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Abelian Π-Algebras and their Projective Dimension

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To Mark Mahowald on his 60th birthday

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ABSTRACT. Derived functors of Π-algebras – which are algebraic models for the homotopy groups of a space, together with the action of the homotopy operations on them – serve as the E^2-terms of a number of spectral sequences; thus the homological properties of Π-algebras are of some interest. As a first approximation we here study the properties of a simpler subcategory – that of *abelian* Π-algebras: we show that any abelian Π-algebra (simply connected, of finite type) is either free, or has infinite projective dimension.

1. Introduction

A Π-*algebra* is an algebraic model for the homotopy groups $\pi_\star X$ of a pointed space X, together with the action of the primary homotopy operations on them, in the same sense that algebras over the Steenrod algebra are models for the cohomology of a space.

Derived functors of Π-algebras serve as the E^2-term of a number of spectral sequences; however, Π-algebras are difficult objects to study, not only because the algebra of unstable operations (i.e., the unstable homotopy groups of spheres) are not fully known, but also because the category $\mathcal{P}$ of Π-algebras is not abelian, so the familiar tools of homological algebra are not available to work with.

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The stable analogue, which is the category $\boldsymbol{\pi}$-*Mod* of (graded) modules over the graded stable homotopy ring $\boldsymbol{\pi}= \pi_\star^S S^0$, is more accessible for this reason (even though $\boldsymbol{\pi}$ is not fully known, either). The homological algebra of $\boldsymbol{\pi}$-modules were studied by Tsau Young Lin (see [**L1, L2**]); his main result is that any $\boldsymbol{\pi}$-module has projective dimension 0, 1, or ∞.

In order to better understand the homological properties of Π-algebras, we start here with a simpler category - namely, that of *abelian* Π-algebras. These are intermediate between ordinary Π-algebras and $\boldsymbol{\pi}$-modules in their accessibility, on the one hand, and their closeness to the homotopy category of topological spaces, on the other hand. In the category of abelian Π-algebras we prove an unstable analogue of Lin's theorems:

Theorem 4.4 *Any abelian* Π*-algebra* X *has projective dimension* 0, 1, *or* ∞. *If* X *is simply-connected of finite type, it is either free, or has infinite projective dimension.*

In section 2 we recall the definition of Π-algebras; in section 3 we discuss abelian Π-algebras; and we prove the Theorem in section 4.

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2. Π-algebras

Recall (e.g., from [**B1**, §3]) that a Π-*algebra* is a graded group, together with an action of the primary homotopy operations, satisfying all the universal relations on such operations. The motivating example is $\pi_\star X$, where X is a pointed connected space.

Remark 2.1. A Π-algebra X may be described explicitly as a graded group $\{X_i\}_{i=1}^{\infty}$, (with X_i abelian for $i \geq 2$), equipped with a *composition operation* $\alpha^\# : X_r \to X_k$ for each $\alpha \in \pi_k S^r$ $(k > r > 1)$, and a *Whitehead product* $[\ ,\ ] : X_i \times X_j \to X_{i+j-1}$ for each pair $i, j \geq 1$. The Whitehead products include (cf. [**W**, X,§3]):

- $[\alpha, \xi] = \alpha^\xi - \alpha \in X_r$, where α^ξ is the result of the "π_1-action" of $\xi \in X_1$ on $\alpha \in X_r$ $(r > 1)$;
- the commutators $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1} \in X_1$, for $\alpha, \beta \in X_1$.

If we restrict attention to the subcategory $\mathcal{P}_1 \subset \mathcal{P}$ of *simply-connected* Π-*algebras* – i.e., those with $X_1 = 0$ – the universal identities on these primary operations can be described explicitly, as follows:

(a) $(\alpha \circ \beta)^\# = \beta^\# \circ \alpha^\#$ and $(\alpha + \beta)^\# = \alpha^\# + \beta^\#$ (cf. [**W**, X,8.1]);

(b) The Whitehead products make X into a graded Lie ring (with a shift in indices) – that is, if $x, x' \in X_{p+1}$, $y \in X_{q+1}$, and $z \in X_{r+1}$, then (cf. [**W**, X,§7]):

i. $[x + x', y] = [x, y] + [x', y]$.

ii. $[y,x] = (-1)^{pq}[x,y]$.
iii. $(-1)^{pr}[[x,y],z] + (-1)^{pq}[[y,z],x] + (-1)^{qr}[[z,x],y] = 0.$

(c) Hilton's formula (cf. [**H1**, (6.1)]):

$$\alpha^{\#}(x_1 + x_2) \ = \ \alpha^{\#}x_1 + \alpha^{\#}x_2 \ + \sum_{j=0}^{\infty} (h_j(\alpha))^{\#} w_j(x_1, x_2) \ ,$$

where $w_j(x_1, x_2)$ is the $(j+3)$-rd basic iterated Whitehead product, for some choice of ordering, and $h_j(\alpha)$ is the corresponding Hilton-Hopf invariant (cf. [**W**, XI,8.5]);

(d) The Barcus-Barratt formula (cf. [**BB**, 7.4] or [**Ba**, II, §3]):

$$[\alpha^{\#}x_1, x_2] \ = \ \sum_{n=0}^{\infty} (E^{|x_2|-1}h_{j_n}(\alpha))^{\#}[x_1^{n+1}, x_2] \ ,$$

where $[x_1^n, x_2] = [x_1, [x_1, [\dots, [x_1, x_2]\dots]$ is the iterated Whitehead product with n x_1's, and h_{j_n} is the Hilton-Hopf invariant corresponding to the basic product $[x_1^{n-1}, x_2]$.

(For a comparison of the various definitions of the Whitehead products and Hopf invariants, and the choices of signs, see [**BoS**, §7].)

A non-simply connected Π-algebra has, in addition, a group X_1 (not necessarily abelian), such that for each $n > 1$, X_n is an X_1-module under a "π_1-action" commuting with compositions (cf. [**BB**, p. 68]), and satisfying the appropriate Jacobi identity with respect to the Whitehead products (cf. [**H2**]).

DEFINITION 2.2. The *free* Π-algebras are those which are isomorphic to $\pi_\star W$, for some (possibly infinite) wedge of spheres W. More precisely, if $T = \{T_j\}_{j=1}^{\infty}$ is a graded set, and $W = \bigvee_{j=1}^{\infty} \bigvee_{x \in T_j} S_x^j$, where each S_x^j is a j-sphere, then we say that $\pi_\star W$ is the *free* Π-*algebra generated by* T, (where T is thought of as a subset of $\pi_\star W$ in the obvious way).

Note that even when T is not of finite type, $\pi_s W$ is a direct sum of cyclic groups for $s \geq 2$ (cf. [**H1**]).

DEFINITION 2.3. For any Π-algebra X, let $I(X) \subseteq X$ denote the sub-Π-algebra generated by all non-trivial primary homotopy operation (in $\mathcal{P}_1$: compositions and Whitehead products). The graded abelian group $Q(X) = X/I(X)$ is called the *module of indecomposables* of X (cf. [**B1**, §2]).

3. abelian Π-algebras

The *abelian* Π-algebras may be thought of as an intermediate stage between arbitrary (unstable) Π-algebras, and modules over the stable homotopy ring:

DEFINITION 3.1. A Π-algebra $X \in \mathcal{P}$ is said to be *abelian* if it is an abelian group object – that is, if $Hom_{\mathcal{P}}(Y, X)$ has a natural abelian group structure for any $Y \in \mathcal{P}$. This is equivalent to requiring that all Whitehead products vanish in X (cf. [**BS**, §5.1.2]): the Whitehead products must vanish in order for the natural addition operation on *Hom*-sets to be well-defined; and this suffices by virtue of Hilton's formula 2.1(c) above. The full subcategory of abelian Π-algebras is denoted $\mathcal{P}_{ab} \subset \mathcal{P}$; it is an abelian category.

Remark 3.2. Note that since the π_1-action and the commutator are Whitehead products (§2.1), any abelian Π-algebra X splits as a direct sum of $X_{\geq 2}$ (i.e., the graded group starting in degree 2), and $K(X_1, 1)$ (i.e., a Π-algebra concentrated in degree 1), with X_1 an abelian group. Therefore, we may restrict attention to simply-connected abelian Π-algebras without losing anything of interest.

DEFINITION 3.3. For each $X \in \mathcal{P}$, let $W(X) \subset X$ be the sub-Π-algebra generated (under composition and sums) by all non-trivial generalized Whitehead products. This is an *ideal of* Π-*algebras* (cf. [**BS**, §5.1.1]) – that is, it is closed under Whitehead products with arbitrary elements of X – so that the graded group $X/W(X)$ inherits a Π-algebra structure from X.

This Π-algebra will be denoted $Ab(X)$. In fact, $Ab : \mathcal{P} \to \mathcal{P}_{ab}$ is the *abelianization* functor, so there is a natural transformation $\theta : Id \to Ab$ with the appropriate universal property.

Remark 3.4. Furthermore, there is a *stabilization* functor $S : \mathcal{P} \to \boldsymbol{\pi}\text{-}Mod$, which is the extension (cf. [**BS**, §2]) of the functor on free Π-algebras taking $\pi_\star W$ to $\pi_\star \Omega^\infty \Sigma^\infty W$, for $W = \bigvee_{i \in I} S^{n_i}$ as in §2.2, so that $S(\pi_\star W) \cong \bigoplus_{i \in I} \Sigma^{n_i} \boldsymbol{\pi}$. The functor S factors through Ab, and the indecomposables functor Q of §2.3 factors through S (and thus through Ab), when applied to free Π-algebras.

For any H-space X, $\pi_\star X$ is an abelian Π-algebra; but there are also non-H-spaces with vanishing Whitehead products, whose homotopy groups thus constitute abelian Π-algebras: an example is $\pi_\star \mathbb{C}P^3$ (cf. [**BJS**]).

There is an interesting subcategory $\mathcal{V} \subset \mathcal{P}_{ab}$, modeled on the Π-algebras of *loop spaces* $\pi_\star \Omega X$: namely, those for which the only non-trivial operations are compositions with suspension elements.

(We observe that the remaining primary structure on $\pi_\star X \cong \pi_{\star-1} \Omega X$ – that is, the Samelson/Whitehead products, and non-suspension compositions – may be expressed in terms of *secondary* structure on $\pi_\star \Omega X$.)

DEFINITION 3.5. The left derived functors $L_n Ab$ of the abelianization, evaluated on a Π-algebra X, perhaps deserve to be called the *homology* of X, in the spirit of [**Q**, §2]. In [**DK**], Dwyer and Kan have given an alternative definition of the homology of X with coefficients in an arbitrary module M over the "universal enveloping algebra" $E(X)$; when X is 1-connected and $M = \mathbb{Z}$, this homology is just the derived functors of the indecomposables functor Q of §2.3, so that in light of §3.4 the two versions of homology are related by a Grothendieck spectral sequence (cf. [**BS**]).

The free objects in $\mathcal{P}_{ab}$ are just the abelianizations of the free Π-algebras of §2.2. If we denote $Ab(\pi_\star S^n)$ by $\mathfrak{S}^n$, then $X \in \mathcal{P}_{ab}$ is free $\Leftrightarrow$ $X \cong \bigoplus_i \mathfrak{S}^{n_i}$.

We shall sometimes write $\bigoplus_i \mathfrak{S}^{n_i}\langle a_i\rangle$, where $a_i \in \mathfrak{S}^{n_i}_{n_i}$, when we want to name the generators.

Since few non-trivial Π-algebras – even free ones – are known explicitly in all dimensions, we cannot expect to compute their abelianizations explicitly, either (although of course $\mathfrak{S}^r_{r+k} \cong \pi^S_k S^0$ for $k \leq r-1$). However, the formulas of §2.1 simplify considerably in the case of spheres, since all n-fold iterated Whitehead products vanish in $\pi_\star S^r$ for $n \geq 4$ (or $n \geq 3$, if r is odd) – see [**W**, ch. XI, Thm 8.8].

For example, for $\alpha \in \pi_p S^r$, $\beta \in \pi_q S^r$, the Barcus-Barratt formula (2.1d.) reads:

$$[\alpha,\beta] = (-1)^{q(p+1)+1}[\iota_r,\iota_r]\circ E^{r-1}\alpha\circ E^{p-1}\beta + (-1)^q[[\iota_r,\iota_r],\iota_r]\circ E^{2r-2}\alpha\circ E^{p-1}h_0\beta$$

(where ι_r generates $\pi_r S^r$).

Since for r odd, $[\iota_r,\iota_r]$ has order 2, and $[[\iota_r,\iota_r],\iota_r] = 0$, we find that $2([\alpha,\beta]\circ\gamma) = 0$ for any $\gamma \in \pi_\star S^{p+q-1}$ (by [**W**, XI, Thm 8.9]), so that $W(\pi_\star S^{2k+1})$ is all of order 2 (§3.3), and thus

Lemma 3.6. *$\mathfrak{S}^{2k+1}$ is the cokernel of $[\iota_{2k+1},\iota_{2k+1}]_\# : \pi_\star S^{4k+1} \to \pi_\star S^{2k+1}$, and $2\cdot\pi_\star S^{2k+1}$ maps monomorphically to $\mathfrak{S}^{2k+1}$, for all $k \geq 0$.*

Of course, when $r = 1, 3$ or 7, S^r is an H-space, so that $\mathfrak{S}^r \cong \pi_\star S^r$ in those three cases.

One can say less about the abelianization of $\pi_\star S^{2k}$, in general, since $[\iota_{2k},\iota_{2k}]$ has infinite order, and (for $k \geq 2$) also $[[\iota_{2k},\iota_{2k}],\iota_{2k}] \neq 0$. However, we have the following

Fact 3.7. *$\mathfrak{S}^2_i \cong \pi_i S^2 \otimes \mathbb{Z}/2$ for $i > 2$, and $\mathfrak{S}^2_2 \cong \mathbb{Z}$.*

Proof. Since $[\iota_2,\iota_2] = 2\eta_2 \in \pi_3 S^2$ and $(\eta_2)_\# : \pi_k S^3 \xrightarrow{\cong} \pi_k S^2$ is an isomorphism for $k \geq 3$, we find that $[\alpha,\beta] = 0$ for $\alpha \in \pi_p S^2$, $\beta \in \pi_q S^2$, $(p, q \geq 3)$.

Thus $W(\pi_\star S^2)$ is generated as a graded group by elements $[\iota_2,\iota_2]\circ\gamma$, for $\gamma \in \pi_k S^3$, and these are precisely $(2\eta_2)\circ\gamma = 2(\eta_2\circ\gamma)$ by Hilton's formula (2.1c.). Therefore, $W(\pi_\star S^2) = 2\pi_{>2}S^2$, so $\mathfrak{S}^2_i \cong \pi_i S^2/2\pi_i S^2 = \pi_i S^2 \otimes \mathbb{Z}/2$ for $i > 2$. □

4. projective dimension of abelian Π-algebras

In this section we prove our main result – namely, that essentially any non-free abelian Π-algebra has infinite projective dimension. For this we need the following

Lemma 4.1. *Any projective object in $\mathcal{P}_{ab}$ is free.*

PROOF. If $P \in \mathcal{P}_{ab}$ is projective, it is a summand in a free abelian Π-algebra F, with $P \xrightarrow{i} F \xrightarrow{f} P$ such that $f \circ i = id_P$. Thus $Q(P)$ (§2.3) is a summand of the graded free abelian group $Q(F)$, so it is also free abelian, and one can choose compatible bases $B \subseteq C$ for $Q(P) \subseteq Q(F)$.

Lifting to $P \hookrightarrow F$, we get an isomorphism $F \cong \bigoplus_{j \in C} \mathfrak{S}^j$, with P already a direct summand $P \overset{i'}{\hookrightarrow} F' = \bigoplus_{j \in B} \mathfrak{S}^j$. Since $Q(i') : Q(P) \to Q(F')$ is an isomorphism, so is i'. □

Remark 4.2. The same fact clearly holds both for $\mathcal{P}$ (or the category $\mathcal{P}_1$ of simply-connected Π-algebras), and for the category $\boldsymbol{\pi}$-*Mod* of modules over the stable homotopy ring $\boldsymbol{\pi}$ (cf. [**L2**, Cor 5.6]) – in fact, for modules over any graded ring R_* with $R_0 = \mathbb{Z}$. It is also true for the p-local versions of each of these categories.

PROPOSITION 4.3. *For any prime p and* $k > 1$ *there is a non-zero element* $\alpha \in (\mathfrak{S}^k_{k+n})_{(p)}$, *for some* $n > 0$, *such that* $p \cdot \alpha = 0$ *and* $\alpha^{\#}\beta = 0$ *for all* $\beta \in (\mathfrak{S}^t_k)_{(p)}$ *with* $t < k$.

PROOF. (a) If p is odd, Serre's isomorphism

$$\begin{array}{ccccc} \pi_{i-1}S^{2m-1}_{(p)} & \oplus & \pi_i S^{4m-1}_{(p)} & \cong & \pi_i S^{2m}_{(p)} \\ (a & , & b) & \mapsto & E(a) + [\iota_m, \iota_m]_{\#}(b) \end{array}$$

(cf. [**S**, IV, §5]) implies that $E : (\mathfrak{S}^{2m-1}_{\star-1})_{(p)} \xrightarrow{\cong} (\mathfrak{S}^{2m}_{\star})_{(p)}$ is an isomorphism. Therefore, $\mathfrak{S}^r_{(p)}$ has exponent $\leq p^m$ at the prime p, where $m = [(r-1)/2]$ (=integral part), by [**CMN**, Cor. 1.3] & [**N**, Cor 4.3]. In fact, the exponent is precisely p^m, since the desuspensions of the elements of $Im(J)$ to $\alpha^{(m-1)}_{p^{m-1}} \in \pi_{p^{m-1}q+2m}S^{2m+1}_{(p)}$ (cf. [**G**, Prop. 13]) yield elements of order p^m in $\mathfrak{S}^r_{(p)}$, too.

Thus we may choose $\alpha = p^{r-1}\alpha^{(r-1)}_{p^{r-1}} \in (\mathfrak{S}^k_{p^{r-1}q+2r})_{(p)}$, where $r = [(k-1)/2]$, and since $t < k$ we may assume $t \leq k - 2(p-1) + 1$, so $(\mathfrak{S}^t_k)_{(p)}$ has exponent $\leq p^{r-1}$, and is therefore annihilated by $\alpha^{\#}$.

(b) If $p = 2$, the situation is analogous: first note that Hilton's formula for the Hopf map on a composition element (cf. [**H3**] or [**Ba**, III, 6.3]):

$$H(\alpha^{\#}\beta) = H(\alpha)^{\#}(E^{r-1}\beta)^{\#}E^{n-1}\beta + \alpha^{\#}H(\beta) \qquad \alpha \in \pi_\star S^r,\ \beta \in \pi_r S^n$$

implies that for $\alpha \in \pi_\star S^{4n-1}_{(2)}$,

$$H(\alpha^{\#}[\iota_{2n}, \iota_{2n}]) = \alpha^{\#}H([\iota_{2n}, \iota_{2n}]) = 2\alpha + H(\alpha)^{\#}[\iota_{4n-1}, \iota_{4n-1}] \tag{4.1}$$

by [**W**, XI,Thm 8.9], since $H([\iota_{2n}, \iota_{2n}]) = 2\iota_{4n-1}$.

Thus, given any $\gamma \in \pi_q S^{2n}_{(2)}$, let $\delta = H(\gamma)^{\#}[\iota_{2n}, \iota_{2n}]$; then (4.1) implies that $H(\delta) = 2H(\gamma)$, since $H(H(\gamma)) = N \cdot j_4(\gamma)$ for some N by [**Ba**, III, 5.2] (where j_n is the n-th James-Hopf invariant, so $H = j_2$), and $j_4(\gamma)^{\#}[\iota_{4n-1}, \iota_{4n-1}] = 0$ by [**Ba**, III, 6.2].

Therefore, $2\gamma - \delta \in E(\pi_{q-1}S^{2n-1}_{(2)})$, and so any $\gamma \in (\mathfrak{S}^{2n}_q)_{(2)}$ has $2\gamma \in E((\mathfrak{S}^{2n-1}_{q-1})_{(2)})$, which implies

$$exp_2\mathfrak{S}^{2n}_{(2)} \leq 2 \cdot exp_2\mathfrak{S}^{2n-1}_{(2)} . \tag{4.2}$$

Selick has shown ([**Se**]) that $\pi_{>4m+k+1}S^{4m+k+1}_{(2)}$ $(k = 0, 2)$ is annihilated by 2^{3m+k}, so by (4.2) $(\mathfrak{S}^{4m+2k}_{>4m+2k})_{(2)}$ $(k = 0, 1)$ is annihilated by 2^{3m+k} – i.e., $(\mathfrak{S}^n_{>n})_{(2)}$ is annihilated by $2^{[3n/4]}$.

On the other hand, Mahowald (cf. [**M**, Thm. 8.4]) shows that for $n = 8a+b \geq 7$, there is an $\alpha_n \in \pi_\star S^n$ which suspends to the $Im(J)$ element of order 2^{k_n}, where $k_n = 4a - 1$ for $b = 0$, $k_n = 4a$ for $1 \leq b \leq 4$, and $k_n = 4a + j$ for $4 < b = 4 + j \leq 7$ – so $k_n \geq n/2 - 2$. (Note that if n is even, $\bar{\alpha}_n$ may have order 2^{k_n+1} in $\mathfrak{S}^n_{(2)}$).

Setting $\alpha'_n = 2^{k_n-1}\bar{\alpha}_n \in \mathfrak{S}^n_{(2)}$, we see that α'_n annihilates $(\mathfrak{S}^r_{>n})_{(2)}$ for $r < 2n/3 - 3$. But if $2n/3 - 3 \leq r < n$, any $\beta \in \pi_n S^r_{(2)}$ desuspends to $\pi_\star S^{[(n+11)/3]}_{(2)}$; and since $3[(n + 11)/3]/4 \leq n/2 - 2$ for $n \geq 17$, we have $(\alpha'_n)^\# \beta = 0$. The cases $n < 17$ are readily checked. □

As in [**B3**, §5], the analogue for $\mathcal{P}_{ab}$ of T.Y. Lin's result on the projective dimension of $\boldsymbol{\pi}$-modules (cf. [**L1**, Thm 1] and [**L2**, Thm 4.4]) follows from this Proposition. Define an abelian Π-algebra X to be *locally cyclic* if each X_s is a direct sum of cyclic groups. (In particular, this will hold if X is either free, or of finite type).

THEOREM 4.4. *Any* $X \in \mathcal{P}_{ab}$ *has projective dimension* 0, 1, *or* ∞. *If* X *is simply-connected and locally cyclic, it has projective dimension* 0 *or* ∞.

PROOF. If X is an abelian Π-algebra which is not free, this will also be true after localizing at some prime p; so consider the p-local version $(\mathcal{P}_{ab})_{(p)}$ of $\mathcal{P}_{ab}$, in which all groups have been localized at a prime p. We can also assume without loss of generality that X is simply-connected, in the light of Remark 3.2.

By induction on the homological dimension in constructing a projective resolution for X, it suffices by Lemma 4.1 to show that if $X \in (\mathcal{P}_{ab})_{(p)}$ is not free, and $f : F \twoheadrightarrow X$ is any epimorphism from a free abelian Π-algebra F, then $Ker(f)$ is not free. (We shall indicate what fails in homological dimension 1 when X is not locally cyclic).

Let $s \geq 2$ be the first degree in which X is not free, and write

$$F = \bigoplus_{i\in I} \mathfrak{S}^{t_i}\langle a_i\rangle \oplus \bigoplus_{j\in J} \mathfrak{S}^{s}\langle b_j\rangle \oplus \text{(higher degrees)},$$

where $t_i < s$.

We may assume the abelian Π-algebra $K = Ker(f)$ is $(s-1)$-connected (otherwise choose a smaller F). Now we must have

$$K_s \cong \bigoplus_{\gamma\in\Gamma} \mathbb{Z}_{(p)}\langle c_\gamma\rangle \oplus \bigoplus_{\delta\in\Delta}(\mathbb{Z}/p^{r_\delta})\langle d_\delta\rangle$$

since K_s is a subgroup of F_s, which is a direct sum of cyclic $\mathbb{Z}_{(p)}$-modules, and thus is itself a sum of cyclic modules (see [**K**, §15, Thm 17]).

If $\Delta \neq \emptyset$, K is clearly not free, so assume $\Delta = \emptyset$. We then we distinguish two cases:

(a) Let X be locally cyclic. (This is automatically guaranteed when F is in homological dimension > 0, in which case X is just the K of the previous step). Then we may assume that F is minimal, and $K_s \cong \bigoplus_{\gamma\in\Gamma} \mathbb{Z}_{(p)}\langle c_\gamma\rangle$, where each element $c_\gamma \in F_s$ has the form

$$c_\gamma = \sum_{i=1}^{L} \zeta_{i,\gamma}(a_i) \;+\; \sum_{j=1}^{M} n_{j,\gamma}(b_j)\ ,$$

with coefficients $n_{j,\gamma} \in \mathfrak{S}^s_s = \mathbb{Z}_{(p)}$ (not all zero) and $\zeta_{i,\gamma} \in \mathfrak{S}^{t_i}_s$ $(t_i < s)$.
The minimality of F implies $p|n_{j,\gamma}$ for all j. Therefore, by Proposition 4.3, there is an $\alpha \in \mathfrak{S}^s$ such that $\alpha^{\#}\zeta_{i,\gamma} = 0$ and $\alpha^{\#}n_{j,\gamma} = 0$, which again shows $K = Ker(f)$ is not free.

(b) When F is in homological dimension 0 and is not locally cyclic, the argument fails, since one can have a short exact sequence of $(s-1)$-connected abelian Π-algebras:

$$0 \to K = Ker(f) \hookrightarrow F \twoheadrightarrow X \to 0$$

where X_s is a p-divisible $\mathbb{Z}_{(p)}$-module and K a free abelian Π-algebra, and thus the projective dimension of X may actually be 1 (see following example).

□

EXAMPLE 4.5. An example of an $X \in \mathcal{P}_{ab}$ with projective dimension 1 is a *rationalized* free abelian Π-algebra, $\mathfrak{S}^r_{\mathbb{Q}}$, for $r \geq 1$ – compare [**L2**, Thm 5.12(4)]:

Note that the indecomposables functor $Q : \mathcal{P}_{ab} \to grAbgp$ of §2.3 has a left adjoint $F : grAbgp \to \mathcal{P}_{ab}$, which takes graded free abelian groups to free abelian Π-algebras. Applying F to a presentation $\oplus_\alpha \mathbb{Z} \hookrightarrow \oplus_{n=1}^{\infty}\mathbb{Z} \twoheadrightarrow \mathbb{Q}$, concentrated in degree r, yields the required projective resolution of $\mathfrak{S}^r_{\mathbb{Q}}$.

In the category $\mathcal{P}$ of all Π-algebras, one does not have an analogous statement to Theorem 4.4. Since $\mathcal{P}$ is not an abelian category, we must interpret the projective dimension of a Π-algebra X to mean the dimension of a free simplicial resolution of X (cf. [**B1**, §3.2.2]), where a resolution $A_\bullet \to X$ is $\leq n$-dimensional if $sk_n A_\bullet = A_\bullet$ (see [**B2**, §5.3.4]). Then $X = \pi_\star(S^{n_1} \times S^{n_2} \times$

$\ldots \times S^{n_N})$ is an example of a Π-algebra of projective dimension N. (The proof reduces to a calculation in the category of Lie rings – or equivalently, that of associative algebras over $\mathbb{Z}$).

Note that in this case $Ab(X)$ is free; it may be conjectured that all simply-connected Π-algebras of finite type with finite projective dimension have free abelianizations.

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