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MAPPING SPACES OF COMPACT LIE GROUPS AND p-ADIC COMPLETION

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ABSTRACT. If $\mathbf{BG}$, $\mathbf{BH}$ are the classifying spaces of compact Lie groups, with $\mathbf{H}$ connected, then the mapping space functor $\mathbf{map}(\mathbf{BG}, -)$ commutes with p-completion on $\mathbf{BH}$: i.e., for each $f\colon \mathbf{BG} \to \mathbf{BH}$ the component $(\mathbf{map}(\mathbf{BG}, \mathbf{BH})_f)^\wedge_p$ is p-complete, and is homotopy equivalent to $\mathbf{map}(\mathbf{BG}, \mathbf{BH}^\wedge_p)_{i\circ f}$.

1. INTRODUCTION

In studying $\mathbf{map}(\mathbf{BG}, \mathbf{BH})$, the space of maps between the classifying spaces of two compact Lie groups, it is often useful to know whether the p-adic completion commutes with the functor $\mathbf{map}(\mathbf{BG}, -)$; special cases where this occurs were used, for example, in [DZ, JMO, N2, NS]. Here we present a more general result in this direction:

1.1. **Theorem.** *Let* $\mathbf{G}$ *and* $\mathbf{H}$ *be compact Lie groups, with* $\mathbf{H}$ *connected; let* p *be a prime, and* $i\colon \mathbf{BH} \to \mathbf{BH}^\wedge_p$ *the natural inclusion. Then for any map* $f\colon \mathbf{BG} \to \mathbf{BH}$, *the corresponding component of the mapping space,* $\mathbf{map}(\mathbf{BG}, \mathbf{BH}^\wedge_p)_{i\circ f}$, *is p-complete, and*

$$(\mathbf{map}(\mathbf{BG}, \mathbf{BH})_f)^\wedge_p \xrightarrow{\simeq} \mathbf{map}(\mathbf{BG}, \mathbf{BH}^\wedge_p)_{i\circ f}$$

is a homotopy equivalence.

The p-adic completion of a space $\mathbf{X}$ that we refer to is the $(\mathbb{F}_p)_\infty\mathbf{X}$ of [BK, I, §4.2], which we denote by $\mathbf{X}^\wedge_p$. However, unless $\mathbf{X}$ is nilpotent (e.g., simply-connected), $\mathbf{X}^\wedge_p$ need not be *p-complete* in the sense of [BK, I, §5 & VII, §2], and so it enjoys few of the properties associated with completion. In particular, unless $\mathbf{X}^\wedge_p$ is p-complete, the natural map $i\colon \mathbf{X} \to \mathbf{X}^\wedge_p$ will not induce an isomorphism in $\mathbb{F}_p$-homology, so $\mathbf{X}^\wedge_p$ will not be the $H_*(-;\mathbb{F}_p)$-localization of $\mathbf{X}$ (cf. [BK, §2.1]) and $(\mathbf{X}^\wedge_p)^\wedge_p \not\simeq \mathbf{X}^\wedge_p$.

In §2 we list some facts about $\mathbb{Z}^\wedge_p$-modules needed to prove the theorem. In §3 we recall from [JMO] the mod-p approximation for $\mathbf{BG}$, using p-toral groups.

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In §4 the Bousfield-Kan spectral sequence is used to prove p-completeness. The required homotopy equivalence is shown in §5.

2. Finitely generated $\mathbb{Z}_p^\wedge$-modules

Let $\mathscr{F}$ denote the class of finitely generated $\mathbb{Z}_p^\wedge$-modules, where $\mathbb{Z}_p^\wedge$ is the ring of p-adic integers, and let $\mathscr{F}' = \mathscr{F} \cup \{ G : G \text{ is a finite } p\text{-group} \}$.

2.1. **Lemma.** *If* $\mathbf{X}$ *is a connected space with* $\pi_k\mathbf{X} \in \mathscr{F}'$ *for each* $k \geq 1$, *then*

(1) $H_\star(\mathbf{X}; \mathbb{F}_p)$ *is of finite type, that is,* $H_k(\mathbf{X}; \mathbb{F}_p)$ *is finite for each* $k \geq 0$;

(2) $\mathbf{X}$ *is p-complete and* $\mathbb{F}_q$*-acyclic for any prime* $q \neq p$, *that is,* $\widetilde{H}_\star(\mathbf{X}; \mathbb{F}_q) = 0$.

Proof. Any $M \in \mathscr{F}$ is isomorphic to $N \otimes \mathbb{Z}_p^\wedge$, where N is a finitely generated abelian group. Thus $\mathbf{K}(M, n) \simeq \mathbf{K}(N, n)_p^\wedge$, which is p-complete (see [BK, VI, 5.2]), and so $H_\star(\mathbf{K}(M, n); \mathbb{F}_p)$ is of finite type for all $n \geq 1$. Therefore, if $\mathbf{Y}$ is a simply-connected space with each $\pi_i\mathbf{Y} \in \mathscr{F}$, by induction on its Postnikov system, we see $H_\star(\mathbf{Y}; \mathbb{F}_p)$ is of finite type.

Now assume $\pi_1\mathbf{X} = G \in \mathscr{F}'$ and consider the universal covering fibration for $\mathbf{X}$;

$$(\star) \qquad \widetilde{X} \to \mathbf{X} \to \mathbf{K}(G, 1).$$

The action of G on the universal covering space $\widetilde{\mathbf{X}}$ makes $H_t(\widetilde{\mathbf{X}}; \mathbb{F}_p)$ into a G-module, and one has a Leray-Cartan spectral sequence (cf. [CE, XVI, §9]), with

$$E^2_{s,t} \cong H_s(G; H_t(\widetilde{\mathbf{X}}; \mathbb{F}_p)) \Rightarrow H_{t+s}(\mathbf{X}; \mathbb{F}_p).$$

Now for fixed t, let $V = H_t(\widetilde{\mathbf{X}}; \mathbb{F}_p)$ and let $\phi\colon G \to \mathrm{Aut}(V)$ describe the π_1-action. $\mathrm{Aut}(V)$ is finite, and if $G \in \mathscr{F}$ then G is q-divisible for any q prime to p, so in any case $\mathrm{Im}(\phi) \subseteq \mathrm{Aut}(V)$ is a finite p-group. Thus G acts nilpotently on V (cf. [BK, II, 5.2]): that is, there is a filtration $0 = V_0 \subset V_1 \subset \cdots V_i \cdots \subset V_n = V$ of G-modules such that G acts trivially on each V_i/V_{i-1}.

Using the short exact sequences $0 \to V_{i-1} \to V_i \to V_i/V_{i-1} \to 0$, we see by induction on i that each $H_s(G; V_i)$—and so in particular $H_s(G; V) \cong E^2_{s,t}$—is finite. Thus $H_\star(\mathbf{X}; \mathbb{F}_p)$ is of finite type.

Furthermore, because G acts nilpotently on V, by the mod-$\mathbb{F}_p$ fiber lemma of [BK,II, 5.1] the universal covering $(\star)$ remains a fibration after p-completion:

$$(\star)_p^\wedge \qquad \widetilde{\mathbf{X}}_p^\wedge \to \mathbf{X}_p^\wedge \to \mathbf{K}(G, 1)_p^\wedge .$$

Since $G = \pi_1\mathbf{X} \in \mathscr{F}'$, $\mathbf{K}(G, 1)$ is p-complete; similarly $\widetilde{\mathbf{X}} \simeq \widetilde{\mathbf{X}}_p^\wedge$ (being nilpotent, with $\pi_k\widetilde{\mathbf{X}} \in \mathscr{F}$). The Five Lemma, applied to the natural map from the long exact sequence of $(\star)$ to that of $(\star)_p^\wedge$, shows $\pi_\star\mathbf{X} \to \pi_\star(\mathbf{X}_p^\wedge)$ is an isomorphism, so $\mathbf{X}$ is p-complete. Since $\widetilde{H}_\star\widetilde{\mathbf{X}} = 0 = \widetilde{H}_\star\mathbf{K}(G, 1)$ for $q \neq p$ by [BK, VI, 5.6], the same holds for $\mathbf{X}$. □

2.2. **Corollary.** *Let* $\mathbf{X}$ *be a pointed connected space such that* $\pi_k\mathbf{X} \in \mathscr{F}$ *for* $k \geq 2$, *and suppose that* $\pi_1\mathbf{X}$ *has a finite normal series*

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_{n-1} \lhd G_n = \pi_1\mathbf{X} ,$$

where each $G_i/G_{i-1} \in \mathscr{F}'$. *Then* $\mathbf{X}$ *is p-complete and* $\mathbb{F}_q$*-acyclic for* $q \neq p$.

Proof. For each $i = 1, \dots, n$, let $\mathbf{X}_{i-1} \to \mathbf{X}_i \to \mathbf{K}(G_i/G_{i-1}, 1)$ be the covering fibration corresponding to the short exact sequence $1 \to G_{i-1} \to G_i \to G_i/G_{i-1} \to 1$ (where $\mathbf{X}_0 = \widetilde{\mathbf{X}}$ and $\mathbf{X}_n = \mathbf{X}$). As above, $\mathbf{K}(G_i/G_{i-1}, 1)$, and by induction also $\mathbf{X}_{i-1}$, are p-complete and $\mathbb{F}_q$-acyclic, with $\mathbb{F}_p$-homology of finite type. The same then holds for $\mathbf{X}_i$, too, by the covering-space argument in the proof of Lemma 2.1, and thus for $\mathbf{X}$. □

2.3. **Lemma.** *For $A, C \in \mathscr{F}$:*

(1) *If $0 \to A \to B \to C \to 0$ is a short exact sequence of abelian groups, then $B \in \mathscr{F}$.*

(2) *Any group homomorphism $f: C \to A$ is $\mathbb{Z}_p^\wedge$-linear.*

Proof. It is enough to show that the forgetful functor induces isomorphisms

$$(1) \qquad \mathrm{Ext}^1_{\mathbb{Z}_p^\wedge}(C, A) \cong \mathrm{Ext}^1_{\mathbb{Z}}(C, A) \quad \text{and} \quad \mathrm{Hom}_{\mathbb{Z}_p^\wedge}(C, A) \cong \mathrm{Hom}_{\mathbb{Z}}(C, A).$$

As above, write $A \cong A' \otimes \mathbb{Z}_p^\wedge$, $C \cong C' \otimes \mathbb{Z}_p^\wedge$, for finitely generated abelian groups A', C'. Since Ext and Hom commute with finite direct sums, it is enough to consider cyclic C and A, that is, each either $\mathbb{Z}_p^\wedge$ or $\mathbb{Z}/p^r$ for some r.

By the Change of Rings Theorem (see [HS, IV, Theorem 12.2]) we know

$$\mathrm{Ext}^n_{\mathbb{Z}_p^\wedge}(C, A) \cong \mathrm{Ext}^n_{\mathbb{Z}_p^\wedge}(C' \otimes \mathbb{Z}_p^\wedge, A) \xrightarrow{\cong} \mathrm{Ext}^n_{\mathbb{Z}}(C', A) \qquad (n \geq 0),$$

so (1) is satisfied when C is torsion and thus $C = C'$.

Now let $C = \mathbb{Z}_p^\wedge$.

(1) If $A = \mathbb{Z}_p^\wedge$ then $\mathrm{Ext}^1_{\mathbb{Z}}(C, A) = 0$ by [Ha, Proposition 2.1].

(2) If $A = \mathbb{Z}/p^r$, tensor $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ with $\mathbb{Z}_p^\wedge$ to get the exact sequence $0 = \mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}_p^\wedge) \to \mathbb{Z}_p^\wedge \to \mathbb{Q} \otimes \mathbb{Z}_p^\wedge \to (\mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z}_p^\wedge \to 0$. Applying $\mathrm{Ext}^1_{\mathbb{Z}}(-, A)$ to this, we see that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_p^\wedge, \mathbb{Z}/p^r) = 0$ since $\mathbb{Z}_p^\wedge \otimes \mathbb{Q}$ is a $\mathbb{Q}$-vector space and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}/p^r) = 0$.

We clearly also have $\mathrm{Ext}^1_{\mathbb{Z}_p^\wedge}(\mathbb{Z}_p^\wedge, A) = 0$ for any A.

Finally, $\mathrm{Hom}_{\mathbb{Z}_p^\wedge}(\mathbb{Z}_p^\wedge, A) \cong A$ for any $A \in \mathscr{F}$ while $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_p^\wedge, \mathbb{Z}/p^r) \cong \mathbb{Z}/p^r$, so

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_p^\wedge, \mathbb{Z}_p^\wedge) &= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_p^\wedge, \varprojlim \mathbb{Z}/p^r) \\ &\cong \varprojlim \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_p^\wedge, \mathbb{Z}/p^r) \cong \varprojlim \mathbb{Z}/p^r \cong \mathbb{Z}_p^\wedge.\end{aligned}$$

Thus $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_p^\wedge, A) \cong A$ for any $A \in \mathscr{F}$, too. The required isomorphism is readily verified. □

3. The mod-p approximation of BG

In order to prove Theorem 1.1 for more general $\mathbf{G}$, we start with the known case when $\mathbf{G}$ is p-toral, i.e., $\pi_0\mathbf{G}$ is a finite p-group and the identity component of $\mathbf{G}$ is a torus. Then we have

3.1. **Lemma.** *If* $\mathbf{P}$ *is a* p*-toral group and* $\mathbf{H}$ *is a connected compact Lie group, then for any* $f : \mathbf{BP} \to \mathbf{BH}$, $(\mathbf{map}(\mathbf{BP}, \mathbf{BH})_f)^\wedge_p \to \mathbf{map}(\mathbf{BP}, \mathbf{BH}^\wedge_p)_{i\circ f}$ *is a homotopy equivalence.*

This is contained in [JMO, Theorem 3.2]; we give an outline of the proof:

By [N1, Theorem 1.1], $f \simeq B\rho$ for some homomorphism $\rho : \mathbf{P} \to \mathbf{H}$; let $\mathbf{C}(\rho)$ denote its centralizer. The homomorphism $\mathbf{C}(\rho) \times \mathbf{P} \to \mathbf{H}$ passes to classifying spaces and has an adjoint $\mathbf{BC}(\rho) \to \mathbf{map}(\mathbf{BP}, \mathbf{BH})_{B\rho}$, or if we first complete,

$$\mathbf{BC}(\rho)^\wedge_p \to \mathbf{map}(\mathbf{BP}, \mathbf{BH}^\wedge_p)_{i\circ B\rho}.$$

The first map induces an $H_*(-;\mathbb{F}_p)$-isomorphism by [N1], and so a homotopy equivalence after completion (see [BK, I, 5.5]), while the second is shown in [JMO, loc. cit.] to be a homotopy equivalence. □

3.2. *Remark.* Since $\mathbf{C}(\rho)$ is compact and $\pi_0\mathbf{C}(\rho)$ is a finite p-group (cf. [JMO, Proposition A.4]), the homotopy groups $\pi_k(\mathbf{map}(\mathbf{BP}, \mathbf{BH}^\wedge_p)_{i\circ B\rho})$ are finitely generated $\mathbb{Z}^\wedge_p$-modules for $k \geq 2$ and a finite p-group for $k = 1$.

We now recall some results of Jackowski, McClure, and Oliver on the mod-p approximation of $\mathbf{BG}$:

For any compact Lie group $\mathbf{G}$, let $\mathscr{O}_p(\mathbf{G})$ denote the full subcategory of the orbit category $\mathscr{O}(\mathbf{G})$ whose objects are homogenous spaces $\mathbf{G}/\mathbf{P}$ where $\mathbf{P}$ is a p-toral group and whose morphisms are $\mathbf{G}$-maps. In [JMO, 1.3], Jackowski, McClure, and Oliver define a full subcategory $\mathscr{R}_p(\mathbf{G}) \subset \mathscr{O}_p(\mathbf{G})$ (containing $\mathbf{G}/\mathbf{P}$ only for certain "p-stubborn" $\mathbf{P}$'s), which has the property that

$$\underset{\mathscr{R}_p(G)}{\underrightarrow{\mathrm{holim}}}\ \mathbf{EG} \times_{\mathbf{G}} (\mathbf{G}/\mathbf{P}) \to \mathbf{BG}$$

is a $H_*(-;\mathbb{F}_p)$-isomorphism. Here $\underrightarrow{\mathrm{holim}}$ denotes the homotopy direct limit of [BK, XII, §2], and $\mathbf{EG} \times_{\mathbf{G}} (\mathbf{G}/\mathbf{P}) \cong \mathbf{EG}/\mathbf{P} \simeq \mathbf{BP}$.

Recall from [BK, I, §4] that for any space $\mathbf{X}$, the p-completion is obtained as the total space (i.e., homotopy inverse limit) of a certain cosimplicial space: $\mathbf{X}^\wedge_p \overset{\mathrm{def}}{=} \mathrm{Tot}(\underset{\sim}{\mathbb{F}}_p\mathbf{X})^\bullet$, where each space $(\underset{\sim}{\mathbb{F}}_p\mathbf{X})^k$ is homotopy equivalent to an $\mathbb{F}_p$-GEM, i.e., a product of $\mathbf{K}(\mathbb{F}_p, n_i)$'s. Therefore, for any space $\mathbf{Z}$, we have

$$\mathbf{map}(\mathbf{Z}, \mathbf{X}^\wedge_p) = \mathbf{map}(\mathbf{Z}, \mathrm{Tot}(\underset{\sim}{\mathbb{F}}_p\mathbf{X})^\bullet) \cong \mathrm{Tot}(\mathbf{map}(\mathbf{Z}, (\underset{\sim}{\mathbb{F}}_p\mathbf{X})^\bullet))$$

(see [BK, XI, 4.4, 7.6]), so the space of maps into a p-completion is the total space of a cosimplicial $\mathbb{F}_p$-GEM, too.

Now if $f : \mathbf{Y} \to \mathbf{Z}$ is an $H_*(-;\mathbb{F}_p)$-isomorphism, it induces a homotopy equivalence $\mathbf{map}(\mathbf{Z}, \mathbf{K}(\mathbb{F}_p, n)) \xrightarrow{f^*} \mathbf{map}(\mathbf{Y}, \mathbf{K}(\mathbb{F}_p, n))$, and so $\mathbf{map}(\mathbf{Z}, (\underset{\sim}{\mathbb{F}}_p\mathbf{X})^k) \xrightarrow{f^*} \mathbf{map}(\mathbf{Y}, (\underset{\sim}{\mathbb{F}}_p\mathbf{X})^k)$ is a homotopy equivalence for each $k \geq 0$. Therefore, by [BK, XI, 5.6] the same is true for the Tot's, and thus $\mathbf{map}(\mathbf{Z}, \mathbf{X}^\wedge_p) \xrightarrow{f^*} \mathbf{map}(\mathbf{Y}, \mathbf{X}^\wedge_p)$ is a homotopy equivalence. Since

$$\mathbf{map}(\underrightarrow{\mathrm{holim}}\, \mathbf{Y}_i, \mathbf{X}) = \underleftarrow{\mathrm{holim}}\ \mathbf{map}(\mathbf{Y}_i, \mathbf{X})$$

for any diagram $\{\mathbf{Y}_i\}$ (cf. [BK, XII, 4.1]), we have a natural homotopy equivalence

$$\mathbf{map}(\mathbf{BG}, \mathbf{BH}_p^\wedge) \to \underset{\mathscr{R}_p(G)}{\underleftarrow{\operatorname{holim}}}\, \mathbf{map}(\mathbf{EG}/\mathbf{P}, \mathbf{BH}_p^\wedge).$$

Thus, if we restrict a map $f : \mathbf{BG} \to \mathbf{BH}$ to $\mathbf{BP} \hookrightarrow \mathbf{BG}$ (for some $\mathbf{G}/\mathbf{P}$ in $\mathscr{R}_p(\mathbf{G})$), we see that

$$\mathbf{map}(\mathbf{BG}, \mathbf{BH}_p^\wedge)_{i\circ f} \to \underset{\mathscr{R}_p(G)}{\underleftarrow{\operatorname{holim}}}\, \mathbf{map}(\mathbf{EG}/\mathbf{P}, \mathbf{BH}_p^\wedge)_{i\circ f|_{\mathbf{BP}}} \tag{2}$$

is the inclusion of a component (the homotopy inverse limit need not be connected!).

4. Cosimplical spaces

Let $\mathrm{sk}\mathscr{R}_p(\mathbf{G})$ be a skeleton of $\mathscr{R}_p(\mathbf{G})$, that is, a full subcategory of $\mathscr{R}_p(\mathbf{G})$, containing a single representative of each isomorphism type of its objects. This is a finite category, since $\mathscr{R}_p(\mathbf{G})$ has finitely many isomorphism types of objects, and finitely many morphisms between them (cf. [JMO, Proposition 1.6]).

Given a map $f \colon \mathbf{BG} \to \mathbf{BH}$ as above, consider the finite diagram of spaces

$$\underline{\mathbf{X}} = \{\mathbf{X}_\mathbf{P}\}_{\mathbf{G}/\mathbf{P} \in \mathrm{sk}\mathscr{R}_p(\mathbf{G})}, \quad \text{where } \mathbf{X}_\mathbf{P} = \mathbf{map}(\mathbf{BP}, \mathbf{BH}_p^\wedge)_{i\circ f|_{\mathbf{BP}}}.$$

By cosimplicial replacement (see [BK, XI, §5]) we obtain a cosimplicial space $\mathbf{Y}^\bullet$, with

$$\mathbf{Y}^n = \prod_{\mathbf{G}/\mathbf{P}_{i_0} \to \cdots \to \mathbf{G}/\mathbf{P}_{i_n}} \mathbf{X}_{\mathbf{P}_{i_0}}$$

(where the product, over all possible sequences of n composable morphisms in $\mathrm{sk}\mathscr{R}_p(\mathbf{G})$, is finite), such that $\underleftarrow{\operatorname{holim}}_{\mathrm{sk}\mathscr{R}_p(\mathbf{G})}\{\mathbf{X}_\mathbf{P}\} \cong \operatorname{Tot}\mathbf{Y}^\bullet$.

Now if $\mathbf{Z}^\bullet$ is the cosimplicial replacement of the analogous infinite diagram of $\mathbf{X}_\mathbf{P}$'s for the full category $\mathscr{R}_p(\mathbf{G})$, then the equivalence of categories $\mathrm{sk}\mathscr{R}_p(\mathbf{G}) \hookrightarrow \mathscr{R}_p(\mathbf{G})$ (with noncanonical inverse $\mathscr{R}_p(\mathbf{G}) \to \mathrm{sk}\mathscr{R}_p(\mathbf{G})$) induces a homotopy equivalence $\operatorname{Tot}\mathbf{Y}^\bullet \xrightarrow{\simeq} \operatorname{Tot}\mathbf{Z}^\bullet$, so that up to homotopy the natural map of (2) above is the inclusion of one component in $\operatorname{Tot}\mathbf{Y}^\bullet$:

$$\mathbf{map}(\mathbf{BG}, \mathbf{BH}_p^\wedge)_{i\circ f} \hookrightarrow \underset{\mathscr{R}_p(G)}{\underleftarrow{\operatorname{holim}}}\{\mathbf{X}_\mathbf{P}\} \simeq \operatorname{Tot}\mathbf{Y}^\bullet.$$

We choose a basepoint $y_0 \in \operatorname{Tot}\mathbf{Y}^\bullet$ corresponding to the map $i \circ f$.

4.1. **Lemma.** *For any $f \colon \mathbf{BG} \to \mathbf{BH}$, the space $\mathbf{map}(\mathbf{BG}, \mathbf{BH}_p^\wedge)_{i\circ f}$ is p-complete and $\mathbb{F}_q$-acyclic for $q \neq p$.*

Proof. Consider the Bousfield-Kan spectral sequence for $\mathbf{Y}^\bullet$ as above (more precisely, for the component of y_0 in $\operatorname{Tot}\mathbf{Y}^\bullet$ (cf. [B2, §2])) with $E_2^{s,t} \cong \pi^s\pi_t\mathbf{Y}^\bullet$.

For $t \geq 2$, the construction of $\mathbf{Y}^\bullet$ and Remark 3.2 imply that $\pi_t\mathbf{Y}^s \in \mathscr{F}$ and all the cosimplicial morphisms of $\pi_t\mathbf{Y}^\bullet$ are $\mathbb{Z}_p^\wedge$-linear by Lemma 2.3(b); hence $E_2^{s,t} \in \mathscr{F}$. For $t = 1$, $E_2^{0,1}$ is a subgroup of $\pi_1\mathbf{Y}^0 \cong \prod \pi_1\mathbf{X}_\mathbf{P}$, and so is itself a finite p-group by Remark 3.2.

Moreover, if $t \geq 2$, the differentials $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$ are homomorphisms, and thus $\mathbb{Z}_p^\wedge$-linear, for $t > s \geq 0$. Therefore, $E_r^{s,t} \in \mathscr{F}$ for $r \leq \infty$, if $t > s \geq 0$ or $t = s \geq r$. For $t = 1$ we have $E_r^{0,1} \subseteq E_{r-1}^{0,1} \subseteq E_2^{0,1}$ (cf. [B2, §2.4]), so $E_r^{0,1}$ is a finite p-group.

Since $E_2^{s,t} \cong \underleftarrow{\lim}^s_{\mathscr{R}_p(G)} \pi_t \underline{\mathbf{X}}$ by [BK, XI, 7.1], Lemma 4.2 below, applied to the functors

$$\pi_t(\mathbf{EG} \times_{\mathbf{G}} -) : \mathscr{R}_p(\mathbf{G}) \to \mathbb{Z}_p^{\wedge}\text{-Mod},$$

shows that there is an N such that $E_2^{s,t} = 0$ for $s > N$ and $t \geq 2$.

This in turn implies the complete convergence of the spectral sequence (see [B2, §4.5]): thus, for each $t \geq 1$ there is a finite tower of epimorphisms

$$\pi_t(\operatorname{Tot} \mathbf{Y}^\bullet, y_0) \cong Q_N\pi_t \twoheadrightarrow \cdots Q_s\pi_t \twoheadrightarrow Q_{s-1}\pi_t \twoheadrightarrow \cdots Q_0\pi_t \twoheadrightarrow Q_{-1}\pi_t = 1,$$

where $Q_s\pi_t = \operatorname{im}\{\pi_t(\operatorname{Tot} \mathbf{Y}^\bullet, y_0) \to \pi_t(\operatorname{Tot}_s \mathbf{Y}^\bullet, y_0)\}$ (cf. [BK, IX, §5.3]), and for each $s \geq 0$ there is a short exact sequence

$$1 \to E_\infty^{s,s+t} \to Q_s\pi_t \to Q_{s-1}\pi_t \to 1 .$$

Now for $t \geq 2$ we have $E_\infty^{s,s+t} \in \mathscr{F}$. Therefore, Lemma 2.3(a) implies (by induction on s) that $Q_s\pi_t \in \mathscr{F}$ for all s, and so $\pi_t(\operatorname{Tot} \mathbf{Y}^\bullet, y_0)$ is in $\mathscr{F}$, too.

For $t = 1$ we obtain a finite normal series

$$0 = G_0 \lhd G_1 \lhd \cdots \lhd G_N \lhd G_{N+1} = \pi_1(\operatorname{Tot} \mathbf{Y}^\bullet, y_0),$$

where $G_i/G_{i-1} = E_\infty^{N-i+1, N-i+2}$ is in $\mathscr{F}$ for $1 \leq i \leq N$ and $G_{N+1}/G_N = E_\infty^{0,1}$ is a finite p-group. Thus Corollary 2.2 applies, and the component of y_0 in $\operatorname{Tot} \mathbf{Y}^\bullet$ is p-complete, and $\mathbb{F}_q$-acyclic for $q \neq p$. □

The following lemma appeared in an earlier version of [JMO].

4.2. **Lemma.** *If* $\mathbf{G}$ *is any compact Lie group and* p *a prime, there is an* N *such that for any contravariant functor*

$$F : \mathscr{R}_p(\mathbf{G}) \to \mathbb{Z}_p^{\wedge}\text{-Mod}$$

we have $\underleftarrow{\lim}^s_{\mathscr{R}_p(G)} F = 0$ *for* $s > N$.

Proof. The homotopy direct limit $E\mathscr{R}_p(\mathbf{G}) = \underrightarrow{\operatorname{holim}}_{\mathscr{R}_p(G)} \mathbf{G}/\mathbf{P}$ is a $\mathbf{G}$-space, and $\underleftarrow{\operatorname{holim}}^s_{\mathscr{R}_p(G)} F \cong H^s_{\mathbf{G}}(E\mathscr{R}_p(\mathbf{G}); F)$ for all $s \geq 0$ by [JMO, Theorem 1.7]. Here $H^*_{\mathbf{G}}(-; F)$ denotes equivariant cohomology with the functor F as coefficient system (see [I, 2.2]).

By [JMO, Proposition 1.2, Theorem 2.14], there exists a finite dimensional $\mathbf{G}$-complex $\mathbf{X}$ with finitely many orbit types and a $\mathbf{G}$-$\mathbb{F}_p$-isomorphism $f : \mathbf{X} \to E\mathscr{R}_p(\mathbf{G})$; that is, a $\mathbf{G}$-equivariant map f such that $f^{\mathbf{H}} : \mathbf{X}^{\mathbf{H}} \to (E\mathscr{R}_p(\mathbf{G}))^{\mathbf{H}}$ is an $H_*(-; \mathbb{F}_p)$-isomorphism on $\mathbf{H}$-fixed point sets for any $\mathbf{H} \subseteq \mathbf{G}$.

Since each $H_k((E\mathscr{R}_p(\mathbf{G}))^{\mathbf{H}}; \mathbb{Z})$ is finitely generated (see [JMO, Proposition 1.1]), $f^{\mathbf{H}}$ is in fact an isomorphism in $\mathbb{Z}_p^{\wedge}$-homology for each $\mathbf{H}$, and therefore f is a $\mathbf{G}$-$\mathbb{Z}_p^{\wedge}$-homology isomorphism; by [JMO, A.13] this implies that $H^*_{\mathbf{G}}(E\mathscr{R}_p(\mathbf{G}); F) \cong H^*_{\mathbf{G}}(\mathbf{X}; F)$ for any $\mathbb{Z}_p^{\wedge}$-module valued coefficient system.

Now one can filter $\mathbf{X}$ by $\mathbf{G}$-skeleta $\mathbf{X}_0 \subset \mathbf{X}_1 \subset \cdots \subset \mathbf{X}_i \subset \cdots \subset \mathbf{X}_k = \mathbf{X}$ so that $\mathbf{X}_i/\mathbf{X}_{i-1}$ contains a single orbit type $\mathbf{G}/\mathbf{P}_i$. If N is the dimension of $\mathbf{X}$, by induction on the $\mathbf{X}_i$ one then shows (as in the proof of [JMO, A.13]) that $H^s_{\mathbf{G}}(\mathbf{X}; F) = 0$ for $s > N$. □

5. The homotopy equivalence

For a connected compact Lie group $\mathbf{H}$, consider the arithmetic square

$$\begin{array}{ccc} \mathbf{BH} & \xrightarrow{i} & \mathbf{BH}^\wedge \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j'} \\ \mathbf{BH}_{\mathbb{Q}} & \xrightarrow{i_{\mathbb{Q}}} & (\mathbf{BH}^\wedge)_{\mathbb{Q}} \end{array} \tag{3}$$

(see [BK,VI, 8.1]), where $\mathbf{X}^\wedge = \prod \mathbf{X}_p^\wedge$ is the product over all primes p of the p-completions and $\mathbf{X}_{\mathbb{Q}}$ is the $\mathbb{Q}$-localization.

Without loss of generality, $i_{\mathbb{Q}}$ is a fibration and (3) is a pullback diagram, so both horizontal maps have the same fiber $\mathbf{F}$. Since $\mathbf{H}$ is compact and $\mathbf{BH}_{\mathbb{Q}}$, $(\mathbf{BH}^\wedge)_{\mathbb{Q}}$ are rational H-spaces, they are even-dimensional rational GEMs (that is, products of even-dimensional rational Eilenberg-Mac Lane spaces) and $\mathbf{F}$ is an odd-dimensional rational GEM.

For any map $f : \mathbf{BG} \to \mathbf{BH}$ (where $\mathbf{G}$ is a compact Lie group), (3) induces another pullback diagram

$$\begin{array}{ccc} \mathbf{map}(\mathbf{BG}, \mathbf{BH})_f & \longrightarrow & \mathbf{map}(\mathbf{BG}, \mathbf{BH}^\wedge)_{i\circ f} \\ \downarrow & & \downarrow \\ \mathbf{map}(\mathbf{BG}, \mathbf{BH}_{\mathbb{Q}})_{j\circ f} & \longrightarrow & \mathbf{map}(\mathbf{BG}, (\mathbf{BH}^\wedge)_{\mathbb{Q}})_{j'\circ i\circ f} \end{array} \tag{4}$$

As for any compact Lie group, $H^{2k-1}(\mathbf{BG}; \mathbb{Q}) = 0$ for all $k \geq 1$ (cf. [Bo, Theorem 19.1]). Since $\mathbf{F} \simeq \prod \mathbf{K}(\mathbb{Q}, 2r_i - 1)$ is an odd-dimensional rational GEM, $\mathbf{map}(\mathbf{BG}, \mathbf{F})$ is an odd-dimensional rational GEM, too, by a direct calculation of its homotopy groups. In particular, $\mathbf{map}(\mathbf{BG}, \mathbf{F})$ is connected, and $\mathbb{F}_p$-acyclic for any prime p.

Thus $\mathbf{map}(\mathbf{BG}, \mathbf{F})$ is the fiber of $\mathbf{map}(\mathbf{BG}, \mathbf{BH}_{\mathbb{Q}})_c \to \mathbf{map}(\mathbf{BG}, (\mathbf{BH}^\wedge)_{\mathbb{Q}})_c$, where c is the constant map. Because $\mathbf{BH}_{\mathbb{Q}}$ is an H-space and $i_{\mathbb{Q}}$ is an H-map, this is in fact the fiber for *all* components and thus for the two horizontal maps in (4).

Therefore, applying the q-completion functor to the top fibration sequence in the diagram

$$\mathbf{map}(\mathbf{BG}, \mathbf{F}) \to \mathbf{map}(\mathbf{BG}, \mathbf{BH})_f \to \mathbf{map}(\mathbf{BG}, \mathbf{BH}^\wedge)_{i\circ f},$$

we get another fibration (by [BK, II, 5.2]):

$$\mathbf{map}(\mathbf{BG}, \mathbf{F})_q^\wedge \to (\mathbf{map}(\mathbf{BG}, \mathbf{BH})_f)_q^\wedge \xrightarrow{g} (\mathbf{map}(\mathbf{BG}, \mathbf{BH}^\wedge)_{i\circ f})_q^\wedge,$$

with g a homotopy equivalence (since the fiber is contractible).

Finally, Lemma 4.1 implies that $(\mathbf{map}(\mathbf{BG}, \mathbf{BH}_p^\wedge)_{i\circ f})_q^\wedge$ is homotopy equivalent to $(\mathbf{map}(\mathbf{BG}, \mathbf{BH}_p^\wedge)_{i\circ f})$ for $q = p$, and is contractible for $q \neq p$, so we get the desired homotopy equivalence

$$(\mathbf{map}(\mathbf{BG}, \mathbf{BH})_f)_p^\wedge \xrightarrow{\simeq} \mathbf{map}(\mathbf{BG}, \mathbf{BH}_p^\wedge)_{i\circ f}.$$

This completes the proof of Theorem 1.1. □

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