Emmanuel told us about Sullivan's localizations. This actually gives us two pull backs coming from arithmetic:

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & \prod_p \mathbb{Z}_p \arrow[d] \\ \mathbb{Q} \arrow[r] & (\prod_p \mathbb{Z}_p)\otimes\mathbb{Q} \end{tikzcd} \quad ; \quad \begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}[1/p] \arrow[d] \\ \mathbb{Z}_p \arrow[r] & \mathbb{Q}_p \end{tikzcd}$$

. These give us corresponding squares in spaces

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod_p X_p^\wedge \arrow[d] \\ X_\mathbb{Q} \arrow[r] & (\prod_p X_p^\wedge)_\mathbb{Q} \end{tikzcd} \quad ; \quad \begin{tikzcd} X \arrow[r] \arrow[d] & X[1/p] \arrow[d] \\ X_p^\wedge \arrow[r] & (X_p^\wedge)[1/p] \end{tikzcd}.$$

We want to do this equivariantly, where we'll have more options.

If V is a G-rep, let $a_V : S^0 \to S^V$ be the zero section.

Lemma 1) a_V is null iff $V^G \neq 0$. No path vs. path

2) $a_{V\oplus W} = a_V \cdot a_W$

3) $\mathrm{Res}^G_H a_V = a_{\mathrm{res}\, V}$.

4) $N_H^G a_V = a_{\mathrm{Ind}\, V}$.

5) If $V^G = \{0\}$, then $a_V^k \neq 0 \ \forall k \in \mathbb{N}$.

In fact, $k \cdot a_V$ is also $\neq 0$ if $a_V \neq 0$.

This provides a great class in $\pi^G_{-V} S^0$, so we can play a Sullivan-esque game:

$$S^0[a_V^{-1}] = \varinjlim (S^{nV}) =: S^{\infty V}.$$

Basic properties:

① If $a_V = 0$, this is $\underset{G}{\simeq} *$.

② If $a_V \neq 0$, then the fixed points are just S^0 (in spaces)

③ So if $H \subseteq G$ has $V^H \neq \{0\}$, then $\left(S[a_V^{-1}]\right)^H \simeq *$.

④ $\mathrm{Map}_*(G/H_+, S[a_V^{-1}]) \simeq *$.

So inverting a_V did 2 things: inverted the map & nullified all G/H_+ s.t. $V^H \neq \{0\}$.

Now $a_V^{-1} S^0$ sits in a cofiber sequence:

$$S(\infty V)_+ \to S^0 \to S^{\infty V} = S^0[a_V^{-1}]$$

$\mathbb{Q}/\mathbb{Z}$ for a_V as an ind thing

$S(\infty V)$ has a few properties: $S(\infty V)^H = \begin{cases} \emptyset & V^H = \{0\}. \\ \simeq * & V^H \neq \{0\} \end{cases}$

Spaces of this form are called "universal spaces": $E\mathcal{F}$. The unreduced suspension is $\tilde{E}\mathcal{F}$.

So lets look at

$$F(S(\infty V)_+, X) = \lim F(S(nV)_+, X) = \varinjlim \underbrace{D(S^{nV}/S^0)}_{\wr\wr\; S^0/a_V^n} \wedge X = X^\wedge_{a_V}.$$ So we have a square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \wedge S[1/a_V] = X[a_V^{-1}] \arrow[d] \\ X^\wedge_{a_V} = F(S(\infty V)_+, X) \arrow[r] & X^\wedge_{a_V}[a_V^{-1}]. \end{tikzcd}$$

So let's consider $G = C_2$ for concreteness

$a_V = a_\sigma^k$. ? $\mathcal{F} = \{e\}$, so $S(\infty\sigma) = EG$.

$$\begin{tikzcd} X^G \arrow[r] \arrow[d] & X[a_\sigma^{-1}]^G \arrow[d] \\ F(EG_+, X)^G \arrow[r] & (X^t)^G \end{tikzcd}$$

is an equivariant pull-back.

Def · $F_G(EG_+, X) = X^{hG}$ ← what is this really? X^G is a limit. Not a holim

· $(X[a_\rho^{-1}])^G = X^{\Phi G} = \phi^G(X)$

· $(X^t)^G =$ "Tate cohom object".

If $X \to Y$ is a G-map and a underlying weak equiv, then we don't automatically have that $X^G \to Y^G$ is a w.e. (if it's a homeomorphism, however, we do). Ex: $S^{\infty V} \to *$ is a G-map & the underlying map is a w.e, but on fixed points, it's $S^0 \to *$. Homotopy fixed points fixes this.

Now our diagram is:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X[a_V^{-1}] \arrow[d] \\ X^\wedge_{a_V} \arrow[r] & a_V^{-1} X^\wedge_{a_V} \end{tikzcd}$$

sees only s.g. H w/ $V^H = \{0\}$

cares only about H s.t. $V^H \neq \{0\}$

, and this is a homotopy pull-back square.

Now a very important corollary. The vertical maps are ring maps if X is a ring. Thus:

Thm If $\phi^H X \simeq *$ for all $\{1\} \neq H \subseteq G$, then $X \to F(EG_+, X)$ is a G-weak equiv & $X^G \xrightarrow{\simeq} X^{hG}$.

This is proved by induction on $|G|$ in the above diagram, essentially.