COHOMOLOGY AND HIGHER HOMOTOPY OPERATIONS

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Abstract. We provide a description of higher homotopy operations as defined in [BM], in cohomological terms.

1. Introduction

In [BM], a "geometric" definition of higher homotopy operations based on the W-construction of Boardman and Vogt, was given in terms of an obstruction theory for rectifying diagrams. On the other hand, in [DKSm2] Dwyer, Kan and Smith gave an obstruction theory for rectifying diagrams in the homotopy category, framed in standard cohomological form. The purpose of the present note is to explain how the geometrical definition can be re-stated in terms of an appropriate cohomology theory.

1.1. **Notation.** The category of compactly generated topological spaces is denoted by $\mathcal{T}$, and that of pointed connected compactly generated spaces by $\mathcal{T}_*$; their homotopy categories are denoted by $\operatorname{ho}\mathcal{T}$ and $\operatorname{ho}\mathcal{T}_*$ respectively. The category of simplical sets will be denoted by $\mathcal{S}$, and that of pointed simplicial sets by $\mathcal{S}_*$. $\mathcal{C}at$ denotes the category of all small categories.

1.2. **Spanier's approach to higher operations.** In [Sp2], Spanier gave a general theory of higher order operations (extending the definition of secondary operations given in [Sp1]), somewhat similar in spirit to the approach we propose here: an $(n+1)$-st order operation is defined as a set of cohomology classes $\mathcal{O}^n \subset H^n(K;\Gamma_n)$, where K is a simplicial complex (corresponding essentially to our $b\mathcal{P}$ – see §2.4 below), and the coefficients Γ_n are a *stack* ("cosheaf of groups") on K, defined

$$\Gamma_n(\sigma) := \pi_n\Phi(\sigma) \qquad \text{for} \quad \sigma \in K,$$

where $\Phi(\sigma)$ is the topological space assigned to the simplex $\sigma \in K$ by a given *carrier* ("cosheaf of spaces"), and π_n is as usual its n-th homotopy group.

2. Lattices and higher homotopy operations

First, we recall the definition of the higher homotopy operations originally given in [BM]:

2.1. **Definition.** A *lattice* is a finite directed non-unital category Γ (that is, we omit the identity maps), equipped with two objects $v_{\text{init}} = v_{\text{init}}(\Gamma)$ and $v_{\text{fin}} = v_{\text{fin}}(\Gamma)$ with a unique map $\phi_{\max} : v_{\text{init}} \to v_{\text{fin}}$, and for every $w \in V := \operatorname{Obj}\Gamma$, there is at least one map from v_{init} to w, and at least one from w to v_{fin}. A composable sequence of k arrows in Γ will be called a k-*chain.*

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2.2. **The W-construction.** Given a lattice Γ, one can define a new category $W\Gamma$ enriched over cubical sets, with the same set of objects V, called the *bar construction* on Γ by Boardman and Vogt (cf. [BV, III, §1]):

2.3. **Definition.** For $u, v \in V$, let $\Gamma_{n+1}(u,v)$ be the set of $(n+1)$-chains from u to v in Γ, and $W\Gamma(u,v) := \coprod_{n\geq 0} \Gamma_{n+1}(u,v) \times I^n / \sim$, where I is the unit interval. Write $f_1 \circ_{t_1} f_2 \cdots f_n \circ_{t_n} f_{n+1}$ for $\langle u \xrightarrow{f_{n+1}} v_n \cdots \to v_1 \xrightarrow{f_1} v\rangle \times (t_n, \ldots, t_1)$ in $\Gamma_{n+1}(u,v) \times I^n$; then the relation $\sim$ is generated by

$$f_1 \circ_{t_1} f_2 \cdots f_n \circ_{t_n} f_{n+1} \ \sim\ f_1 \circ_{t_1} \cdots \circ_{t_{i-1}} (f_i f_{i+1}) \circ_{t_{i+1}} \cdots \circ_{t_n} \circ f_{n+1} \qquad \text{if } \ t_i = 0$$

for $1 \leq i \leq n$, where $(f_i f_{i+1})$ denotes f_i composed with f_{i+1}.

The categorial composition in $W\Gamma$ is given by the concatenation:

$$(f_1 \circ_{t_1} \cdots \circ_{t_l} f_{l+1}) \circ (g_1 \circ_{u_1} \cdots \circ_{u_k} g_{k+1}) := (f_1 \circ_{t_1} \cdots \circ_{t_l} f_{l+1} \circ_1 g_1 \circ_{u_1} \cdots \circ_{u_k} g_{k+1}).$$

We write $\mathcal{P} := W\Gamma(v_{\text{init}}, vf)$.

2.4. **Definition.** The *basis* category $bW\Gamma$ for a lattice Γ is defined to be the cubical subcategory of $W\Gamma$ with the same objects, and with morphisms given by $bW\Gamma(u,v) := W\Gamma(u,v)$ if $(u,v) \neq (v_{\text{init}}, v_{\text{fin}})$, while

$$bW\Gamma(v_{\text{init}}, v_{\text{fin}}) := \bigcup \{\alpha \circ \beta \mid \beta \in W\Gamma(v_{\text{init}}, w),\ \alpha \in W\Gamma(w, v_{\text{fin}}),\ v_{\text{init}} \neq w \neq v_{\text{fin}}\}\,,$$

so that $b\mathcal{P} := bW\Gamma(v_{\text{init}}, v_{\text{fin}})$ consists of all decomposable morphisms.

2.5. **Fact** ([BM, Proposition 2.15]). *For any lattice Γ, $W\Gamma(v_{\text{init}}, v_{\text{fin}})$ is combinatorially isomorphic to the cone $CbW\Gamma(v_{\text{init}}, v_{\text{fin}})$ on its basis, with the vertex of the cone corresponding to the unique maximal 1-chain $\langle v_{\text{init}} \xrightarrow{\phi_{\max}} v_{\text{fin}}\rangle$.*

2.6. **Higher homotopy operations.** We can use the W-construction to define a higher homotopy operations, as follows:

2.7. **Definition.** *Initial data* for a higher homotopy operation is a lattice Γ, together with a functor $\mathcal{A} : \Gamma \to \operatorname{ho}\mathcal{T}_*$. A *rectification* of the initial data $\mathcal{A} : \Gamma \to \operatorname{ho}\mathcal{T}_*$ is then a strict Γ-diagram realizing $\mathcal{A}$ – i.e., a functor $F : \Gamma \to \mathcal{T}_*$ such that $\pi \circ F$ is naturally isomorphic to $\mathcal{A}$, where $\pi : \mathcal{T}_* \to \operatorname{ho}\mathcal{T}_*$ is the obvious projection functor.

Recall that the (right) *half-smash* $X \rtimes K$ of topological spaces X and K, where X is pointed, is defined to be $(X \times K)/(\{*\} \times K) = X \wedge K_+$, where K_+ is K with a disjoint basepoint added. $X \rtimes K$ is again a pointed space, with the class of $\{*\} \times K$ as the distinguished point.

2.8. **Definition.** Given initial data $\mathcal{A} : \Gamma \to \operatorname{ho}\mathcal{T}_*$, *complete data* for the corresponding higher homotopy operation consists of a continuous functor $C\mathcal{A} : bW\Gamma \to \mathcal{T}_*$ such that $\pi \circ C\mathcal{A} = \mathcal{A} \circ (\varepsilon|_{bW\Gamma})$.

The corresponding *higher order homotopy operation* is the subset

$$\langle\!\langle \mathcal{A} \rangle\!\rangle \subset [\mathcal{A}(v_{\text{init}}) \rtimes b\mathcal{P}, \mathcal{A}(v_{\text{fin}})]_{\operatorname{ho}\mathcal{T}_*}$$

consisting of the homotopy equivalence classes of maps

$$C\mathcal{A}|_{bW\Gamma(v_{\text{init}}, v_{\text{fin}})} : bW\Gamma(v_{\text{init}}, v_{\text{fin}}) = b\mathcal{P} \longrightarrow \mathcal{T}_*(\mathcal{A}(v_{\text{init}}), \mathcal{A}(v_{\text{fin}}))$$

induced by all possible complete data $C\mathcal{A}$ for $\mathcal{A}$.

2.9. **Definition.** The higher operation $\langle\!\langle \mathcal{A} \rangle\!\rangle$ is said to *vanish* if it contains the homotopy class of a constant map $b\mathcal{P} \longrightarrow \mathcal{T}_*(\mathcal{A}(v_{\text{init}}), \mathcal{A}(v_{\text{fin}}))$.

2.10. **Fact** ([BM, Theorem 3.8]). *The homotopy operation $\langle\!\langle \mathcal{A} \rangle\!\rangle$ vanishes (and in particular, is defined) if and only if $\mathcal{A}$ has a rectification, so it is precisely the last obstruction to rectifying $\mathcal{A}$.*

3. Simplicial model categories

Even though the cubical mapping spaces of $W\Gamma$ are more economical, for our purposes it will be more convient to work with simplicial sets, since certain facts about them that we shall need are readily available in the literature.

3.1. **A simplicial version of $W\Gamma$.** In [CP, §2], Cordier and Porter described a version of $W\Gamma$ which is enriched over *simplicial*, rather than cubical, sets:

Given a lattice Γ as above, the category $W_s\Gamma$ has the same objects as Γ, and for each pair of nodes (u, v) of Γ, $W_s\Gamma(u, v) \in \mathcal{S}$ is a simplicial set with one r-simplex $\sigma_{(\mathcal{U},\Phi)}$ for each chain $\Phi = \langle u = v_{n+1} \xrightarrow{f_{n+1}} v_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} v_1 \xrightarrow{f_1} v_0 = v\rangle = f_1 f_2 \cdots f_n f_{n+1}$ and each partition $\mathcal{U} = (U_1, \ldots, U_r)$ of a subset $U \subset \{v_1, \ldots, v_n\}$ of the set of internal nodes of Φ, with each of the sets U_i nonempty. The faces of σ are defined by:

(i) $d_0(\sigma) := \sigma_{(\mathcal{U}^{(0)},\Phi')}$, where Φ' is obtained from Φ by carrying out the compositions at each node $v_i \in U_1$, and $\mathcal{U}^{(0)} := (U_2, \ldots, U_r)$.
(ii) $d_j(\sigma) := \sigma_{(\mathcal{U}^{(j)},\Phi)}$, where $\mathcal{U}^{(j)} := (U_1, \ldots, U_{j-1}, U_j \cup U_{j+1}, U_{j+2}, \ldots, U_r)$ for $0 < j < r$.
(iii) $d_r(\sigma) := \sigma_{(\mathcal{U}^{(r)},\Phi)}$, where $\mathcal{U}^{(r)} := (U_1, \ldots, U_{r-1})$.

The degenerate simplices are obtained by allowing partitions with $U_j = \emptyset$, and the simplicial composition map is defined by concatentation of chains in the obvious way.

3.2. *Remark.* Note that $W_s\Gamma(u, v)$ provides a canonical triangulation of the cubical set $W\Gamma(u, v)$; moreover, because of the way products of simplicial sets are defined, the simplicial composition map $\circ : W_s\Gamma(u, v) \times W_s\Gamma(v, w) \to W_s\Gamma(u, w)$ defines levelwise maps

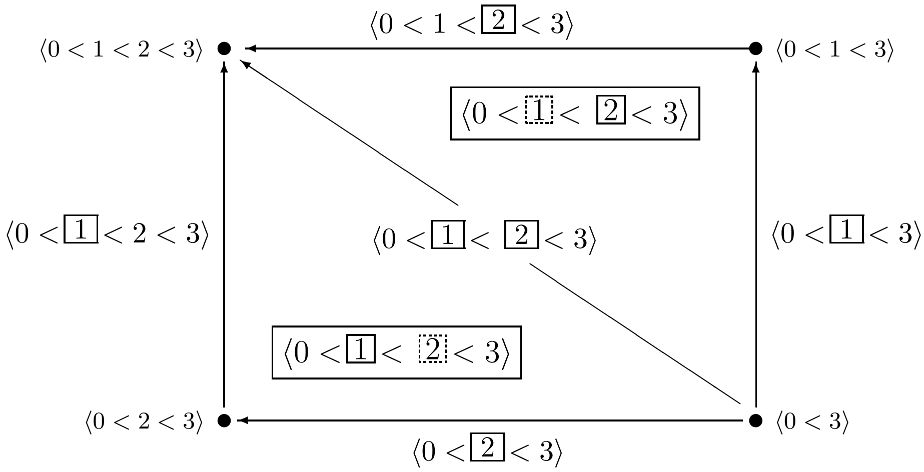
$$\circ : W_s\Gamma(u, v)_n \times W_s\Gamma(v, w)_n \to W_s\Gamma(u, w)_n \qquad \text{for each } n \geq 0,$$

so we can think of $W_s\Gamma$ as a simplicial category – that is, a simplicial object over $\mathcal{C}at$ – with (dimensionwise) fixed objects. On the other hand, $W\Gamma$ is not a cubical category in this sense.

3.3. **Example.** If the lattice Γ is the linearly ordered chain $0 < 1 < 2 < 3$, then $W_s\Gamma = W_s\Gamma(0, 3)$ is a triangulated square:

Here the notation $\langle 0 < \boxed{1} < \boxed{2} < 3\rangle$, for example, means that we have partitioned the internal nodes $\{1, 2\}$ of the (maximal) chain $0 < 1 < 2 < 3$ with $U_1 = \{2\}$ (the unbroken box), and $U_2 = \{1\}$ (the dashed box).

3.4. **Definition.** For a lattice Γ as above, with $V = \mathrm{Obj}\ \Gamma$, let $\mathcal{V} = (V, \prec)$ denote the partially ordered set with $u \prec v \Leftrightarrow \mathrm{Hom}_\Gamma(u, v) \neq \emptyset$, and V-$\mathcal{C}at$ the category of all small non-unital (directed pointed) categories $\mathcal{C}$ with $\mathrm{Obj}\ \mathcal{C} = V$, and $\mathrm{Hom}_\mathcal{C}(u, v) \neq \{*\}$ *Rightarrow* $u \prec v$ in $\mathcal{V}$. Let sV-$\mathcal{C}at$ (cf. [DK1, §1.4]) be the catgegory of simplicial objects over V-$\mathcal{C}at$.



Thus each $\mathcal{M}_\bullet \in sV\text{-}\mathcal{C}at$ is a simplicial category with fixed object set V in each dimension, and all face and degeneracy functors are the identity on objects. Equivalently (see §3.2), one can think of $\mathcal{M} = \mathcal{M}_\bullet$ as a category enriched over $\mathcal{S}_*$, with $\mathrm{Obj}\,\mathcal{M} = V$ (i.e, for each ordered pair $u \prec v$ of $\mathcal{V}$ we have a pointed simplicial set $\mathcal{M}(u,v) \in \mathcal{S}_*$, and to each ordered triple $u \prec v \prec w$ a map $\circ : \mathcal{M}(u,v) \times \mathcal{M}(v,w) \to \mathcal{M}(u,w)$, which is associative in the obvious sense.

3.5. **Fact.** *If $\mathcal{C}$ is the category of non-unital small directed categories, there is a forgetful functor $U : C \to \mathcal{D}i\mathcal{G}$ to the category of directed graphs, whose left adjoint $F : \mathcal{D}i\mathcal{G} \to \mathcal{C}$ is the "free category" of* [Ha] *(see also* [DK1, §2.4]*).*

This pair of adjoint functors defines a comonad $FU : \mathcal{C} \to \mathcal{C}$, and thus an augmented simplicial category $\mathcal{E}_\bullet \to \mathcal{C}$ with $\mathcal{E}_n := (FU)^{n+1}\mathcal{C}$, as in [Go, App., §3]. If $\mathcal{C} \in V\text{-}\mathcal{C}at$, then $\mathcal{E}_\bullet \in sV\text{-}\mathcal{C}at$.

3.6. **Example.** Both Γ and $W_s\Gamma$ can be thought of as being in $sV\text{-}\mathcal{C}at$ (the first being trivial). However, if we think of Γ as a (non-unital small directed) category, and of $W_s\Gamma$ as a *simplicial* category , then the above comonad construction, when applied to Γ, yields $\mathcal{E}_\bullet = W_s\Gamma$.

The augmentation morphism $\varepsilon : W_s\Gamma \to \Gamma$ is defined by $d_0^{n+1} : W_s\Gamma_n(u,v) \to \Gamma(u,v)$ (where $d_0 : W_s\Gamma_0(u,v) \to \Gamma(u,v)$ is the iterated composition on any chain).

3.7. **Definition.** A simplicial category $\mathcal{E}_\bullet \in sV\text{-}\mathcal{C}at$ is *free* if each category E_n, and each degeneracy functor $s_j : E_n \to E_{n+1}$, is in the image of the functor $F : V\text{-}\mathcal{C}at \to V\text{-}\mathcal{C}at$.

In particular, the "spheres" in $sV\text{-}\mathcal{C}at$ are objects the form $\mathcal{M}_\bullet := \mathbf{S}^n_{(u,v)}$ for $n \geq 1$ and $u \prec v$ in $\mathcal{V}$, defined by:

$$\mathcal{M}(u',v') = \begin{cases} \mathbf{S}^n & \text{for } u' = u \text{ and } v' = v \\ * & \text{otherwise,} \end{cases}$$

and evidently each $\mathbf{S}^n_{(u,v)}$ is free, and in fact the free objects are just arbitrary coproducts (in $sV\text{-}\mathcal{C}at$) of spheres.

A map $\Phi : \mathcal{M}_\bullet \to \mathcal{N}_\bullet$ in $sV\text{-}\mathcal{C}at$ is *free* if for each $n \geq 0$, there is

a) a coproduct of spheres $\mathcal{E}_n \in V\text{-}\mathcal{C}at$,
b) a map $\varphi_n : W_n \to Y_n$ in $\mathcal{C}$ which induces an isomorphism

$$(M_n \amalg_{L_n\mathcal{M}_\bullet} L_n\mathcal{N}_\bullet) \amalg E_n \to N_n.$$

The "latching objects" for a simplicial category $\mathcal{M}_\bullet$ are $L_n\mathcal{M}_\bullet := \coprod_{0\leq i\leq n-1} M_{n-1}/\sim$, where for any $x \in M_{n-2}$ and $0 \leq i \leq j \leq n-1$ we set $s_j x$ in the i-th copy of M_{n-1} equivalent to $s_i x$ in the $(j+1)$-st copy of M_{n-1}.

3.8. **The model category structure on sV-$\mathcal{C}at$.** For $\mathcal{V} = (V, \prec)$ as above, sV-$\mathcal{C}at$ has a simplicial E^2-model category (in the sense of [DKSt, §5]), in which the simplicial function complexes $\underline{\mathrm{Hom}}(\mathcal{M}_\bullet, \mathcal{N}_\bullet)$ is the limit (under the composition maps) of the diagram $\langle \underline{\mathrm{Hom}}_{\mathcal{S}_*}(M(u,v), N(u,v))\rangle_{u\prec v}$, and $\mathcal{M}_\bullet \otimes K$ and $\mathcal{M}_\bullet^K$ (for $K \in \mathcal{S}$) are defined as in [Q, Ch. II, §1-2]).

A map (simplicial functor) $\Phi : \mathcal{M}_\bullet \to \mathcal{N}_\bullet$ is

(i) a *weak equivalence* if $\Phi(u,v) : M(u,v) \to N(u,v)$ is a weak equivalence of simplicial sets for each $u \prec v$;
(ii) a *fibration* if each $\Phi(u,v)$ is a fibration of simplicial sets;
(iii) a *cofibration* if it is a retract of a free map.

(Compare [DK1, Proposition 7.2]).

3.9. **Fact** ([CP, §1]). *The augmentation $\varepsilon : W_s\Gamma \to \Gamma$ induces a weak equivalence of simplicial sets $\varepsilon_{(u,v)} : W_s\Gamma(u,v) \to K(\Gamma(u,v), 0)$ for each $u \prec v$, where $K(A,0)$ is the constant simplicial set on $A \in \mathcal{S}et$.*

3.10. **Definition.** For any $X_\bullet \in sV$-$\mathcal{C}at$, let $Y_\bullet$ be a fibrant replacement for $X_\bullet$, and let

$$\pi_n^{(u,v)}(X_\bullet) := [\mathbf{S}^n_{(u,v)}, Y_\bullet] := \pi_0\underline{\mathrm{Hom}}(\mathbf{S}^n_{(u,v)}, Y_\bullet)$$

for each $u \prec v$ in $\mathcal{V}$.

3.11. *Remark.* Note that $\pi_n X_\bullet := \langle \pi_n^{(u,v)}(X_\bullet)\rangle_{u\prec v}$ constitutes a $\Gamma_{X_\bullet}$-diagram of groups (abelian, if $n \geq 2$), where $\Gamma_{X_\bullet} := \pi_0 X_\bullet$ is the lattice of components of the mapping spaces of $X_\bullet$. Moreover, a map $\Phi : \mathcal{M}_\bullet \to \mathcal{N}_\bullet$ in sV-$\mathcal{C}at$ is a weak equivalence if an only if it induces an isomorphism in $\pi_n X_\bullet$ for all $n \geq 0$.

4. Cohomology for sV-$\mathcal{C}at$

In [DKSm1, §2.1], Dwyer, Kan, and Smith defined the cohomology with local coefficients for a small category $\mathcal{D}$, in which lie the obstructions to rectifying a homotopy commutative D-diagram constructed in [DKSm2, §3.5]. We now show how these cohomology groups can be reinterpreted in more familiar terms. First, note that Postnikov towers may be defined for $Y_\bullet \in sV$-$\mathcal{C}at$ as in [DK2, §1.2]:

4.1. **Definition.** For each $n \geq 0$ define $Y_\bullet^{(n)}$ by setting $Y_k^{(n)} := M_k$ for $k \leq n+1$ and $Y_k^{(n)} := M_k(Y_\bullet^{(n)})$ for $k \geq n+2$, where the n-th "matching object" for $Y_\bullet$ is defined for $u \prec v$ in $\mathcal{V}$ and $n \geq 1$ by $M_n Y_\bullet(u,v) := \{(x_0, \ldots, x_n) \in (Y_{n-1}(u,v))^{n+1} \mid d_i x_j = d_{j-1}x_i$ for all $k \leq i < j \leq n\}$. For any $X_\bullet \in sV$-$\mathcal{C}at$, choose some fibrant replacement $Y_\bullet$ for $X_\bullet$, and set $P^n X_\bullet := Y_\bullet^{(n)}$.

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