\mathbf{a}(1,2) + \mathbf{a}(1,3)))\\
\partial_{\phi_1\theta_1} f \;&= \partial_{\phi_1\psi} f \;=\; 0\\
\partial_{\psi\theta_1} f \;&= 2\mathbf{v}^{(1)} \cdot (\mathbf{a}(1,2) + \mathbf{a}(1,3))\\
\partial_{\phi_1\phi_1} f \;&= \mathbf{v}_1^{(1)} \cdot \mathbf{v}_2^{(1)} \, \frac{\mathbf{v}^{(1)} \cdot (4\,\mathbf{v}^{(1)} + \mathbf{a}(1,2) + \mathbf{a}(1,3) - \mathbf{x}^{(2)} - \mathbf{x}^{(3)})}{\|\mathbf{v}^{(1)}\|^2}\\
\partial_{\psi\psi} f \;&= \mathbf{a}(1,2) \cdot (\mathbf{x}^{(2)} - \mathbf{v}^{(1)}) \;+\; \mathbf{a}(1,3) \cdot (\mathbf{x}^{(3)} - \mathbf{v}^{(1)})
\end{aligned}$$

Again, generically the critical point is non-degenerate. □

3.2. Identifying the critical points.

Since we usually have no explicit description of the configuration space $\mathcal{C}$ as a manifold, it is hard to calculate the Morse function $f : \mathcal{C} \to \mathbb{R}$ directly. However, in the course of proving Theorem 2.2 we gave a geometric description of each of the possible critical points of f – which are the main ingredient needed for analyzing the topology of $\mathcal{C}$ – in terms of the work space $\mathcal{W}$. We can use this geometric information in order to identify all possible candidates for critical points, and then we need only calculate df in local coordinates at these points (also provided in the proof above) to check if they are indeed critical, and find their indices.

Recall that $\mathcal{W}$ (Definition 1.6) is the space of all possible locations of the moving platform $\mathcal{P}$, whose vertices must be situated in the respective work spaces $\mathcal{W}_i$ $(i = 1, 2, 3)$ of (the end points of) the three branches. Each $\mathcal{W}_i$ is an annulus centered at the i-the vertex $\mathbf{x}^{(i)}$ of the fixed triangle.

Also recall the concept of the *coupler curve* γ of a planar four-bar linkage - that is, a degenerate polygonal mechanism with $k = 2$ linear branches $(n^{(1)} = n^{(2)} = 1)$, but having a triangular platform $\mathcal{P}$: the coupler curve is the work space for the third (unattached) vertex of $\mathcal{P}$. See [Hal, Ch. 4]. We consider the coupler curves for two vertices $\mathbf{x}^{(i)}$ (say $i = 1, 2$) of a triangular mechanism $\Gamma = (\mathcal{L}, \mathcal{X}, \mathcal{P})$ as above, in which the two corresponding branches are aligned, so that each can be replaced by a single linear branch of length $\ell^{(i)} := \ell_1^{(i)} + \ell_2^{(i)}$ or $\ell_1^{(i)} - \ell_2^{(i)}$, as the case may be.

(1) The critical points of Type I (all three branches aligned) correspond to placements of $\mathcal{P}$ with all three vertices on the (inner or outer) boundary circles of these annuli. Determining these is a straightforward geometric problem, which can be described as intersecting the coupler curve for the first two vertices, say, with the two boundary circles of $\mathcal{W}_3$.
(2) For critical points of Type II, we need also a line field V along the coupler curve γ, where $V(\gamma(t))$ is the line from $\gamma(t)$ to the intersection point $P(t)$ of $\mathrm{Line}^{(1)}$ with $\mathrm{Line}^{(2)}$. This line field is readily calculated from γ. The critical points are then those configurations for which $V(\gamma(t))$ passes through $\mathbf{x}^{(3)}$.
(3) For critical points of Type III, the first vertex $\mathbf{v}^{(1)}$ of $\mathcal{P}$ must lie on one of the two boundary circles of $\mathcal{W}_1$. Given $\mathbf{v}^{(1)}$, the possible positions of $\mathcal{P}$ are determined by its rotation angle θ around its first vertex, and at most two values θ', θ'' of θ satisfy condition (38). Thus we can define on $\partial\mathcal{W}_1$ two line fields V', V'' which associate to $\mathbf{v}^{(1)}$ the line between the midpoints of the $(2, 3)$-side of the fixed and moving triangles in the positions corresponding to θ', θ'' respectively. The critical points are those for which the vector $\mathbf{v}^{(1)}$ lies on one of these two lines.

3.3. **Example.** In general, the critical points of a Morse function on a manifold do not determine its topology, though together with their indices they impose certain restrictions on its homology, via the Morse inequalities. However, in the simplest cases the geometric considerations described above limit the possible critical points so severely that the configuration space $\mathcal{C}$ can be recovered in full. Note that there are two connected components in $\mathcal{C}$, determined by the orientation of the moving platform.

For example, consider a triangular mechanism with one branch (say, $k = 3$) having one very large link, so that the work space for the vertex $\mathbf{p}^{(3)}$ contains those for all points of the moving platform, and thus imposes no restriction on the allowed configurations. We assume the moving platform is a small triangle, and that the work space for (the vertex of) the first branch is a small annulus, intersecting that of the second branch in a crescent-shaped lune, which is the approximate "work space" for the moving platform (i.e., for its barycenter). Finally, assume that the fixed vertex $\mathbf{x}^{(3)}$ is far to the left (see Figure 9).

Now we may analyze the possible critical points as follows:

(1) Since the two small annuli above are wholly contained in the large one, and the moving platform is small, there are no critical points of Type I.
(2) Note that there are exactly two cases where $\mathrm{Line}^{(1)}$ meets $\mathrm{Line}^{(2)}$ on the inner boundary circle of the work space for vertex 2. Since $g^{(23)}$ is very small, any critical points of Type II must occur nearby, so that the edges $\mathbf{a}(12)$ and $\mathbf{a}(13)$ (which nearly coincide) are aligned with $\mathbf{v}^{(1)}$ (see Figure 10).

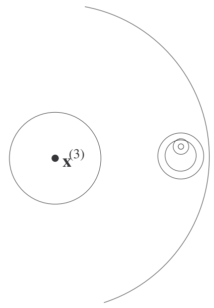

FIGURE 9. Work spaces for the three moving vertices

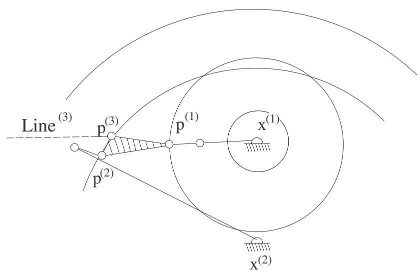

FIGURE 10. A critical point of Type II

By choosing appropriate generic values for the parameters, we can ensure that there are exactly two critical of Type II in each component of $\mathcal{C}$.

(3) Consider the three dashed lines $L^{(k)}$ in Figure 11, each connecting $\mathbf{x}^{(k)}$ with the midpoint of opposite (fixed) edge (for $k = 1, 2, 3$). Because the moving triangle is so small, the vector $\mathbf{v}^{(k)}$ must approximate the direction of $L^{(k)}$ in order to obtain a critical point of Type III – but since these lines do not pass near the approximate work space for the moving platform, no such critical points can occur.

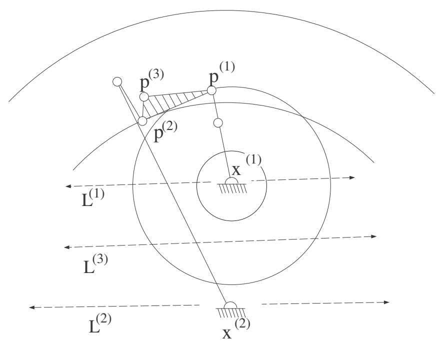

FIGURE 11. Potential critical points of Type III

Thus each component of the configuration space $\mathcal{C}(\Gamma)$ has exactly two critical points in this case (both of Type II), so it is homeomorphic to S^3.

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