A NOTE ON THE COVER NUMBER AND INDEPENDENCE NUMBER IN HYPERGRAPHS

ELI BERGER AND RAN ZIV

Abstract. Consider a hypergraph of rank $r > 2$ with m edges, independence number α and cover number ρ. We prove the inequality

$$\rho \leq \frac{(r-2)m + \alpha}{r-1} .$$

One application of this inequality is a special case of a conjecture of Aharoni and the first author extending Ryser's Conjecture to matroids.

1. introduction

A *hypergraph* is a nonempty set of sets. The union of these sets is called the *ground set* of the hypergraph. If $\mathcal{H}$ is a hypergraph we call the elements of its ground set the *vertices* of $\mathcal{H}$ and the element of $\mathcal{H}$ the *edges* of $\mathcal{H}$. The rank $R(\mathcal{H})$ of $\mathcal{H}$ is the maximal cardinality of an edge in it. If X is a set of vertices we write $R_{\mathcal{H}}(X)$ for the maximal cardinality of an edge contained in X. If X contains no edges we write $R_{\mathcal{H}}(X) = 0$. The *independence* number $\alpha(\mathcal{H})$ of $\mathcal{H}$ (in some papers called the *strong independence* number) is the maximal size of an *independent* set of vertices, namely a set of vertices no two of which are in the same edge. The *cover number* $\rho(\mathcal{H})$ of $\mathcal{H}$ is the minimal number of edges whose union is the ground set. The *matching number* $\nu(\mathcal{H})$ of $\mathcal{H}$ is the maximal number of disjoint edges. The *transversal number* $\tau(\mathcal{H})$ of $\mathcal{H}$ is the nimimal cardinality of a set intersecting all the edges.

A *partition* of a set S is a set of disjoint sets whose union is S. Every function f from S to some range C defines a partition of S to all sets of the form $\{s \in S : f(s) = c\}$ for some $c \in C$. We shall usually identify f with the partition it defines.

A set X of vertices in a hypergraph is called an *exact transversal* if $|X \cap e| = 1$ for every edge e. An exact transversal X of a hypergraph $\mathcal{H}$ yields a partition of the edges defined by the function $f : \mathcal{H} \to X$, where $\{f(e)\} = X \cap e$. We say that $\mathcal{H}$ is r-partite if there exists a partition of the ground set of $\mathcal{H}$ to r exact transversals.

A hypergraph $\mathcal{M}$ is called a *matroid* if

- Every subset of an edge is an edge.
- For every two edges e_1, e_2, where $|e_2| > |e_1|$, there exists some vertex $v \in e_2 \setminus e_1$ so that $e_1 \cup \{v\} \in \mathcal{H}$.

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Let V be a set with some partition defined on it. The *partition matroid* associated with the partition is the set of sets intersecting each set in the partition at most once. The *dual partition matroid* associated with the partition is the set of sets not containing any set in the partition. These two matroids are dual in the usual notion of matroid duality that the maximal edges in the partition matroid are axactly the comlements of the maximal edges in the dual partition matroid.

A well known conjecture of Ryser states the following:

Conjecture 1.1. *Let $r > 1$ be an integer and let $\mathcal{H}$ be an r-partite hypergraph. Then*

$$\tau(\mathcal{H}) \leq (r-1)\nu(\mathcal{H}) \ .$$

For $r = 2$ this is a well known theorem of König. For $r = 3$ this was proved in [1].

Let $\mathcal{H}$ be an r-partite hypergraph, with partition $V = V_1 \cup \cdots \cup V_r$ of the ground set to exact transversals. Each exact transversal V_i yields a partition of the edges of $\mathcal{H}$ associated with some partition matroid $\mathcal{M}_i$. We can formulate $\tau(\mathcal{H})$ and $\nu(\mathcal{H})$ in terms of the matroids $\mathcal{M}_1, \ldots, \mathcal{M}_r$.

Observation 1.2.

$$\nu(\mathcal{H}) = R(\mathcal{M}_1 \cap \cdots \cap \mathcal{M}_r)$$

$$\tau(\mathcal{H}) = \min \sum_{i=1}^{r} R_{\mathcal{M}_i}(X_i) \ ,$$

where the minimum is taken over all partitions $X_1 \cup \cdots \cup X_r$ of the edge set.

This gives rise to the following generalization of Ryser's conjecture, which was formulated and proved for the case $r = 3$ in [2]

Conjecture 1.3. *Let $\mathcal{M}_1, \ldots, \mathcal{M}_r$ be r matroids on the same ground set. Then*

$$(r-1)R(\mathcal{M}_1 \cap \cdots \cap \mathcal{M}_r) \geq \min \sum_{i=1}^{r} R_{\mathcal{M}_i}(X_i) \ ,$$

where the minimum is taken over all partitions $X_1 \cup \cdots \cup X_r$ of the ground set.

If true for partition matroids, this conjecture would imply Ryser's conjecture. We attack this conjecture "from the other direction", proving it for dual partition matroids.

Let us consider once again an r-partite hypergraph $\mathcal{H}$, with partition $V = V_1 \cup \cdots \cup V_r$ of the ground set to exact transversals. But now for each exact transversal V_i, let us consider the dual partition matroid $\mathcal{M}_i^*$. In this case we can formulate $\rho(\mathcal{H})$ and $\alpha(\mathcal{H})$ in terms of the matroids $\mathcal{M}_1^*, \ldots, \mathcal{M}_r^*$ and the number of edges $m = |\mathcal{H}|$.

Observation 1.4.

$$\rho(\mathcal{H}) = m - R(\mathcal{M}_1 \cap \cdots \cap \mathcal{M}_r)$$

$$\alpha(\mathcal{H}) = m - \min \sum_{i=1}^{r} R_{\mathcal{M}_i}(X_i) \ ,$$

where the minimum is taken over all partitions $X_1 \cup \cdots \cup X_r$ of the edge set.

Thus in order to prove Conjecture 1.3 for dual partition matroid, we need to prove $\rho(\mathcal{H}) \leq \frac{(r-2)m+\alpha(\mathcal{H})}{r-1}$ for every r-partite hypergraph $\mathcal{H}$.

It turns up that this is true for hypergraphs of rank r in general, not necessarily r-partite.

Theorem 1.5. *Let $r > 2$ be an interger. In a hypergraph of rank at most r with m edges, independence number α and cover number ρ, the following inequality holds:*

$$\rho \leq \frac{(r-2)m+\alpha}{r-1} .$$

2. Proof of the main theorem

Assume the theorem is false and let $\mathcal{H}$ be a counterexample with the minimal possible number of edges. As costomery in hypergraph theory, for every vertex v we define the degree $d(v)$ of v to be the number of edges in which it appears.

Assertion 2.1. *There is no vertex of degree one in $\mathcal{H}$.*

Proof. Suppose the vertex v appears only in one edge e, and let

$$\mathcal{H}' = \{f \setminus e : \ e \neq f \in \mathcal{H}\} .$$

Then we have $|\mathcal{H}'| \leq |\mathcal{H} - 1|$, $R(\mathcal{H}') \leq R(\mathcal{H})$, $\alpha(\mathcal{H}) \geq \alpha(\mathcal{H}') + 1$ and $\rho(\mathcal{H}) \leq \rho(\mathcal{H}') + 1$. Thus $(r-1)\rho(\mathcal{H}) \leq r-1+(r-1)\rho(\mathcal{H}') \leq r-1+(r-2)|\mathcal{H}'|+\alpha(\mathcal{H}') = (r-2)(|\mathcal{H}'|+1)+(\alpha(\mathcal{H}')+1) \leq (r-2)|\mathcal{H}|+\alpha(\mathcal{H})$, so $\mathcal{H}$ is not a counterexample. $\square$

Assertion 2.2. *Every two edges intersect in at most one vertex.*

Proof. Let e, f be two edges and suppose $|e \cap f| > 1$ and let $\mathcal{H}' = \mathcal{H} \setminus \{e\}$

Then we have $|\mathcal{H}'| = |\mathcal{H} - 1|$, $R(\mathcal{H}') \leq R(\mathcal{H})$, and $\rho(\mathcal{H}) \leq \rho(\mathcal{H}')$. Note that every independent set in $\mathcal{H}'$ has at most $r-1$ vertices from e. Removing $r-2$ of them obtains a set independent in $\mathcal{H}$ and thus $\alpha(\mathcal{H}) \geq \alpha(\mathcal{H}') + r - 2$. This gives $(r-1)\rho(\mathcal{H}) \leq (r-1)\rho(\mathcal{H}') \leq (r-2)|\mathcal{H}'|+\alpha(\mathcal{H}') = (r-2)(|\mathcal{H}'|+1)+(\alpha(\mathcal{H}')+r-2) \leq (r-2)|\mathcal{H}| + \alpha(\mathcal{H})$, so $\mathcal{H}$ is not a counterexample. $\square$

Definition 2.3. *Let $k > 2$ be integer and let $e_1, \dots, e_k$ be distinct edges in $\mathcal{H}$. We say that $(e_1, \dots, e_k)$ is a* cycle *if $e_i \cap e_{i+1 \mod k} \neq \emptyset$ and $e_i \cap e_j = \emptyset$ for all other pairs. We say that the cycle is* even *or* odd *if k is even or odd respectively. We write $e_i \cap e_{i+1 \mod k} = \{v_i\}$. If some v_i has degree 2 then we say that the cycle is* weak. *Otherwise we say that the cycle is* strong.

Assertion 2.4. *There are no strong cycles.*

Proof. Let $(e_1, \dots, e_k)$ be a strong cycle in $\mathcal{H}$. For each $i = 1, \dots, k$ if e_i has some vertex u_i not appearing in any edge e_j with $j \neq i$, we define $X_i = e_i \setminus \{u_i\}$. Otherwise we write $X_i = e_i$. Note that the latter happens only when $|e_i| = 2$ and we assume $r > 2$ so in both cases we get $|X_i| \leq r - 1$. Write $X = \bigcup_{i=1}^{k} X_i$ and note that $|X| = \sum_{i=1}^{k} |X_i| - k \leq (r-2)k$.

Let now $\mathcal{H}' = \mathcal{H} \setminus \{e_1, \dots, e_k\}$

We have $|\mathcal{H}'| = |\mathcal{H} - k|$, $R(\mathcal{H}') \leq R(\mathcal{H})$, and $\rho(\mathcal{H}) \leq \rho(\mathcal{H}')$. Note that for every independent set S in $\mathcal{H}'$, the set $S \setminus X$ is an independent set in $\mathcal{H}$, and thus $\alpha(\mathcal{H}) \geq \alpha(\mathcal{H}') - (r-2)k$. This gives $(r-1)\rho(\mathcal{H}) \leq (r-1)\rho(\mathcal{H}') \leq (r-2)|\mathcal{H}'| + \alpha(\mathcal{H}') = (r-2)(|\mathcal{H}'|+k) + (\alpha(\mathcal{H}') + (r-2)k) \leq (r-2)|\mathcal{H}| + \alpha(\mathcal{H})$, so $\mathcal{H}$ is not a counterexample. $\square$

Assertion 2.5. *There are no even cycles.*

Proof. Let $k = 2j$, let $(e_1, \ldots, e_k)$ be an even cycle in $\mathcal{H}$ with $e_i \cap e_{i+1 \mod k} = \{v_i\}$ and write $K = \{v_1, \ldots, v_k\}$. For each $i = 1, \ldots, j$ if e_{2i} has some vertex u_i not in K, we define $X_i = e_{2i} \setminus \{u_i\}$. Otherwise we write $X_i = e_{2i}$. Once again in both cases we get $|X_i| \leq r - 1$. Write $X = \bigcup_{i=1}^{j} X_i$ and note that $|X| = \sum_{i=1}^{j} |X_i| \leq (r-1)j$.

Let now $\mathcal{H}' = \mathcal{H} \setminus \{e_2, e_4, \ldots, e_k\}$

We have $|\mathcal{H}'| = |\mathcal{H} - j|$, $R(\mathcal{H}') \leq R(\mathcal{H})$, and $\rho(\mathcal{H}) \leq \rho(\mathcal{H}')$. Note that for every independent set S in $\mathcal{H}'$, the set $S \setminus X$ is an independent set in $\mathcal{H}$. Also note that $|S \cap K| \leq j$ and therefore $|S \setminus X| \geq |S| - (r-2)j$. This gives $\alpha(\mathcal{H}) \geq \alpha(\mathcal{H}') - (r-2)j$ and we have $(r-1)\rho(\mathcal{H}) \leq (r-1)\rho(\mathcal{H}') \leq (r-2)|\mathcal{H}'| + \alpha(\mathcal{H}') = (r-2)(|\mathcal{H}'| + j) + (\alpha(\mathcal{H}') + (r-2)j) \leq (r-2)|\mathcal{H}| + \alpha(\mathcal{H})$, so $\mathcal{H}$ is not a counterexample. $\square$

Assertion 2.6. *There are no odd weak cycles.*

Proof. Let $k = 2j + 1$, let $(e_1, \ldots, e_k)$ be an odd weak cycle in $\mathcal{H}$ with $e_i \cap e_{i+1 \mod k} = \{v_i\}$ and write $K = \{v_1, \ldots, v_k\}$. We may assume $d(v_1) = 2$. For each $i = 1, \ldots, j$ if e_{2i} has some vertex u_i not in K, we define $X_i = e_{2i} \setminus \{u_i\}$. Otherwise we write $X_i = e_{2i}$. Once again in both cases we get $|X_i| \leq r - 1$. Write $X = \bigcup_{i=1}^{k} X_j$ and note that $|X| = \sum_{i=1}^{j} |X_i| \leq (r-1)j$.

Let now $\mathcal{H}' = \mathcal{H} \setminus \{e_2, e_4, \ldots, e_{2j}\}$

We have $|\mathcal{H}'| = |\mathcal{H} - j|$, $R(\mathcal{H}') \leq R(\mathcal{H})$, and $\rho(\mathcal{H}) \leq \rho(\mathcal{H}')$.

We now wish to show that

$$(*) \qquad \alpha(\mathcal{H}) \geq \alpha(\mathcal{H}') - (r-2)j\ .$$

Consider an independent set S in $\mathcal{H}'$, the set $R = S \setminus X$ is an independent set in $\mathcal{H}$. If $|R| \geq |S| - (r-2)j$ we have $(*)$. The only case where this does not occur is if $v_2, v_k \in S$ and $v_1 \notin S$, in which case it is possible that $|R| = |S| - (r-2)j - 1$. But then $R \cup \{v_1\}$ is independent in $\mathcal{H}$ and we have $(*)$.

We can now conclude the proof as before: $(r-1)\rho(\mathcal{H}) \leq (r-1)\rho(\mathcal{H}') \leq (r-2)|\mathcal{H}'| + \alpha(\mathcal{H}') = (r-2)(|\mathcal{H}'| + j) + (\alpha(\mathcal{H}') + (r-2)j) \leq (r-2)|\mathcal{H}| + \alpha(\mathcal{H})$, so $\mathcal{H}$ is not a counterexample. $\square$

The last 3 assertions clearly give

Corollary 2.7. *There are no cycles in $\mathcal{H}$.*

Let G be the line graph of $\mathcal{H}$. Namely, the vertices of G are the edges of $\mathcal{H}$ and two vertices of G are adjacent if and only if they correspond to intersecting edges. By the corollary, G is *chordal*, i.e., every induced cycle in it is a triangle. Furtheremore, every clique in G corresponds to a set of edges in $\mathcal{H}$ meeting at a vertex.

We say that a vertex in a graph is *simplicial* if its neighbors form a clique. A classic theorem (cf. [3]) says

Theorem 2.8. *Every chordal graph has a simplicial vertex.*

Let now e be an edge in $\mathcal{H}$ corresponding to a simplicial vertex of G and let $e_1, \ldots, e_d$ be the edges corresponding to its neighbors. Since $e, e_1, \ldots, e_d$ correspond to a clique in G, they share some common vertex v. Let u be a vertex in e other than v. Since $d(u) > 1$ it must appear in some edge $f \neq e$. We have $f \cap e \neq \emptyset$, so f is a neighbor of e in G. In other words $f = e_i$ for some i and $v \in f$. But this implies $|f \cap e| > 1$, in contradiction to Assertion 2.2. This means that a minimal counterexample for Theorem 1.5 can not exist, and Theorem 1.5 is thus proved.

3. Further questions

Definition 3.1. *Let r be a positive integer. We define $f(r)$ to be the minimal number so that every hypergraph $\mathcal{H}$ of rank at most r satisfies*

$$\rho(\mathcal{H}) \leq \frac{f(r)|\mathcal{H}| + \alpha(\mathcal{H})}{f(r)+1} .$$

In the previous section we showed that for $r \geq 3$ we have $f(r) \leq r-2$. The following theorem gives a lower bound for $f(r)$

Theorem 3.2.

$$f(r) \geq \lfloor \frac{r}{2} \rfloor$$

Proof. Let $\mathcal{H}$ be the hypergraph of $r+1$ lines in general position. In other words, $\mathcal{H} = \{e_0, e_1, \ldots, e_r\}$ and each e_i is the set of all unordered pairs $\{i,j\}$ with $j \in \{0,1,\ldots,r\}$ and $j \neq i$. Thus $\mathcal{H}$ has rank r.

We note that the union of all but any two edges e_i and e_j misses the unordered pair $\{i,j\}$ and therefore $\rho(\mathcal{H}) = r$. An independent set here is just a set of disjoint unordered pair and hence $\alpha(\mathcal{H}) = \lfloor \frac{r+1}{2} \rfloor$.

The number $f(r)$ must satisfy

$$\rho(\mathcal{H}) \leq \frac{f(r)|\mathcal{H}| + \alpha(\mathcal{H})}{f(r)+1} ,$$

which in this case gives

$$r \leq \frac{f(r)(r+1) + \lfloor \frac{r+1}{2} \rfloor}{f(r)+1} ,$$

and we get

$$f(r) \geq r - \lfloor \frac{r+1}{2} \rfloor = \lfloor \frac{r}{2} \rfloor .$$

□

Putting these two bound together for every $r \geq 3$ we have

$$\lfloor \frac{r}{2} \rfloor \leq f(r) \leq r-2 .$$

For $r=3$ and $r=4$ this gives the tight values $f(3)=1$ and $f(4)=2$. For $r=5$ we have $2 \leq f(5) \leq 3$. We failed to find a hypergraph showing that $f(5) > 2$ and thus the question whther $f(5)=2$ is still open. It is also unknown whther the limit $\lim_{r\to\infty} \frac{f(r)}{r}$ exists and where it lies between $\frac{1}{2}$ and 1.

References

[1] R. Aharoni, Ryser's conjecture for 3-partite 3-graphs, *Combinatorica* **21**(2001), 1–4.
[2] R. Aharoni and E. Berger, The intersection of a matroid and a simplicial complex *Trans. Am. Math. Soc.* to appear.
[3] G. Dirac, On rigid circuit graphs, *Abh. Math. Sem. Univ. Hamburg* **25(1961) 71–76.**

Institue of Advanced Study, Princeton, NJ, USA
E-mail address, Eli Berger: `eberger@ias.edu`

E-mail address, Ran Ziv: `ranziv@telhai.ac.il`