THE NUMBER OF EDGES IN CRITICAL STRONGLY CONNECTED GRAPHS

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Abstract. We prove that the maximal number of directed edges in a vertex-critical strongly connected simple digraph on n vertices is $\binom{n}{2} - n + 4$.

1. Introduction

A directed graph (or *digraph*) without loops or multiple edges is called *strongly connected* if each vertex in it is reachable from every other vertex. It is called (vertex) *critical strongly connected* if, in addition to being strongly connected, it has the property that the removal of any vertex from it results in a non-strongly connected graph. We denote by $M(n)$ the maximal number of edges in a critical strongly connected digraph on n vertices. Schwarz [?] conjectured (and proved for $n \leq 5$) that $M(n) \leq \binom{n}{2}$. This conjecture was proved by London in [?]. In this paper we determine the precise number of $M(n)$, showing that it is $\binom{n}{2} - n + 4$. (The corresponding number for edge-critical strongly connected graphs is $2n-2$, see e.g. [?], pp 65-66.)

Here is some notation we shall use. Given a digraph D we denote by $V(D)$ the set of its vertices, and by $E(D)$ the set of edges. Throughout the paper the notation n will be reserved for the number of vertices in the digraph named D. For a vertex v of D we write $E^+_D(v)$ for the set of vertices u for which $(v,u) \in E(D)$ and $E^-_D(v)$ for the set of vertices u for which $(u,v) \in E(D)$. We write $d_D(v)$ for the degree of v, namely $|E^+_D(v)|+|E^-_D(v)|$. For a subset A of $V(D)$ we write $D - A$ for the graph obtained from D by removing all vertices in A, together with all edges incident with them. If A consists of a single vertex a, we write $D-a$ for $D - \{a\}$. By D/A we denote the digraph obtained from D by contracting A, namely replacing all vertices of A by a single vertex a, and defining $E^+_{D/A}(a) = \bigcup\{E^+_D(v) : v \in A\} \setminus A$ and $E^-_{D/A}(a) = \bigcup\{E^-_D(v) : v \in A\} \setminus A$.

2. The number of edges in vertex-critical graphs

Theorem 2.1. *For $n \geq 4$*

$$M(n) = \binom{n}{2} - n + 4$$

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A vertex-critical graph with $\binom{n}{2} - n + 4$ edges is the following. Take a directed cycle $(v_1, v_2, \ldots, v_n)$, and add the directed edges (v_i, v_j), $3 \le j < i \le n$ and the edge (v_2, v_1). Thus, what remains to be proved is that in a vertex-critical graph the number of edges does not exceed $\binom{n}{2} - n + 4$.

The proof will be based on two lemmas.

Lemma 2.2. *Let D be a strongly connected digraph and $v \in V(D)$ a vertex satisfying $d(v) \ge n$. Then there exists a vertex $z \in V(D) \setminus \{v\}$ such that $D - z$ is strongly connected.*

Proof The proof is by induction on n. For $n < 2$ the lemma is vaccuously true, since its conditions are impossible to fulfil. For $n = 2$ take z to be the vertex of the graph different from v. Let now $n > 2$ and suppose that the lemma is true for all graphs with fewer than n vertices. Let v be as in the lemma. There exists then a vertex u such that between u and v there is a double-arc (that is, two oppositely directed eges). Let $C = D/\{u, v\}$, and name w the vertex of C replacing the shrunk pair $\{u, v\}$. By a negation hypothesis, we may assume that $D - u$ is not strongly connected. We claim then that $d_C(w) \ge n - 1$. This will prove the lemma, since by the induction hypothesis it will follow that C has a vertex z different from w whose removal leaves C strongly connected. But then, clearly, also $D - z$ is strongly connected.

To prove the claim note, first, that $d_C(w) \ge n - 2$. This follows from the fact that each edge in D incident with v and different from the two edges joining v with u, has its copy in C. Since by our assumption $D - u$ is not strongly connected, there are two edges in D, say (x, u) and (u, y), such that y is not reachable from x in $D - u$. If $x = v$ then the edge (w, y) is an edge in C not having a copy (v, y) in D, and thus can be added to the $n - 2$ edges incident with v counted above, and thus $d_C(w) \ge n - 1$, as desired. Similarly, if $y = v$ then the edge (x, w) shows that $d_C(w) \ge n - 1$. If, on the other hand, $x \ne v$ and $y \ne v$, then one of (x, w) or (w, y) is an edge in C not counted above. □

Note that the lemma proves the original conjecture of Schwarz, namely that $M(n) \le \binom{n}{2}$.

Lemma 2.3. *Let D be a critical digraph and C a chordless cycle in it, such that $V(C) \ne V(D)$. Then $d(v) \le n - |V(C)| + 2$ for all $v \in V(C)$, with strict inequality holding for at least two vertices.*

Proof Let $J = D/V(C)$, and denote by c the vertex of J obtained from the contraction of C. Write k for $|V(C)|$. The graph J has $n-k+1$ vertices, and therefore, by Lemma **??**, $d_J(c) \le n - k$. This implies that $d(v) \le n - k + 2$ for every $v \in V(C)$.

Suppose now that, for some vertex w of C, there obtains $d(v) = n - k + 2$ for all vertices $v \in V(C) \setminus \{w\}$. Then $d_J(c) = n - k$, and all sets $E^+_D(v) \setminus V(C)$

($v \in V(C) \setminus \{w\}$) are equal, and the same goes for the sets $E^-_D(v) \setminus V(C)$. Moreover, $(E^+_D(w) \setminus V(C)) \subseteq E^+_D(v)$ and $(E^-_D(w) \setminus V(C)) \subseteq E^-_D(v)$ for all $v \in V(C)$. But then $D - w$ must be strongly connected, since if (x, w) and (w, y) are edges in D, then y is reachable from x in $D - w$ through vertices of $V(C) \setminus \{w\}$. □

Proof of Theorem **??** The proof is by induction on n. Write s_n for the value claimed by Theorem **??** for $M(n)$, namely

$$s_n = \binom{n}{2} - n + 4$$

Since D is critically strongly connected, it contains a chordless cycle C. Let $|V(C)| = k$. If $V(C) = V(D)$ then we are done because then $|E(D)| = n \le s_n$. Thus we may assume that $V(C) \neq V(D)$, and since D is critical, this implies that $n \ge k + 2$.

Let $J = D/V(C)$, and denote by c the vertex of J obtained from the contraction of C.

Assertion 2.4.

$$|E(D)| - |E(J)| \le s_n - s_{n-k+1}$$

Consider first the case $k = 2$. Let v be one of the two vertices of C. The graph J, being the contraction of a strongly connected graph, is itself strongly connected, and since $D - v$ is not strongly connected, we have $J \neq D - v$. This implies that $|E(J)| > |E(D - v)|$, and hence

$$|E(D)| - |E(J)| \le d_D(v) - 1 \le n - 2 = s_n - s_{n-1}$$

Assume now that $k \ge 3$. Let v be a vertex of C having maximal degree, namely $d_D(v) \ge d_D(u)$ for all $u \in V(C)$. Let r be the number of edges in D not incident with any vertex of C. Then

$$|E(D)| = r - k + \sum_{u \in V(C)} d_D(u)$$

and

$$|E(J)| = d_J(c) + r \ge d_D(v) - 2 + r$$

and therefore

$$\begin{aligned} |E(D)| - |E(J)| &\le 2 - k + \sum_{u \in V(C)\setminus\{v\}} d_D(u) \\ &\le 2 - k + 2(n - k + 1) + (k - 3)(n - k + 2) \\ &= (k - 1)n - (k^2 - 2k + 2) \le s_n - s_{n-k+1} \end{aligned}$$

which proves the assertion.

If J is critical, then the theorem follows from Assertion **??** and the induction hypothesis. So, we may assume that J is not critical. But, for every vertex u different from c, the graph $J - u$ is not strongly connected, since the graph $D - u$ is not strongly connected. Hence, by Lemma **??**, we have

$$d_J(c) \leq n - k \tag{1}$$

On the other hand, the fact that J is not critical means that $J - c = D - V(C)$ is strongly connected.

We next show:

Assertion 2.5.

$$\sum_{v \in V(C)} d_D(v) \leq (n-1)k - n + 4$$

Proof of the assertion By Lemma **??** $d(v) < n$ for all vertices v. Hence, if $d_D(v) = 2$ for some $v \in V(C)$ then the assertion is true. So, we may assume that $d_D(v) \neq 2$ for all $v \in V(C)$. This means that $(E_D^+(v) \cup E_D^-(v)) \setminus V(C) \neq \emptyset$ for every $v \in V(C)$. Let $V(C) = \{v_1, v_2 \dots v_k\}$ and $E(C) = \{(v_i, v_{i+1}) : 1 \leq i \leq k\}$ (where, as usual, the indices are taken modulo k). Without loss of generality we may assume that $E_D^+(v_1) \setminus V(C) \neq \emptyset$. If $E_D^-(v_3) \setminus V(C) \neq \emptyset$ then $D - v_2$ is strongly connected. Thus we may assume that $E_D^-(v_3) \setminus V(C) = \emptyset$ and $E_D^+(v_3) \setminus V(C) \neq \emptyset$. Applying this argument again and again, we conclude that k is even and that $E_D^-(v_i) \setminus V(C) = \emptyset$ for all odd i and $E_D^+(v_i) \setminus V(C) = \emptyset$ for all even i. By (**??**) it follows that for every two adjacent vertices on C the total number of edges incident with them and not belonging to C does not exceed $n - k$. This implies that:

$$\sum_{v \in V(C)} d_D(v) \leq \frac{k}{2}(n-k) + 2k \leq (n-1)k - n + 4$$

proving the assertion.

Recall now that $n \geq k + 2$ and that $D - V(C)$ is strongly connected. Hence $D - V(C)$ contains a chordless cycle C'. Let $k' = |V(C')|$. The same arguments as above hold when C is replaced by C', and thus we may assume that

$$\sum_{v \in V(C')} d_D(v) \leq (n-1)k' - n + 4$$

This, together with Lemma **??**, yields:

$$\sum_{v \in V(D)} d_D(v) \leq (n-1)k - n + 4 + (n-1)k' - n + 4 + (n-1)(n - k - k') = 2s_n$$

which means that

$$|E(D)| \leq s_n$$

□

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