I'm very excited to be here, talking about the Kervaire problem and our solution. The proof we'll discuss, and the one in the paper are a... simplification of our original story. In this talk, I'll give an outline and some (personal) history.

Thm There are manifolds of Kervaire invariant one only in dims $\underbrace{2, 6, 14, 30, 62}_{\text{all known}}, \overset{?}{\&}\ 126$

The Kervaire Invariant ← Inbar Klang is the obstruction to framed surgery in the middle dim, so

Cor: Almost all framed manifolds are frame bordant to an exotic sphere.

Already I've used Pontryagin's dictionary: $\pi_{n+k}(S^n) \cong \{M^k \subseteq \mathbb{R}^{n+k} + \text{framing}\}/\sim$

Theorem of Browder: M^m Kervaire invariant $1 \leftrightarrow \Theta \in \pi_m(S^0) \Rightarrow m = 2^{i+1}-2$, in the ASS, h_i^2 reps Θ.

Boris Chorny

So what do we do?

$$\begin{tikzcd} \boxed{\cancel{\beta t}?\colon \text{ANSS}} \arrow[r, squiggly, "②"] \arrow[d, squiggly, "①"] & \boxed{H^*(C_8; \Omega_{\mathbb{O}*})} \arrow[d, leftrightarrow] \\ \boxed{h_i^2 : \text{ASS}} & \boxed{\pi_* \Omega_{\mathbb{O}}^{C_8}} \end{tikzcd}$$

1. ASS is a mess. Lift to ANSS ← David Blanc

   This is already more geometric. ANSS is built on complex bordism ← Yonatan Harpaz

2. Replace with a simpler computation: homotopy fixed points (David & me)

3. Rigidify: go equivariant ← Debasis Sen

$$\begin{tikzcd} x \arrow[r, mapsto, "f"] \arrow[d, "\text{big bar}"] & f(x) \\ h_j^2 & \end{tikzcd}$$

↰ not a perm cycle!

maps of SS: $d_i(f(x)) = f(d_i(x))$

Proof: ① If Θ exists, it is detected in $\pi_{2^{i+1}-2} \Omega_{\mathbb{O}}^{hC_8}$ (Detection)
② $\pi_k \Omega_{\mathbb{O}}^{hC_8} = \pi_{k+256} \Omega_{\mathbb{O}}^{hC_8}$ (Periodicity Theorem)
③ $\pi_{-2} \Omega_{\mathbb{O}}^{C_8} = 0$ (Gap Theorem)
④ $\Omega_{\mathbb{O}}^{C_8} \simeq \Omega_{\mathbb{O}}^{hC_8}$ (Homotopy fixed points thm)

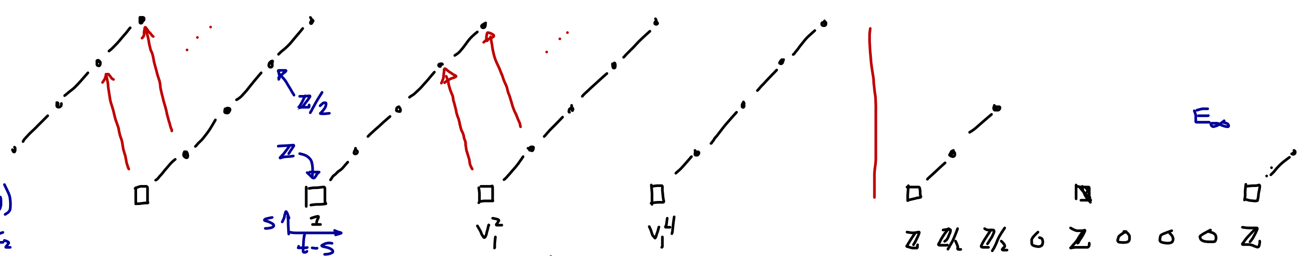
Ok. Let's unpack this a little more. Why go equivariant? Slice Tower. ← Emmanuel For $\Omega_{\mathbb{O}}$, we can actually compute some equivariant homotopy groups very easily!

$\Omega_{\mathbb{O}}$ is a perpetual source of confusion, so I'll give some background: $K_{\mathbb{R}}$, KU, & KO

What does $K_{\mathbb{R}}$ classify? If X is a C_2-equiv. space, $K\mathbb{R}^0(X) =$ iso classes of $V \downarrow X$ w/ $\sigma: V \to V$ conj. linear.

KU gets an action of C_2 via complex conjugation. Atiyah showed that $KO = KU^{hC_2}$. We can play similar games with $K_{\mathbb{R}}$!

The HFPSS:



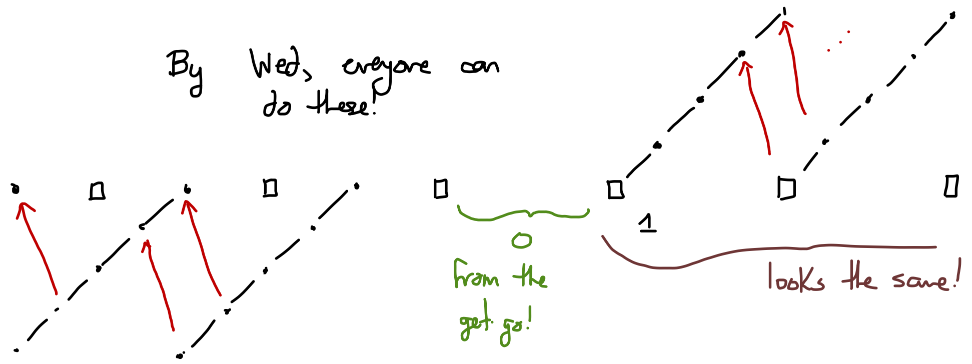
$E_2^{st} = H^s(C_2; \pi_t KU) \Rightarrow \pi_{t-s} KU^{hC_2}$

Periodic = obvious

$\pi_{-2} = 0$ requires d_3s.

- KO_* is 8-periodic & $\pi_{-2} KO = 0$.
- The map from the Adams-Novikov 2-line to this is very far from injective! First important class we miss is ν^2.

For KU, Atiyah also showed that $KU^{C_2} = KO$. So we can mirror the slice story (due to Dugger)



$$\boxed{Gr(KU) = \bigvee_{i \in \mathbb{Z}} S^{i\rho} \wedge H\mathbb{Z}.}$$

So we want something simple, like KU. Before this, the three of us had studied some generalizations of KU, the Hopkins-Miller higher real K-theories. Doug's odd-primary method showed that a simple C_8 action would suffice. Why not?

Spectrum	$E_? = E_\infty$	diff on π_{-2}
KU	E_4	d_3
TMF(5)	E_{14}	d_{13}
$EO_4(C_8)$	? (prob ~ E_{120})	?? (hugely long)

The proof is designed to dodge all of this messiness.

So what do we do instead? Take a geometric object (our MU^{G}) and then "invert the Bott class".