THE INTERSECTION OF A MATROID AND A SIMPLICIAL COMPLEX

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ABSTRACT. A classical theorem of Edmonds provides a min-max formula relating the maximal size of a set in the intersection of two matroids to a "covering" parameter. We generalize this theorem, replacing one of the matroids by a general simplicial complex. One application is a solution of the case $r = 3$ of a matroidal version of Ryser's conjecture. Another is an upper bound on the minimal number of sets belonging to the intersection of two matroids, needed to cover their common ground set. This, in turn, is used to derive a weakened version of a conjecture of Rota. Bounds are also found on the dual parameter - the maximal number of disjoint sets, all spanning in each of two given matroids. We study in detail the case in which the complex is the complex of independent sets of a graph, and prove generalizations of known results on "independent systems of representatives" (which are the special case in which the matroid is a partition matroid). In particular, we define a notion of k-matroidal colorability of a graph, and prove a fractional version of a conjecture, that every graph G is $2\Delta(G)$-matroidally colorable.

The methods used are mostly topological.

1. INTRODUCTION

The point of departure of this paper is a notion which has been recently developed and studied, that of an *independent system of representatives* (ISR), which is a generalization of the notion of a system of distinct representatives (SDR). Like in the case of SDR's, an ISR is a choice function of elements from a system of sets $V_1, V_2, \ldots, V_m$, namely a choice of elements $x_1 \in V_1, x_2 \in V_2, \ldots, x_m \in V_m$. In the case of SDR's, the elements x_i are assumed to be distinct. In the case of ISR's they are not necessrily distinct, but there is another element added, that of a graph G on $V = \bigcup_{1 \leq i \leq m} V_i$. The system of representatives is then called and ISR if x_i, x_j are not adjacent in G for $i \neq j$. All graphs considered in this paper are assumed to be loopless (namely, (v, v) is not an edge), and thus a vertex is not adjacent to itself.

Hall's theorem [16] provides a necessary and sufficient condition for the existence of an SDR: the union of every k sets V_i should be of size at least k. By contrast, testing for the existence of an ISR is NP-complete, and hence a non-trivial necessary and sufficient condition for it is not expected. What can be expected are non-trivial sufficient conditions. Recently topological methods have been applied to

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this end, and a sufficient condition has been found of topological nature: for every $I \subseteq \{1, 2, \ldots, m\}$, the simplicial complex of the independent sets of the graph induced on G by $\bigcup_{i \in I} V_i$ should be of connectivity at least $|I| - 2$ (the definitions of the above notions are given below). This criterion has been applied in various ways to provide sufficient conditions of combinatorial nature.

The notion of an ISR does not lose generality by assuming that all sets V_i are disjoint (in several papers on the subject this assumption is indeed made). The reason is that there is a simple transformation reducing the general case to this case: if a vertex v appears in two sets V_i and V_j, replace it by two copies of it, put one of these copies in V_i and the other in V_j, and connect the two copies in G. More generally, a vertex belonging to many V_i's is replaced by a clique, each vertex in the clique belonging to a different set V_i.

In the case that the sets V_i are disjoint there is another way of viewing ISR's. Let $\mathcal{P}$ be the partition matroid defined by the sets V_i, namely $A \in \mathcal{P}$ if $|A \cap V_i| \leq 1$ for every $i \leq m$. An ISR is then a base of $\mathcal{P}$ which is independent in G. This formulation calls for two generalizations: replacing $\mathcal{P}$ by a general matroid on V, and replacing the set of independent sets in G by a general simplicial complex $\mathcal{C}$ (the notions of "matroid", "simplicial complex" and related concepts are all defined in subsequent sections). The case in which $\mathcal{C}$ is a matroid as well is the subject of the celebrated theorem of Edmonds on the intersection of two matroids, which (in one of its formulations) provides a necessary and sufficient condition for the existence of a base in one matroid that is independent in the other. In the case that $\mathcal{C}$ is a general simplicial complex, such a condition is again not to be expected, and only non-trivial sufficient conditions may be sought. The main aim of this paper is to find such a condition, formulated in topological terms. We shall find that Edmonds' theorem remains true when the rank function in the matroid is replaced by a connectivity parameter of the complex (the two coinciding in the case of matroids).

The rest of the paper is devoted to applications and generalizations of this theorem, and to combinatorially-formulated lower bounds on the connectivity of a complex. One application is related to Ryser's conjecture. The latter is a generalization of Kőnig's theorem to r-partite r-graphs, and it can be given a matroidal generalization in the same way that Edmonds' theorem generalizes Kőnig's theorem. We give a proof of the case $r = 3$ of this matroidal conjecture, generalizing the recent solution of the same case of Ryser's conjecture. The main tool here, apart from the main theorem of the paper, is a lower bound on the connectivity of the intersection of matroids.

Another application is to coloring-type problems, namely problems on the minimal number of simplices from a given complex needed to cover its ground set. In particular, we shall be interested in the case that the complex is the intersection of given matroids. Kőnig's line-coloring theorem treats this question in the case of partition matroids. It says that that if the common ground set of two partition matroids $\mathcal{M}$ and $\mathcal{N}$ is decomposable into k sets belonging to $\mathcal{M}$ and is also decomposable into k sets belonging to $\mathcal{N}$, then it is decomposable into k sets in $\mathcal{M} \cap \mathcal{N}$. This fails for general matroids, but we prove that for general matroids the same conditions imply that the ground set can be decomposed into $2k$ sets in the intersection of the matroids. This yields a weakened version of a conjecture of Rota.

In a more general setting, we wish to decompose the ground set into sets belonging to the intersection of a matroid and a general complex. A conjecture which has drawn some attention of late is concerned with the case that the matroid is a partition matroid $\mathcal{P}$, and the complex is that of the independent sets of a graph. In such a case it is conjectured that it is possible to decompose the ground set into no more than $\max(\Delta(\mathcal{P}), 2\Delta(G))$ sets in the intersection of the complex and the matroid, where $\Delta(G)$ denotes, as usual, the maximal degree of a vertex in G, and $\Delta(\mathcal{M})$ of a matroid $\mathcal{M}$ is defined as the maximum, over all subsets A of the ground set, of $|sp_{\mathcal{M}}(A)|/|A|$ (in the case of a partition matroid, this is just the size of the largest part). We conjecture that the same is true also for general matroids, and prove a fractional version of this conjecture. For this purpose we prove a weighted version of the main theorem, in the case when the complex in question is that of the independent sets of a graph (in fact, we suspect that this weighted version is valid for any complex).

Wishing to make the paper accessible to a wide audience, we shall not assume familiarilty with the topological notions used. For this reason, the next section is devoted to topological preliminaries. A similar section is included on concepts from matroid theory.

2. Topological preliminaries

A non-empty hypergraph $\mathcal{C}$ is called a *simplicial complex* (or plainly a *complex*) if it is hereditary, meaning that $\sigma \in \mathcal{C}$ and $\tau \subseteq \sigma$ imply $\tau \in \mathcal{C}$. The edges of $\mathcal{C}$ are called its *simplices*. We choose a set containing all the simplices of the complex and call it the *ground set* of $\mathcal{C}$. This set is denoted by $V(\mathcal{C})$. The elements of $V(\mathcal{C})$ are called *vertices*. Throughout the paper we assume that the ground set is finite. The maximal size of a simplex in $\mathcal{C}$ is denoted by $\mu(\mathcal{C})$. For a subset X of $V(\mathcal{C})$ we denote by $\mathcal{C} \restriction X$ the set of simplices in $\mathcal{C}$ contained in X. The complex of independent sets in a graph G, namely the sets containing no edges, is denoted by $\mathcal{I}(G)$.

There is also a geometric definition of the notion of simplicial complex. A *geometric simplex* is the convex hull of k points in $\mathbb{R}^n$, where $k \leq n+1$, and the points are in general position. A *face* of a simplex is a simplex spanned by a subset of its vertex set. A collection $\mathcal{F}$ of simplices in $\mathbb{R}^n$ is called a *geometric simplicial complex* if every two simplices meet, if at all, in a face common to both. The union of all these simplices is denoted by $||\mathcal{F}||$. The *support* $supp(x)$ of a point x in the complex is the smallest (with respect to inclusion) face containing x. Given two complexes $\mathcal{C}$ and $\mathcal{D}$, a function $f : ||\mathcal{C}|| \to ||\mathcal{D}||$ is called *simplicial* if f sends every vertex to a vertex and f is linear on each simplex.

It is well known and easy to show that every combinatorial simplicial complex can be realised as a geometric complex in $\mathbb{R}^n$ for some n, and that this can be done in a unique way, up to isomorphism. By "realisation" we mean the existence of a bijection between the simplices of the two complexes. We shall usually not distinguish between a complex and its realisation.

The *link* $lk_{\mathcal{C}}(\sigma)$ of a simplex σ in a complex $\mathcal{C}$ is the complex consisting of all simplices $\tau \in \mathcal{C}$ such that $\tau \cap \sigma = \emptyset$ and $\tau \cup \sigma \in \mathcal{C}$. We shall use this notion only for singleton simplices, $\{x\}$, and write then $lk_{\mathcal{C}}(x)$ for $lk_{\mathcal{C}}(\{x\})$. For a vertex x we denote the complex $\{\sigma \in \mathcal{C} : \ x \notin \sigma\}$ by $\mathcal{C} - x$.

We shall use two notions of "union" of complexes $\mathcal{C}$ and $\mathcal{D}$. One is just $\mathcal{C} \cup \mathcal{D}$. The second notion, denoted by $\mathcal{C} \vee \mathcal{D}$, is the complex $\{\sigma \cup \tau : \ \sigma \in \mathcal{C}, \ \tau \in \mathcal{D}\}$. The union $\mathcal{C} \vee \mathcal{C} \ldots \vee \mathcal{C}$ of a complex $\mathcal{C}$ with itself k times is denoted by $\bigvee^k \mathcal{C}$. If the ground sets of two complexes $\mathcal{C}$ and $\mathcal{D}$ are disjoint, then $\mathcal{C} \vee \mathcal{D}$ is denoted by $\mathcal{C} * \mathcal{D}$, and is called the *join* of $\mathcal{C}$ and $\mathcal{D}$. The join is defined also for general complexes, by first taking copies of the complexes on disjoint ground sets.

The basic concept used in ISR theory is that of *connectivity* of a complex. It indicates which "holes" in the complex can be filled, and which cannot. A "hole" here is a homeomorph of a sphere.

Here is a more rigorous definition. A *piecewise linear n-sphere* (or PL-n-sphere, for short) is a geometric simplicial complex $\mathcal{C}$ homeomorphic to S^n, the boundary of the $n+1$-dimensional ball B^{n+1}. *Piecewise linear balls* are similarly defined.

A complex $\mathcal{C}$ is said to be *k-connected* if for every $i \leq k$ and every simplicial function ψ from a PL-i-sphere S into $\mathcal{C}$ there exists a simplicial extension of ψ to a PL-$i+1$-ball with boundary S. As a matter of definition, -1-connectedness means being non-empty.

We denote by $\eta(\mathcal{C})$ the largest k for which $\mathcal{C}$ is k-connected, plus 2. (The addition of 2 simplifies the formulations of certain results. Combinatorially, it signifies the number of vertices of the simplices used to fill the spheres.) If $\mathcal{C}$ is k-connected for every k, we write $\eta(\mathcal{C}) = \infty$.

Here are some basic facts about the connectivity parameter η. Recall that a subspace X of a topological space Y is said to be a *retract* of Y if there exists a continuous mapping from Y to X that sends each point of X to itself. We say that a simplicial complex $\mathcal{C}$ is a retract of a simplicial complex $\mathcal{D}$ if the topological space $||\mathcal{C}||$ is a retract of the topological space $||\mathcal{D}||$. (The retraction function does not have to be simplicial.)

Observation 2.1. *If $\mathcal{C}$ is a retract of $\mathcal{D}$ then $\eta(\mathcal{C}) \geq \eta(\mathcal{D})$.*

A basic fact about connectivity relates the connectivity of the join to the connectivity of its factors:

Lemma 2.2. *For every pair $\mathcal{C}, \mathcal{D}$ of complexes,*

$$\eta(\mathcal{C} * \mathcal{D}) \geq \eta(\mathcal{C}) + \eta(\mathcal{D})$$

Let us give here an outline of the proof of this theorem. Let $c = \eta(\mathcal{C})$ and $d = \eta(\mathcal{D})$. It is not hard to reduce the problem to the case where $c = \mu(\mathcal{C})$ and $d = \mu(\mathcal{D})$. (In other words, we can ignore simplices with "too many" vertices.) Now let $\mathcal{U}_c^{\mathcal{C}}$ be the simplicial complex on the same ground set as $\mathcal{C}$ whose simplices are all sets with size at most c and similarly, let $\mathcal{U}_d^{\mathcal{D}}$ be the simplicial complex on the same ground set as $\mathcal{D}$ whose simplices are all sets with size at most d. It can be shown that $\mathcal{C}$ is a retract of $\mathcal{U}_c^{\mathcal{C}}$ and $\mathcal{D}$ is a retract of $\mathcal{U}_d^{\mathcal{D}}$, and hence $\mathcal{C} * \mathcal{D}$ is a retract of $\mathcal{U}_c^{\mathcal{C}} * \mathcal{U}_d^{\mathcal{D}}$. Finally, it is not hard to calculate that $\eta(\mathcal{U}_c^{\mathcal{C}} * \mathcal{U}_d^{\mathcal{D}}) = c + d$. This proves the lemma. (The proof of the cases that $c = \infty$ or $d = \infty$ is a bit different, but the main idea is the same.)

The following properties of the connectivity function follow directly from basic algebraic topological results such as the Hurewicz theorem, Van-Kampen's theorem and the Mayer-Vietoris theorem.

Lemma 2.3. *For any pair $\mathcal{A}, \mathcal{B}$ of complexes,*
(1) $\eta(\mathcal{A} \cup \mathcal{B}) \geq \min(\eta(\mathcal{A}), \eta(\mathcal{B}), \eta(\mathcal{A} \cap \mathcal{B}) + 1)$

(2) $\eta(\mathcal{A} \cap \mathcal{B}) \geq \min(\eta(\mathcal{A}), \eta(\mathcal{B}), \eta(\mathcal{A} \cup \mathcal{B}) - 1)$
(3) $\eta(\mathcal{A}) \geq \min(\eta(\mathcal{A} \cap \mathcal{B}), \eta(\mathcal{A} \cup \mathcal{B}))$

(Note that the "$\cup$" operation is identical when viewed in the simplicial sense, that is - the union of the sets of simplices, and in the topological sense, namely the union of the topological spaces. The same goes for the "$\cap$" operation.)

We shall mainly use the following corollary of part (1) of Lemma 2.3:

Corollary 2.4. *For every vertex x of a simplicial complex $\mathcal{C}$*

$$\eta(\mathcal{C}) \geq \min(\eta(\mathcal{C} - x), \eta(lk_{\mathcal{C}}(x)) + 1)$$

To keep this paper as elementary as possible, we give this corollary also a triangulative (that is, simplicial) proof:

Proof. Let $k = \min(\eta(\mathcal{C} - x), \eta(lk_{\mathcal{C}}(x)) + 1) - 2$, let S be a PL-k-sphere in $\mathcal{C}$ and let $\psi : S \to \mathcal{C}$ be a simplicial function. Suppose that there exists some vertex v of S with $\psi(v) = x$. Then the complex $lk_S(v)$ is a PL-$k-1$-sphere and $\psi(lk_S(v))$ is contained in $lk_{\mathcal{C}}(x)$. Since $\eta(lk_{\mathcal{C}}(x)) \geq k + 1$, there exist a PL-$k$-ball B with boundary $lk_S(v)$ and a continuous extension of ψ to B with $\psi(B) \subseteq lk_{\mathcal{C}}(x)$. In fact, we can linearly extend ψ to be defined on all of $v * B$. By this we triangulated a "cap", in which ψ can be extended. Consider now the PL-k-sphere $(B \cup S) - v$, and repeat the same procedure with it. In this way we excise caps from S, until we obtain a PL-k-sphere T not containing any vertex in $\psi^{-1}(x)$. Since $\eta(\mathcal{C} - x) \geq k+2$, there exists a PL-$k+1$-ball D to which ψ can be extended. Adding all excised caps to D yields a PL-$k+1$-ball filling S in which ψ can be extended. □

In some cases we shall need another connectivity parameter, which behaves like η, but is always finite. For that end we define $\overline{\eta}(\mathcal{C}) = \min(\eta(\mathcal{C}), \mu(\mathcal{C}))$. There is still another parameter that serves that goal, which is more natural in some sense. Replace each vertex $v \in V(\mathcal{C})$ by two copies of it, v' and v'', and each simplex σ by all simplices in which for every $v \in \sigma$ precisely one of v', v'' appears. In other words, every simplex in $\mathcal{C}$ is replaced by the join of the appropriate number of copies of S^0. (Note that this is the same operation used to make the sets V_i in the definition of an ISR disjoint.) The resulting complex is called in the literature the *deleted join* of the complex. In this paper we denote the deleted join of a complex $\mathcal{C}$ by $\hat{\mathcal{C}}$.

We write $\hat{\eta}(\mathcal{C}) = \eta(\hat{\mathcal{C}})$. The parameter $\hat{\eta}$ pinpoints the "essence" of the connectivity, getting rid of connectivity arising for fortuitous reasons. For example, we have:

Lemma 2.5.

$$\hat{\eta}(\mathcal{C}) \leq \overline{\eta}(\mathcal{C})$$

Proof. A simplex of maximal size $v_1, v_2, \ldots, v_k$ in $\mathcal{C}$ gives rise to a generalized octahedron Ω in $\hat{\mathcal{C}}$, whose vertices are $v_1', v_1'', v_2', v_2'', \ldots, v_k', v_k''$, which is a PL-$k$-sphere. It cannot be filled in $\hat{\mathcal{C}}$, since the latter does not contain any simplices of size $k+1$, which are needed for the filling. This shows that $\hat{\eta}(\mathcal{C}) \leq \mu(\mathcal{C})$. The inequality $\hat{\eta} \leq \eta$ is easy. □

Another way in which $\hat{\eta}$ is "tamer" than η is the following continuity-type property:

Lemma 2.6. $\hat{\eta}(\mathcal{C} - x) \geq \hat{\eta}(\mathcal{C}) - 1$ *for any* $x \in V(\mathcal{C})$

Proof. Put in part (2) of Lemma 2.3 $\mathcal{A} = \hat{\mathcal{C}} - x'$, $\mathcal{B} = \hat{\mathcal{C}} - x''$. Then $\mathcal{A} \cup \mathcal{B} = \hat{\mathcal{C}}$, $\mathcal{A} \cap \mathcal{B} = (\mathcal{C} \hat{-} x)$. Note that $\mathcal{A}$ can be viewed as obtained from $\hat{\mathcal{C}}$ by identifying the points x' and x'', hence $\eta(\mathcal{A}) = \eta(\mathcal{B}) \geq \hat{\eta}(\mathcal{C})$. This observation, together with Lemma 2.3, yield the desired result. □

3. Concepts from matroid theory

For easy reference, we repeat the basic definitions and some of the basic facts on matroids. For facts not proved here, the reader is referred to [22]. A non-empty simplicial complex $\mathcal{M}$ is called a *matroid* if whenever $\sigma, \tau \in \mathcal{M}$ and $|\tau| > |\sigma|$ there exists $x \in \sigma \setminus \tau$ such that $\tau + x \in \mathcal{M}$. A set belonging to $\mathcal{M}$ is also called *independent* in it. The *rank* $\rho_{\mathcal{M}}(A)$ of a set A is the maximal size of a set $\sigma \in \mathcal{M}$, $\sigma \subseteq A$. A maximal set in $\mathcal{M}$ is called a *base*, and a minimal dependent set a *circuit*. The rank of the entire ground set is denoted by $\rho(\mathcal{M})$. The *span* $sp_{\mathcal{M}}(A)$ of a set A is the set of all elements x such that $\sigma + x \notin \mathcal{M}$ for some $\sigma \in \mathcal{M}$, $\sigma \subseteq A$. If $sp_{\mathcal{M}}(A) = A$ then A is called a *flat*. The *dual* $\mathcal{M}^*$ of $\mathcal{M}$ is the matroid whose bases are the complements of the bases of $\mathcal{M}$. We write ρ^* for the rank function in $\mathcal{M}^*$. For a subset X of V we denote by $\mathcal{M}.X$ the matroid consisting of those subsets τ of X such that $\tau \cup \sigma \in \mathcal{M}$ for all $\sigma \in \mathcal{M}$, $\sigma \cap X = \emptyset$. By M/X we denote the matroid $M.(V \setminus X)$.

Remark 3.1. *For $\sigma \in \mathcal{M}$ we have $lk(\sigma) = \mathcal{M}/\sigma$.*

A *partition matroid* $\mathcal{P}$ is defined by a partition $P_1, P_2, \ldots, P_m$ of the ground set, the definition being that $\sigma \in P$ if and only if $|\sigma \cap P_i| \leq 1$ for all $i \leq m$.

An element x is called a *loop* if $\{x\} \notin \mathcal{M}$. It is called a *co-loop* if it is a loop in $\mathcal{M}^*$, which means that it belongs to every base of $\mathcal{M}$. Similarly, a set is called a *co-circuit* in $\mathcal{M}$ if it is a circuit in $\mathcal{M}^*$. Note that a set spans the ground set if and only if it intersects all co-circuits.

If $\mathcal{M}, \mathcal{N}$ are matroids on the same ground set, then $\mathcal{M} \vee \mathcal{N}$ is also a matroid.

Here are some straightforward corollaries of the definitions which we shall need (here V is the ground set of the matroid and X is any subset of V):

$$(\mathcal{M} \restriction X)^* = \mathcal{M}^*.X \tag{1}$$

$$\rho^*(X) = |X| + \rho_{\mathcal{M}}(V \setminus X) - \rho(\mathcal{M}) \tag{2}$$

$$\rho(\mathcal{M}.X) = \rho(\mathcal{M}) - \rho_{\mathcal{M}}(V \setminus X) \tag{3}$$

For matroids, the connectivity parameter is particularly simple. In fact, it is more or less the rank of the matroid:

Lemma 3.2 ([7])**.** *If there exists a co-loop in $\mathcal{M}$, then $\eta(\mathcal{M}) = \infty$. Otherwise $\eta(\mathcal{M}) = \rho(\mathcal{M})$.*

Let us just show the first part: if x is a co-loop then $\mathcal{M} = (\mathcal{M} - x) * \{x\}$, and since η of a singleton is ∞ this implies by Lemma 2.2 that $\eta(\mathcal{M}) = \infty$.

4. The intersection of a matroid and a simplicial complex

As mentioned in the introduction, the aim of this paper is to generalize the notion of an ISR in two directions - introducing a general matroid rather than a partition matroid, and a general complex instead of the complex of independent sets in a graph. Let us start with a one-step generalization, namely replacing the complex. We choose to do it here in the equivalent (and often more convenient) terminology of bipartite graphs:

Definition 4.1. *Let Γ be a bipartite graph with sides A and B, and let $\mathcal{C}$ be a simplicial complex on B. A $\mathcal{C}$-ISR is then a function using only edges of Γ, whose domain is A, and whose range belongs to $\mathcal{C}$. If the range of the function is only assumed to be a subset of A, the function is called a* partial $\mathcal{C}$-ISR.

To formulate the basic result relating ISR's and connectivity, we need the following notation (which will also be used for other purposes). For a graph G and a vertex x in it, we denote by $N(x)$ the closed neighborhood of x, namely the set of vertices adjacent to x, together with x itself. By $\tilde{N}(x)$ we denote $N(x) \setminus \{x\}$. For a set X of vertices we write $N(X)$ for $\bigcup_{x \in X} N(x)$, and $\tilde{N}(X)$ for $\bigcup_{x \in X} \tilde{N}(x)$.

The following was proved for $\mathcal{I}(G)$-type complexes implicitly (using Sperner's lemma) in [6] and explicitly (using homology theory) in [19]. Both proofs did not use the special nature of the complex, and thus both yield:

Theorem 4.2. *For any simplicial complex $\mathcal{C}$ on B, if $\eta(\mathcal{C} \restriction \tilde{N}_\Gamma[X]) \geq |X|$ for every $X \subseteq A$ then there exists a $\mathcal{C}$-ISR.*

Next we come to the second generalization, the introduction of a general matroid on the ground set. Let us first recall Edmonds' two matroids intersection theorem [11]:

Theorem 4.3. *Given two matroids $\mathcal{M}$ and $\mathcal{N}$ on the same ground set V,*

$$\mu(\mathcal{M} \cap \mathcal{N}) = \min\{\rho_{\mathcal{M}}(X) + \rho_{\mathcal{N}}(V \setminus X) : \ X \subseteq V\}$$

This theorem has also a Hall-like counterpart, easily derived from it and vice versa (see, e.g. [5]). To formulate it, we shall need the following terminology. Let $\mathcal{M}, \mathcal{C}$ be a matroid and a complex, respectively, on the same ground set V. The pair $\Pi = [\mathcal{M}, \mathcal{C}]$ is said to be *matchable* if there exists a base of $\mathcal{M}$ belonging to $\mathcal{C}$. Write $\nu(\mathcal{M}, \mathcal{C})$ for $\mu(\mathcal{M} \cap \mathcal{C})$, and $\tau(\mathcal{M}, \mathcal{C})$ for the minimum, over all subsets X of the ground set V, of $\rho_{\mathcal{M}}(X) + \eta(\mathcal{C} \restriction V \setminus X)$.

Theorem 4.4. *A pair $\mathcal{M}, \mathcal{N}$ of matroids on the same ground set V is matchable if and only if $\rho_{\mathcal{N}}(X) \geq \rho(\mathcal{M}.X)$ for every $X \subseteq V$.*

As already explained, we wish to generalize these theorems to the case in which one of the matroids (say $\mathcal{N}$) is replaced by a simplicial complex $\mathcal{C}$. We shall show that the same theorems remain valid also in this case, upon replacing the function $\rho_{\mathcal{N}}(X)$ by $\eta(\mathcal{C} \restriction X)$.

Theorem 4.5. *Let $\mathcal{M}$ be a matroid and $\mathcal{C}$ a simplicial complex on the same ground set V. If $\eta(\mathcal{C} \restriction X) \geq \rho(\mathcal{M}.X)$ for every $X \subseteq V$ then $[\mathcal{M}, \mathcal{C}]$ is matchable. In fact, it suffices to assume the condition for sets X that are complements of flats in $\mathcal{M}$.*

Theorem 4.6. $\nu(\mathcal{M}, \mathcal{C}) \geq \tau(\mathcal{M}, \mathcal{C})$

Remark: In view of Lemma 3.2, Theorem 4.6 is a generalization of Edmonds' Theorem. To see how Theorem 4.2 follows from Theorem 4.5, consider first the case in which the sets $\tilde{N}(a),\ a \in A$ are disjoint, and take as $\mathcal{M}$ the partition matroid on B whose parts are $\tilde{N}(a),\ a \in A$. The theorem now follows from the observations that the complement of a flat in this matroid is of the form $\tilde{N}[X]$ for some $X \subseteq A$, and $\rho(\mathcal{M}.\tilde{N}[X]) = |X|$. The case in which the sets $\tilde{N}(a)$ are not disjoint follows by the standard technique of splitting vertices, described in the introduction.

We shall give two proofs of Theorem 4.5. The first is modelled after a proof by Welsh [23] of Edmonds' theorem. It uses the following well known construction, which will serve us throughout the paper. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ be simplicial complexes on the same ground set V. By $\Gamma[\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m]$ we denote a bipartite graph together with a simplicial complex on one of its sides. The first side of this graph is just V, and the second the union of m disjoint copies of V. Every vertex $v \in V$ in the first side is connected to all its copies in the other side. On the second side we put the complex $\mathcal{D} = \mathcal{C}_1 * \mathcal{C}_2 * \ldots * \mathcal{C}_m$. A $\mathcal{D}$-ISR in $\Gamma[\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m]$ corresponds to a partition of V into sets $X_i \in \mathcal{C}_i$. In the case $\mathcal{C}_1 = \mathcal{C}_2 = \ldots = \mathcal{C}_m = \mathcal{C}$ we write $\Gamma[\mathcal{C}^m]$ for $\Gamma[\mathcal{C}_1, \ldots, \mathcal{C}_m]$.

Proof of Theorem 4.5.

Let $\Gamma = \Gamma[\mathcal{C}, \mathcal{M}^*]$. The matchability of Π means the existence of a set $A \in \mathcal{C}$ such that $V \setminus A \in \mathcal{M}^*$. But this is equivalent to saying that Γ has a $\mathcal{C} * \mathcal{M}^*$-ISR. By Theorem 4.2 and Lemma 2.2 this will follow if we prove that every subset X of V satisfies $\eta(\mathcal{C} \restriction X) + \eta(\mathcal{M}^* \restriction X) \geq |X|$. If X is not the complement of a flat in $\mathcal{M}$, then it contains an element $x \in sp_{\mathcal{M}}(V \setminus X)$. Thus x is a loop in $\mathcal{M}.X$, in other words a co-loop in $\mathcal{M}^* \restriction X$. This, by Lemma 3.2, implies that $\eta(\mathcal{M}^* \restriction X) = \infty$, hence the required inequality is satisfied. For X which is the complement of a flat, Equations 3 and 2 and Lemma 3.2 imply that the inequality is equivalent to the condition on X assumed in the theorem. □

Remark: The above proof of the last statement of the theorem (i.e. that it is enough to consider complements of flats) is due to Roy Meshulam.

Proof of Theorem 4.6.

We repeat the proof of Theorem 4.5, applying this time a standard deficiency argument. Let $d = \max\{\rho(\mathcal{M}.X) - \eta(\mathcal{C} \restriction X) :\ X \subseteq V\}$. Let X be the set at which this maximum is attained. Then $\tau(\mathcal{C}, \mathcal{M}) \leq \eta(\mathcal{C} \restriction X) + \rho_{\mathcal{M}}(V \setminus X) = \eta(\mathcal{C} \restriction X) + \rho(\mathcal{M}) - \rho(\mathcal{M}.X) = \rho(\mathcal{M}) - d$.

To obtain a bound on $\nu(\mathcal{C}, \mathcal{M})$, note first that if $d = 0$ then by Theorem 4.5 $\nu(\mathcal{M}, \mathcal{C}) = \rho(\mathcal{M})$, while $\tau(\mathcal{M}, \mathcal{C}) \leq \rho(\mathcal{M})$. Hence we may assume that $d > 0$. Let $\Gamma = \Gamma[\mathcal{M}^*, \mathcal{C}, \mathcal{U}]$, where $\mathcal{U}$ is the d-uniform matroid on V, namely a subset X of V is in $\mathcal{U}$ if $|X| \leq d$. Clearly, $\eta(\mathcal{U}) = d$. Hence, by Lemma 2.2, Γ satisfies the conditions of Theorem 4.2, and thus has an ISR, say I'. Then $I' \setminus V(\mathcal{U}) \in \mathcal{C} \cap \mathcal{M}$, proving that $\nu(\mathcal{C}, \mathcal{M}) \geq \rho(\mathcal{M}) - d$. □

Since $\hat{\eta}(\mathcal{D}) \leq \eta(\mathcal{D})$ for every complex $\mathcal{D}$, Theorem 4.5 remains valid when η is replaced everywhere by $\hat{\eta}$. But for $\hat{\eta}$ we can prove a version which is stronger in another sense:

Corollary 4.7. *Let $\Pi = [\mathcal{M}, \mathcal{C}]$ be a pair of a matroid and a complex on the same ground set V. If $\hat{\eta}(\mathcal{C} \restriction X) \geq \rho(\mathcal{M}.X)$ for every $X \subseteq V$ which is a flat in $\mathcal{M}^*$ then Π is matchable.*

Proof. It suffices to show that the minimum of $\hat{\eta}(\mathcal{C} \restriction X) - \rho(\mathcal{M}.X)$ is attained at a set X which is a flat of $\mathcal{M}^*$. For this purpose, it suffices to show that if X is not a flat in $\mathcal{M}^*$, then there exists $v \in V \setminus X$ such that $\hat{\eta}(\mathcal{C} \upharpoonright X+v) - \rho(\mathcal{M}.X+v) \le \hat{\eta}(\mathcal{C} \upharpoonright X) - \rho(\mathcal{M}.X)$. But X not being a flat in $\mathcal{M}^*$ means that there exists $v \in V \setminus X$ such that $\rho^*(X+v) = \rho^*(X)$. Since $\rho^*(X) = |X| - \rho(\mathcal{M}.X)$ (this can be deduced, e.g., from (2) and (3)), this implies that $\rho(\mathcal{M}.X+v) = \rho(\mathcal{M}.X)+1$. Using Lemma 2.6 the desired inequality follows. □

5. Another proof of Theorem 4.5

In this section we present yet another proof of Theorem 4.5. It follows a different approach, not using Theorem 4.2.

The *flat complex* $\mathcal{F}(\mathcal{M})$ of a matroid $\mathcal{M}$ is defined as follows: the vertices of $\mathcal{F}(\mathcal{M})$ are the flats of $\mathcal{M}$ (including the empty set but not including the entire ground set V), and the simplices of $\mathcal{F}(\mathcal{M})$ are chains of flats, namely sets $\{F_1, F_2, \dots F_k\}$ such that $F_1 \subset F_2 \subset \dots \subset F_k$.

Lemma 5.1. *Let $\mathcal{M}, \mathcal{C}$ be a matroid and a simplicial complex satisfying the conditions of Theorem 4.5. Then there exists a continuous map $\xi : ||\mathcal{F}(\mathcal{M})|| \to ||\mathcal{C}||$ such that for every $x \in ||\mathcal{F}(\mathcal{M})||$ there exists $F \in supp_{\mathcal{F}(\mathcal{M})}(x)$ satisfying $F \cap supp_{\mathcal{C}}(\xi(x)) = \emptyset$.*

Proof. Let $r = \rho(\mathcal{M})$. We define ξ for points of the interior of each simplex at a time. For every singleton $\{F\}$ we set $\xi(F)$ to be some vertex in $V - F$ (recall that V itself is not a vertex in $\mathcal{F}(\mathcal{M})$). For any non-singleton simplex $\sigma = \{F_1, F_2, \dots F_k\}$ of $\mathcal{F}(\mathcal{M})$, if ξ is already defined for the boundary of σ, we may assume $F_1 \supset F_2 \supset \dots \supset F_k$. Thus $\rho(F_1) \le r-1$, $\rho(F_2) \le r-2$, $\dots$, $\rho(F_k) \le r-k$, and we have $\eta(\mathcal{C} \upharpoonright (V - F_k)) \ge \rho(\mathcal{M}.(V-F_k)) = r - \rho(F_k) \ge k$. Therefore the function ξ defined on the boundary of σ (homeomorphic to S^{k-2}) can be extended to a continuous function from $||\sigma||$ to $||\mathcal{C} \upharpoonright (V - F_k)||$. □

For a simplicial complex $\mathcal{C}$ let $\beta(\mathcal{C})$ be the baricenteric subdivision of $\mathcal{C}$, i.e., the vertices of $\beta(\mathcal{C})$ are the simplices of $\mathcal{C}$, and the simplices of $\beta(\mathcal{C})$ are all sets $\{s_1, s_2, \dots s_k\}$ that can be ordered in a way that $s_1 \subset s_2 \subset \dots \subset s_k$. The following is a basic result on the baricentric subdivision:

Theorem 5.2. *There exists an homeomorphism $\iota : ||\mathcal{C}|| \to ||\beta(\mathcal{C})||$ such that $supp_{\mathcal{C}}(x)$ contains every simplex in $supp_{\beta(\mathcal{C})}(\iota(x))$.*

Recall the Knaster-Kuratowski- Mazurkiewicz (KKM) Theorem [14]:

Theorem 5.3 (The KKM Theorem)**.** *Let Δ be a simplex and for each vertex v of Δ let A_v be an open subset of $||\Delta||$. Suppose that for every $x \in ||\Delta||$ there exists some v in the support of x such that $x \in A_v$. Then $\bigcap_v A_v$ is not empty.*

Proof of Theorem 4.5.

Let Δ be the simplex whose vertices are all cocircuits of $\mathcal{M}$, let $\iota : ||\Delta|| \to ||\beta(\Delta)||$ be as in Theorem 5.2, and let $\xi : ||\mathcal{F}(\mathcal{M})|| \to ||\mathcal{C}||$ be as in Lemma 5.1. For every vertex s of $\beta(\Delta)$ (i.e., for every set of cocircuits of $\mathcal{M}$) we define $\pi(s) = V \setminus (\bigcup_{D \in s} D)$. Note that $\pi(s)$ is a flat of $\mathcal{M}$ and hence can be regarded as a vertex of $\mathcal{F}(\mathcal{M})$. Also note that if $s_1 \subset s_2$ then $\pi(s_2) \subseteq \pi(s_1)$. Hence using linear continuation we can extend π to be a function from $||\beta(\Delta)||$ to $||\mathcal{F}(\mathcal{M})||$.

Let x be any point in $||\Delta||$ and let $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ be the supports of x, $\iota(x)$, $\pi(\iota(x))$, $\xi(\pi(\iota(x)))$, respectively. We claim that for some $D \in \sigma_1$, the intersection $D \cap \sigma_4$ is not empty. Indeed, there exists $F \in \sigma_3$ with $F \cap \sigma_4 = \emptyset$ and there exists $s \in \sigma_2$

with $\pi(s) = F$. Thus $\sigma_4 \subseteq V \setminus F = \bigcup_{D \in s} D$ and σ_4 has nonempty intersection with some $D \in s$. But $s \subseteq \sigma_1$ and hence the claim is proved.

For each cocircuit D We now define A_D to be the set of all $x \in ||\Delta||$ such that the support of $\xi(\pi(\iota(x)))$ has nonempty intersection with D. By the claim proved above we know that the sets A_D satisfy the conditions of the KKM theorem and hence there exists some $x_0 \in \bigcap_D A_D$. The support of $\xi(\pi(\iota(x_0)))$ is a simplex of $\mathcal{C}$ whose intersection with every cocircuit of $\mathcal{M}$ is non-empty, and hence it contains a base of $\mathcal{M}$. This proves that $[\mathcal{M}, \mathcal{C}]$ is matchable. □

6. Lower bounds on the connectivity of a complex

In order to apply Theorems 4.5 and 4.6 in combinatorial settings, we need combinatorially formulated lower bounds on the connectivity of complexes. For graphic complexes such lower bounds have indeed been found (see e.g. [6, 19, 4, 20]). All known bounds in this case are given in terms of domination parameters. Here are two such parameters: $\tilde{\gamma}(G)$ is the minimal size of a set X such that $\tilde{N}(X) = V(G)$, and $i\gamma(G)$ is the maximum, over all independent sets I in G, of the minimal size of a set X such that $I \subseteq N(X)$. (Note that if G contains an isolated vertex, then $\tilde{\gamma}(G) = \infty$.)

Theorem 6.1. [19]

$$\eta(\mathcal{I}(G)) \geq \tilde{\gamma}(G)/2$$

Theorem 6.2. [6]

$$\eta(\mathcal{I}(G)) \geq i\gamma(G)$$

The above notions of domination can be extended to general complexes. Since of the two theorems only the first is valid in the general case, we shall define here only $\tilde{\gamma}$ for general complexes. A set A in a complex $\mathcal{C}$ is said to *span* a vertex v if there exists a simplex $\sigma \subseteq A$ in $\mathcal{C}$, such that $\sigma + v \notin \mathcal{C}$. Let $\tilde{sp}_{\mathcal{C}}(A)$ denote the set of all vertices spanned by A, and let $sp_{\mathcal{C}}(A) = A \cup \tilde{sp}_{\mathcal{C}}(A)$. Note that this agrees with the definition of spanning in a matroid. Also note that $sp_{\mathcal{I}(G)}(A) = N_G(A)$. The *domination number* $\tilde{\gamma}(\mathcal{C})$ of $\mathcal{C}$ is the minimal size of a set A such that $\tilde{sp}_{\mathcal{C}}(A) = V$.

Theorem 6.3.

$$\eta(\mathcal{C}) \geq \frac{\tilde{\gamma}(\mathcal{C})}{2}$$

To prove this we need the following Lemma from [4]:

Lemma 6.4. *For every positive integer m and every PL-m-sphere S there exists a PL-m+1-ball $\mathcal{B}$ having S as a sub-complex, where*
(1) $||\mathcal{B}||$ is homeomorphic to B^{m+1}, by a homeomorphism mapping $||S||$ to the boundary of the ball.
(2) The vertices in $\mathcal{B}$ not belonging to S can be ordered, as, say, $V(\mathcal{B}) - V(S) = \{x_1, x_2, \ldots, x_k\}$ so that for every x_i there are at most $2m+2$ vertices $v \in V(S) \cup \{x_1, \ldots, x_{i-1}\}$ such that $\{x_i, v\} \in \mathcal{B}$.

Proof of Theorem 6.3.

Write $m = \eta(\mathcal{C}) - 1$ and suppose for contradiction that $\tilde{\gamma}(\mathcal{C}) > 2m+2$. Let S be a PL-m-sphere and let ψ be a simplicial function from S into $\mathcal{C}$. Let $\mathcal{B}$ and $x_1, \ldots, x_k$ be as in the lemma. We now define $\psi(x_i)$ for each $i = 1, \ldots, k$ in its turn to be a vertex of $\mathcal{C}$ not spanned by $\{\psi(v) : v \in V(S) \cup \{x_1, \ldots, x_{i-1}\},\ \{x_i, v\} \in \mathcal{B}\}$. Let σ be a simplex of $\mathcal{B}$ not contained in $V(S)$, and suppose that $\{\psi(v) : v \in \sigma\}$ is a circuit in $\mathcal{C}$. Consider the maximal i with $x_i \in \sigma$. The vertex $\psi(x_i)$ is then spanned by $\{\psi(v) : v \in \sigma - x_i\}$. This contradicts the choice of $\psi(x_i)$. Thus $\{\psi(v) : v \in \sigma\}$ is not a circuit. This is true for all the simplices of $\mathcal{B}$ and hence we can define ψ to be linear on every simplex and get a continuous function from $\mathcal{B}$ to $\mathcal{C}$. This shows that $\mathcal{C}$ is m-connected, which, recalling that $m = \eta(\mathcal{C}) - 1$, yields a contradiction. $\square$

Lower bounds on the connectivity can be obtained also in the case that the complex is the intersection of matroids.

Theorem 6.5. *Let $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_k$ be matroids on the same ground set V. Then*

$$\eta(\mathcal{M}_1 \cap \mathcal{M}_2 \cap \ldots \cap \mathcal{M}_k) \geq \nu(\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_k)/k$$

The case $k = 1$ yields the main statement in Lemma 3.2, namely that in a matroid $\eta \geq \rho$.

The proof uses the following lemma:

Lemma 6.6. *Let $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_k$ be as in the theorem, and let $x \in V$ be a vertex which is not a loop in any of the matroids. Then $\nu(\mathcal{M}_1/x, \mathcal{M}_2/x, \dots, \mathcal{M}_k/x) \geq \nu(\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_k) - k$.*

Proof. Let $I \in \mathcal{M}_1 \cap \mathcal{M}_2 \cap \ldots \cap \mathcal{M}_k$ be of size $\nu(\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_k)$. For each $1 \leq i \leq k$ such that $I + x \notin \mathcal{M}_i$ let C_i be the $\mathcal{M}_i$-circuit contained in $I + x$ and containing x. Choose any $x_i \in C_i \cap I$ for each such i, and let I' be obtained from I by removing all x_i's. Then $I' \in \mathcal{M}_1/x \cap \mathcal{M}_2/x \cap \ldots \cap \mathcal{M}_k/x$, and $|I'| \geq |I| - k = \nu(\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_k) - k$, as desired. $\square$

Proof of Theorem 6.5.

Denote the complex $\mathcal{M}_1 \cap \mathcal{M}_2 \cap \ldots \cap \mathcal{M}_k$ by $\mathcal{H}$. Write $\nu = \nu(\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_k) = \mu(\mathcal{H})$, and choose some $I \in \mathcal{H}$ of size ν. The proof is by induction on $|V| - |I|$. If $I = V$ then $\eta(\mathcal{H}) = \infty$. Otherwise, let x be any vertex outside I. Since $I \subseteq V \setminus \{x\}$, we have $\nu(\mathcal{H} - x) = \nu$, hence by the induction hypothesis $\eta(\mathcal{H} - x) \geq \nu/k$. Note also that $lk_{\mathcal{H}}(x) = \mathcal{M}_1/x \cap \mathcal{M}_2/x \cap \ldots \cap \mathcal{M}_k/x$, and by Lemma 6.6 and the induction hypothesis we have $\eta(lk_{\mathcal{H}}(x)) \geq \nu/k - 1$. This, together with Corollary 2.4, yield the desired inequality. $\square$

In the case of the intersection of two matroids, more can be said:

Theorem 6.7. *Let $\mathcal{M}, \mathcal{N}$ be two matroids on the same ground set and let k be a positive integer. Then $\eta(\mathcal{M} \cap \mathcal{N}) \geq \frac{\nu(\mathcal{M}, \bigvee^k \mathcal{N})}{k+1}$.*

The proof follows the same outline as that of Theorem 6.5, but the lemma used there is replaced by:

Lemma 6.8. *Let $\mathcal{M}, \mathcal{N}, k$ be as in the theorem, let X be a set in $\mathcal{M} \cap \bigvee^k \mathcal{N}$ and let v be any vertex outside X, which is not a loop in any of the matroids. Then there exists a subset Y of X satisfying $|Y| \geq |X| - k - 1$ and $Y \in (\mathcal{M}/v) \cap \bigvee^k (\mathcal{N}/v)$.*

Proof. Write $X = X_1 \cup X_2 \cup \ldots \cup X_k$, where $X_1, X_2, \dots, X_k$ are independent in $\mathcal{N}$. Let v_0 be a vertex in X such that $X - v_0 + v$ is independent in $\mathcal{M}$. For $i = 1, \dots, k$, let v_i be a vertex in X_i so that $X_i - v_i + v$ is independent in $\mathcal{N}$. The set $Y = X - v_0 - v_1 - \ldots - v_k$ has the desired properties. $\square$

Proof of Theorem 6.7.

The proof is by induction on the size of the ground set. Let $X \in \mathcal{M} \cap \bigvee^k \mathcal{N}$ be a set with $|X| = \nu(\mathcal{M}, \bigvee^k \mathcal{N})$. If X is the entire ground set, then $\eta(\mathcal{M} \cap \mathcal{N}) = \eta(\mathcal{N}) \geq \rho(\mathcal{N}) \geq \frac{|X|}{k} > \frac{\nu(\mathcal{M}, \bigvee^k \mathcal{N})}{k+1}$. Otherwise, let v be any vertex outside X. By the induction hypothesis $\eta((\mathcal{M} \cap \mathcal{N}) - v) \geq \frac{|X|}{k+1} = \frac{\nu(\mathcal{M}, \bigvee^k \mathcal{N})}{k+1}$, and by the lemma $\eta(lk_{\mathcal{M}\cap\mathcal{N}}(v)) \geq \frac{|X|-k-1}{k+1} = \frac{\nu(\mathcal{M}, \bigvee^k \mathcal{N})}{k+1} - 1$. Now Corollary 2.4 gives the desired result. □

Another bound on the connectivity of the intersection of matroids that we shall use is given in terms of its distance from the size of the ground set.

Lemma 6.9. *For any two matroids $\mathcal{M}, \mathcal{N}$ on the same ground set V*

$$\eta(\mathcal{M} \cap \mathcal{N}) \geq |V| - \rho(\mathcal{M}^*) - \rho(\mathcal{N}^*)$$

(Thinking of $|V(\mathcal{C})| - \overline{\eta}(\mathcal{C})$ as a measure of the "topological deficiency" of a complex $\mathcal{C}$, and bearing in mind that $\rho(\mathcal{M}^*) = |V| - \rho(\mathcal{M})$, the lemma says that "the deficiency of the intersection of two matroids is no larger than the sum of their deficiencies".)

Proof of Lemma 6.9.

By induction on $n = |V|$. For $n = 0$ there is nothing to prove. Assume that the result is true for all values of $|V|$ smaller than n. Write $\mathcal{C} = \mathcal{M} \cap \mathcal{N}$, and let $p = |V| - \rho(\mathcal{M}^*) - \rho(\mathcal{N}^*)$. Choose any $x \in V$. If x is a co-loop in both $\mathcal{M}$ and $\mathcal{N}$ then $\mathcal{C} = (\mathcal{C} - x) * \{x\}$, and hence, by Lemma 2.2, $\eta(\mathcal{C}) = \infty$. Thus we may assume that x is not a co-loop in one of the two matroids, say $\mathcal{M}$. Then $\rho((\mathcal{M}-x)^*) = \rho(\mathcal{M}^*/x) = \rho(\mathcal{M}^*)-1$, while $\rho((\mathcal{N}-x)^*) \leq \rho(\mathcal{N}^*)$. By the induction hypothesis $|V(\mathcal{C} - x)| - \eta(\mathcal{C} - x) \leq \rho((\mathcal{M} - x)^*) + \rho((\mathcal{N} - x)^*) < \rho(\mathcal{M}^*) + \rho(\mathcal{N}^*)$, implying that $\eta(\mathcal{C} - x) \geq p$. By the induction hypothesis, $\eta(lk_{\mathcal{C}}(x)) = \eta(\mathcal{M}/x \cap \mathcal{N}/x) \geq |V \setminus \{x\}| - \rho((\mathcal{M}/x)^*) - \rho((\mathcal{N}/x)^*) \geq p - 1$. Applying Corollary 2.4, we get $\eta(\mathcal{C}) \geq p$, as desired. □

7. Ryser's conjecture for matroids

An r-uniform hypergraph H is said to be *r-partite* if $V(H)$ is the disjoint union of sets $U_1, \ldots, U_r$ and each edge of H meets each U_i at exactly one vertex. A famous conjecture of Ryser from the early seventies states that in an r-partite hypergraph $\tau \leq (r-1)\nu$. If true, it would be a generalization of Kőnig's theorem. The case $r = 3$ of the conjecture was proved in [1].

It is possible to generalize Ryser's conjecture to matroids in the same way that Edmonds' theorem generalizes Kőnig's theorem:

Conjecture 7.1. *For any family of matroids $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_r$ on the same ground set we have*

$$\tau(\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_r) \leq (r-1)\nu(\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_r)$$

Ryser's conjecture has a stronger version, made at about the same time (and independently) by Lovász: in an r-partite hypergraph there exist $r-1$ vertices whose removal reduces ν by at least 1. The corresponding matroidal conjecture is:

Conjecture 7.2. *Given matroids $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_r$ on the same ground set V there exist (possibly empty) subsets $X_1, X_2, \ldots, X_r$ of V such that $\sum |X_i| \leq r-1$ and $\nu(\mathcal{M}_1/X_1, \mathcal{M}_2/X_2, \ldots, \mathcal{M}_r/X_r) < \nu(\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_r)$.*

In fact, we suspect that Conjecture 7.2 is true in a much stronger form: we can require that either (1) all sets X_i, apart from one (which is empty), are the same singleton, or (2) only one set X_i is non-empty.

Here we shall prove Conjecture 7.1 for the case $r = 3$, namley:

Theorem 7.3. *Any three matroids $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ on the same ground set V satisfy*

$$\tau(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3) \leq 2\nu(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$$

Proof. Let $\mathcal{C} = \mathcal{M}_2 \cap \mathcal{M}_3$. Then $\nu(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3) = \nu(\mathcal{M}_1, \mathcal{C})$. By Theorem 4.6 there exists a subset X of V such that $\nu(\mathcal{M}_1, \mathcal{C}) \geq \rho_{\mathcal{M}_1}(X) + \eta(\mathcal{C} \restriction (V \setminus X))$. By Theorem 6.5 we have $\eta(\mathcal{C} \restriction (V \setminus X)) \geq \frac{1}{2}\nu((\mathcal{M}_2 \restriction (V \setminus X)), (\mathcal{M}_3 \restriction (V \setminus X)))$. By Theorem 4.3 $\nu((\mathcal{M}_2 \restriction (V \setminus X)), (\mathcal{M}_3 \restriction (V \setminus X))) = \tau(\mathcal{M}_2 \restriction (V \setminus X), \mathcal{M}_3 \restriction (V \setminus X))$, meaning that there exists a subset Y of $V \setminus X$ such that $\nu((\mathcal{M}_2 \restriction (V \setminus X)), (\mathcal{M}_3 \restriction (V \setminus X))) = \rho_{\mathcal{M}_2}(Y) + \rho_{\mathcal{M}_3}(V \setminus X \setminus Y))$. Combining all these we get $\nu(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3) \geq \rho_{\mathcal{M}_1}(X) + \frac{1}{2}(\rho_{\mathcal{M}_2}(Y) + \rho_{\mathcal{M}_3}(V \setminus X \setminus Y)) \geq \frac{1}{2}\tau(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$, as desired. □

8. The expansion number and chromatic number of a complex

As from this section, the focus of our attention will shift to "coloring-type" problems, namely problems concering the decomposition of the ground set of a complex into simplices belonging to the complex. For this purpose, we start by defining two related notions. The first of these is:

Definition 8.1. *The* expansion number $\Delta(\mathcal{C})$ *of a simplicial complex $\mathcal{C}$ is the maximum, over all sets A of vertices, of $\frac{|A|}{\eta(\mathcal{C} \restriction A)}$.*

Thus for a matroid $\mathcal{M}$ we have $\Delta(\mathcal{M}) = \max_{A \in \mathcal{M}} \frac{|sp_{\mathcal{M}}(A)|}{|A|}$. For a partition matroid, this is just the largest size of a part in the partition.

The following trivial observation will be used later.

Observation 8.2. *Let $\mathcal{C}$ be a simplicial complex, let $d > \frac{|V(\mathcal{C})|}{\Delta(\mathcal{C})}$ be an integer and let $\mathcal{U}$ be the d-uniform matroid. Then $\Delta(\mathcal{C} \cap \mathcal{U}) = \Delta(\mathcal{C})$.*

Let us note a simple corollary of Theorem 4.6:

Theorem 8.3. *Every pair $\mathcal{M}, \mathcal{C}$ of a matroid and a complex on the same ground set V satisfies $\nu(\mathcal{M}, \mathcal{C}) \geq \frac{|V|}{\max(\Delta(\mathcal{M}), \Delta(\mathcal{C}))}$.*

Proof. By Theorem 4.6 there exists a set $I \in \mathcal{M} \cap \mathcal{C}$ and a subset X of V such that $|I| \geq \rho(\mathcal{M} \restriction X) + \eta(\mathcal{C} \restriction (V \setminus X))$. By the definition of Δ for matroids and complexes, we then have: $|I \cap X| \geq \frac{|X|}{\Delta(\mathcal{M})}$ and $|I \setminus X| \geq \frac{|V \setminus X|}{\Delta(\mathcal{C})}$. □

.

Lemma 8.4. *A graphic complex $\mathcal{I}(G)$ satisfies $\Delta(G) + 1 \leq \Delta(\mathcal{I}(G)) \leq 2\Delta(G)$.*

Proof. To prove the left inequality, let v be a vertex of G of degree $\Delta(G)$, and let $A = N(v)$. Then G induces on A a complete bipartite graph, meaning that $\mathcal{I}(G) \restriction A$ is not connected, namely $\eta(\mathcal{I}(G) \restriction A) = 1$. Thus $\Delta(\mathcal{I}(G)) \leq \frac{|A|}{\eta(\mathcal{I}(G) \restriction A)} = \Delta + 1$.

The right inequality follows by applying Theorem 6.1 to the induced subgraphs of G, using the observation that $\tilde{\gamma}(G) \geq \frac{|V(G)|}{\Delta(G)}$. □

The second notion we study in this section is that of the chromatic number of a complex:

Definition 8.5. *The* chromatic number $\chi(\mathcal{C})$ *of a simplicial complex* $\mathcal{C}$ *is the smallest number of sets belonging to* $\mathcal{C}$ *whose union is the ground set of the complex.*

Note that $\chi(\mathcal{C})$ may be infinite. Like previous notation, this one is borrowed from graphs, since for a graph G we have $\chi(\mathcal{I}(G)) = \chi(G)$. (Remark: this parameter is sometimes denoted in the literature by $\rho(\mathcal{C})$.)

A theorem of Edmonds ([10], see also [22], Section 8.4) states that in a matroid $\chi = \lceil \Delta \rceil$. The generalization to simplicial complexes is a straightforward corollary of Theorem 4.2:

Corollary 8.6. *In any simplicial complex* $\mathcal{C}$

$$\chi(\mathcal{C}) \leq \lceil \Delta(\mathcal{C}) \rceil$$

Proof. Apply Theorem 4.2 to the graph $\Gamma(\mathcal{C}^m)$, where $m = \lceil \Delta(\mathcal{C}) \rceil$. □

By combining Observation 8.2, Lemma 8.4 and Corollary 8.6 we get the following weakened version of the Hajnal-Szemerédi theorem:

Theorem 8.7. *Let G be a graph with n vertices and maximal degree Δ. Then G has a legal 2Δ-coloring (i.e., a partition to 2Δ disjoint sets), where each color includes at most $\lceil \frac{n}{2\Delta} \rceil$ vertices. If G is chordal then G has a legal $\Delta + 1$-coloring where each color includes at most $\lceil \frac{n}{\Delta+1} \rceil$ vertices.*

Kőnig's line-coloring theorem [15] says that the chromatic index (that is, the edge-chromatic number) of a bipartite graph is equal to the maximal degree of a vertex in the graph. As already mentioned, a bipartite graph induces two partition matroids on its edge set, and hence the theorem can be formulated as follows:

Theorem 8.8. *If $\mathcal{M}, \mathcal{N}$ are partition matroids, then* $\chi(\mathcal{M} \cap \mathcal{N}) = \max(\Delta(\mathcal{M}), \Delta(\mathcal{N}))$.

This theorem is true not only for partition matroids but also for a larger class of matroids called *strongly base orderable* matroids [8]. For general matroids, however, the theorem fails, as the following example shows (cf. [21], Section 42.6c): $\mathcal{M}$ is the graphic matroid on the edge set of K_4, and $\mathcal{N}$ is the partition matroid on this set whose parts are the three pairs of non-adjacent edges. Then $\max(\Delta(\mathcal{M}), \Delta(\mathcal{N})) = 2$, while $\chi(\mathcal{M} \cap \mathcal{N}) = 3$.

Still, we can prove a bound on $\chi(\mathcal{M} \cap \mathcal{N})$:

Theorem 8.9. *For matroids $\mathcal{M}, \mathcal{N}$ on the same ground set*

$$\chi(\mathcal{M} \cap \mathcal{N}) \leq 2\max(\Delta(\mathcal{M}), \Delta(\mathcal{N}))$$

Proof. Write $\mathcal{D} = \mathcal{M} \cap \mathcal{N}$, and $m = \max(\Delta(\mathcal{M}), \Delta(\mathcal{N}))$. By Theorem 8.3, for every subset X of V we have $\nu(\mathcal{D} \restriction X) \geq \frac{|X|}{\max(\Delta(\mathcal{N} \restriction X), \Delta(\mathcal{M} \restriction X))} \geq \frac{|X|}{m}$. By Theorem 6.5 $\eta(\mathcal{D} \restriction X) \geq \nu(\mathcal{D} \restriction X)/2$. Thus $\eta(\mathcal{D} \restriction X) \geq \frac{|X|}{2m}$ for every $X \subseteq V$, meaning that $\Delta(\mathcal{D}) \leq 2m$. The theorem now follows by Corollary 8.6. □

A well known conjecture of Rota (see, e.g., [24]) can be stated in our terminology as follows:

Conjecture 8.10. *Let $\mathcal{M}$ be a matroid of rank r, and $\mathcal{N}$ a partition matroid on the same ground set, having r parts, each of which is a base of $\mathcal{M}$. Then $\chi(\mathcal{M} \cap \mathcal{N}) = r$.*

Assertion 8.11. *Under the conditions of the conjecture* $\chi(\mathcal{M} \cap \mathcal{N}) \le 2r$.

Proof. By the assumption of the conjecture, $\Delta(\mathcal{M}) = \Delta(\mathcal{N}) = r$, and thus the assertion follows from Theorem 8.9 □

Conjecture 8.12. *For any two matroids* $\mathcal{M}$, $\mathcal{N}$ *sharing the same ground set it is true that* $\chi(\mathcal{M} \cap \mathcal{N}) \le \Delta(\mathcal{M}) + \Delta(\mathcal{N})$.

We prove this conjecture only in the case that one of $\Delta(\mathcal{M}), \Delta(\mathcal{N})$ is an integer multiple of the other. This is a direct corollary of the following:

Theorem 8.13. *Let* $\mathcal{M}, \mathcal{N}$ *be two matroids on the same ground set and let* p, q *be two positive integer numbers. If* $\Delta(\mathcal{M}) \le p$ *and* $\Delta(\mathcal{N}) \le pq$ *then* $\chi(\mathcal{M} \cap \mathcal{N}) \le p + pq$.

Proof. Since $\chi(\mathcal{N}) \le pq$ we get $\chi(\bigvee^q \mathcal{N}) \le p$, and thus also $\Delta(\bigvee^q \mathcal{N}) \le p$. By the definitions of Δ and τ, this implies that $\tau(\mathcal{M} \restriction X, \bigvee^q \mathcal{N} \restriction X) \ge \frac{|X|}{p}$ for any subset X of the ground set, which by Theorem 4.3 means that $\nu(\mathcal{M} \restriction X, \bigvee^q \mathcal{N} \restriction X) \ge \frac{|X|}{p}$. By Theorem 6.7 this implies that $\eta(\mathcal{M} \restriction X \cap \mathcal{N} \restriction X) \ge \frac{|X|}{p(q+1)}$. By the definition of Δ of a complex, this means that $\Delta(\mathcal{M} \cap \mathcal{N}) \le p(q+1)$. The result now follows by Corollary 8.6. □

9. Weighted matroids

The proof of certain fractional results that are to follow will use the notion of *weighted matroids*. Let $\mathcal{M}$ be a matroid on the ground set V, and let $w : V \to \mathbb{R}^+$ be a function on V. For $A \subseteq V$ we write $w[A] = \sum_{a \in A} w(a)$. Also define $|w| = w[V]$. The *span function* $w^{\mathcal{M}}$ of w is defined by $w^{\mathcal{M}}(x) = \max_Y \{\min_{y \in Y} w(y) : x \in sp_{\mathcal{M}}(Y)\}$. Note that the span of the characteristic function of a set is the characteristic function of the span of that set. Clearly:

Assertion 9.1. $w^{\mathcal{M}}(x) \ge k$ *if and only if* $x \in sp_{\mathcal{M}}(\{y : w(y) \ge k\})$

This implies the following inequality:

Lemma 9.2. $|w^{\mathcal{M}}| \le \Delta(\mathcal{M})|w|$

Proof. Let $0 \le a_1 < a_2 < \ldots < a_k$ be the values in the image of w and let $a_0 = 0$. Then

$$|w| = \sum_{i=1}^{k}(a_i - a_{i-1})|\{x : w(x) \ge a_i\}| \ge \frac{1}{\Delta(\mathcal{M})}\sum_{i=1}^{k}(a_i - a_{i-1})|sp\{x : w(x) \ge a_i\}| = \frac{|w^{\mathcal{M}}|}{\Delta(\mathcal{M})}$$

□

Denote by $\mathcal{M}_w$ the matroid whose bases are the bases B of $\mathcal{M}$ for which $w[B]$ is maximal. It is easy to see that $\mathcal{M}_w$ is indeed a matroid. In fact,

$$\mathcal{M}_w = \bigoplus_{k \in image(w)} \mathcal{M}/\{v : f(v) > k\} \setminus \{v : f(v) < k\}$$

Here $\bigoplus$ of matroids denotes their direct sum, which in terms of complexes is just their join.

The following lemma links the circuits in $\mathcal{M}$ with those in $\mathcal{M}_w$:

Lemma 9.3. *The set of elements of minimal* w*-value in a given circuit in* $\mathcal{M}$ *is the union of circuits in* $\mathcal{M}_w$.

Proof. Let C be a circuit in $\mathcal{M}$, and let D be the set of elements of minimal w-value in C. We have to show that every $d \in D$ lies in some circuit of $\mathcal{M}_w$ contained in D. Suppose for contradiction that there is no such circuit. Let A be a maximal subset of $D - d$ independent in $\mathcal{M}_w$. If $A + d$ contains a circuit of $\mathcal{M}_w$ then d is in the circuit, and we are done. Thus we may assume that $A + d$ is independent in $\mathcal{M}_w$. Let B be a base of $\mathcal{M}_w$ containing $A + d$. By the definition of $\mathcal{M}_w$, the set B is also a base of $\mathcal{M}$. Now observe that $B - d$ cannot span $C - d$, because $C - d$ spans d while $B - d$ does not span d. Let c be an element of $C - d$ not spaned by $B - d$. This means that $B - d + c$ is a base of $\mathcal{M}$. But $w(c) \geq w(d)$ and therefore $w[B - d + c] \geq w[B]$, and by the maximality of $w[B]$ we have $w(c) = w(d)$. Thus $B - d + c$ must be a base of $\mathcal{M}_w$ as well and $c \in D$. But then $A + c$ is a subset of $D - d$ independent in $\mathcal{M}_w$. This contradicts the maximality of A. □

Given a weight function w on V and a pair $\mathcal{M}, \mathcal{N}$ of matroids on V, we write $\nu_w(\mathcal{M}, \mathcal{N}) = \max\{w[I] : I \in \mathcal{M} \cap \mathcal{N}\}$ and $\tau_w(\mathcal{M}, \mathcal{N}) = \min\{|f| + |g| : f^{\mathcal{M}} + g^{\mathcal{N}} \geq w\}$.

Edmonds' theorem can be generalized to weighted matroids [13] :

Theorem 9.4. $\tau_w = \nu_w$

10. Fractional colorability

The coloring number χ of a complex has a fractional version. A function $f : \mathcal{C} \to \mathbb{R}^+$ is a *fractional coloring* of V in $\mathcal{C}$ if $\sum_{\sigma \ni v} f(\sigma) \geq 1$ for every vertex v. We write $\chi^*(\mathcal{C})$ for the minimum of $\sum_{\sigma \in \mathcal{C}} f(\sigma)$ over all fractional colorings f in $\mathcal{C}$. By LP-duality $\chi^*(\mathcal{C})$ is equal to the the maximum over all functions $w : V \to \mathbb{R}^+$ ($w \not\equiv 0$) of

$$\frac{\sum_{v \in V} w(v)}{\max_{\sigma \in \mathcal{C}} \sum_{u \in \sigma} w(u)} .$$

Theorem 10.1. *If $\Delta(\mathcal{C}) = \frac{p}{q}$, where p, q are integers, then there exist (not necessarily distinct) simplices $\sigma_1, \ldots, \sigma_p \in \mathcal{C}$, such that every vertex belongs to q simplices σ_i.*

Proof. Let $\mathcal{D} = \mathcal{C}^p$ (the join of $\mathcal{C}$ with itself p times), and let $\mathcal{P}$ be the partition matroid on $V(\mathcal{D})$, whose parts are the copies of each vertex $v \in V(\mathcal{C})$. Let $\mathcal{N} = \bigvee^q \mathcal{P}$, meaning that a set belongs to $\mathcal{N}$ if it contains at most q copies of each vertex $v \in V(\mathcal{C})$. Then $\Delta(\mathcal{N}) = \frac{p}{q}$, and also $\Delta(\mathcal{D}) = \Delta(\mathcal{C}) = \frac{p}{q}$. By Theorem 8.3 it follows that $\nu(\mathcal{D}, \mathcal{N}) \geq p|V(\mathcal{C})|/\frac{p}{q} = q|V(\mathcal{C})| = \rho(\mathcal{N})$. Thus there exists a base B of $\mathcal{N}$ belonging to $\mathcal{D}$. The fact that B is a base of $\mathcal{N}$ means that it contains q copies of each vertex v, and the fact that it belongs to $\mathcal{D}$ means that it is the union of p copies of (not necessarily distinct) simplices in $\mathcal{C}$, as required. □

Corollary 10.2. *For any simplicial complex $\mathcal{C}$*

$$\chi^*(\mathcal{C}) \leq \Delta(\mathcal{C})$$

Proof. Write $\Delta(\mathcal{C}) = \frac{p}{q}$. Let $\sigma_1, \ldots, \sigma_p$ be as in the conclusion of the theorem. Putting weight $\frac{1}{q}$ on each of these simplices yields a fractional coloring of total weight $\frac{p}{q}$. □

Corollary 10.3. $\chi^*(\mathcal{M}) = \Delta(\mathcal{M})$ *for any matroid $\mathcal{M}$.*

Proof. In view of Corollary 10.2 it is only necessary to show that $\chi^*(\mathcal{M}) \geq \Delta(\mathcal{M})$. Let A be a set such that $\frac{|sp_{\mathcal{M}}(A)|}{|A|} = \Delta(\mathcal{M})$. Every $\sigma \in \mathcal{M}$ contained in A is then of size at most $\frac{|A|}{\Delta(\mathcal{M})}$, which means that the total weight put on such sets needed to cover each element of A with weight at least 1 is at least $\Delta(\mathcal{M})$. □

Next we use the results on weighted matchings from the previous section to prove a fractional version of Theorem 8.8 for general matroids (which is at the same time a strengthening of Corollary 10.3).

Theorem 10.4. *For any two matroids $\mathcal{M}, \mathcal{N}$ it is true that $\chi^*(\mathcal{M} \cap \mathcal{N}) = \max(\Delta(\mathcal{M}), \Delta(\mathcal{N}))$.*

Proof. The inequality $\chi^*(\mathcal{M} \cap \mathcal{N}) \geq \max(\Delta(\mathcal{M}), \Delta(\mathcal{N}))$ was shown in the proof of Corollary 10.3 . Assume for contradiction that $\chi^*(\mathcal{M} \cap \mathcal{N}) > \max(\Delta(\mathcal{M}), \Delta(\mathcal{N}))$. Then for some function $w : V \to \mathbb{R}^+$ ($w \not\equiv 0$) we have

$$\frac{|w|}{\max_{\sigma \in \mathcal{M} \cap \mathcal{N}} w[\sigma]} > \max(\Delta(\mathcal{M}), \Delta(\mathcal{N})) .$$

But $\max_{\sigma \in \mathcal{M} \cap \mathcal{N}} w[\sigma] = \nu_w(\mathcal{M}, \mathcal{N}) = \tau_w(\mathcal{M}, \mathcal{N})$. Thus there exist two functions $f, g : V \to \mathbb{R}^+$ with $f^{\mathcal{M}} + g^{\mathcal{N}} \geq w$ and $(|f| + |g|)\max(\Delta(\mathcal{M}), \Delta(\mathcal{N})) < |w|$. By Lemma 9.2 we have $|w| \leq |f^{\mathcal{M}}| + |g^{\mathcal{N}}| \leq |f|\Delta(\mathcal{M}) + |g|\Delta(\mathcal{N}) \leq (|f| + |g|)\max(\Delta(\mathcal{M}), \Delta(\mathcal{N})) < |w|$. A contradiction. □

11. Packing jointly spanning sets

Given a pair of matroids $\mathcal{M}, \mathcal{N}$ on the same ground set V, a dual problem to that of covering the ground set by sets in $\mathcal{M} \cap \mathcal{N}$ is that of packing sets which are spanning in both $\mathcal{M}$ and $\mathcal{N}$.

Let us start with one matroid. For a given matroid $\mathcal{M}$ write $\delta(\mathcal{M}) = \min\{\frac{|A|}{\rho(\mathcal{M}.A)} : A \subseteq V\}$. In a partition matroid, this is the minimal size of a part in the partition (this is also the origin of the notation: in the partition matroid $\mathcal{P}$ on the edge set of a bipartite graph, induced by the vertices in one side, $\delta(\mathcal{P})$ is the minimal degree of a vertex in that side). Also let $\pi(\mathcal{M})$ be the maximal number of disjoint bases of $\mathcal{M}$. For a basis B of $\mathcal{M}$ and a set A of vertices it is true that $\rho(\mathcal{M}.A) \leq |A \cap B|$. Hence there cannot be more than $\frac{|A|}{\rho(\mathcal{M}.A)}$ disjoint bases in $\mathcal{M}$, meaning that $\pi(\mathcal{M}) \leq \delta(\mathcal{M})$. Edmonds [11] proved that, up to integrality, equality obtains:

Theorem 11.1. $\pi(\mathcal{M}) = \lfloor \delta(\mathcal{M}) \rfloor$.

The parameters Δ and δ of a matroid are related by:

Observation 11.2. $\frac{1}{\delta(\mathcal{M})} + \frac{1}{\Delta(\mathcal{M}^*)} = 1$.

A simple special case of Corollary 4.7 can be formulated using the parameter δ as follows:

Theorem 11.3. *Let $\Pi = [\mathcal{M}, \mathcal{C}]$ be a pair of a matroid and a complex on the same ground set V. If $\delta(\mathcal{M}) \geq \Delta(\mathcal{C})$ then Π is matchable.*

Let $\mathcal{M}$, $\mathcal{N}$ be two matroids on the same ground set V. Define $\pi(\mathcal{M}, \mathcal{N})$ to be the maximal number of disjoint sets which are spanning in both $\mathcal{M}$ and $\mathcal{N}$. Obviously, $\pi(\mathcal{M}, \mathcal{N}) \leq \min\{\lfloor \delta(\mathcal{M}) \rfloor, \lfloor \delta(\mathcal{N}) \rfloor\}$. In general, equality does not obtain here. To see this, take the same example given for the dual parameter χ, appearing after Theorem 8.8. In that example $\pi(\mathcal{M}) = \pi(\mathcal{N}) = 2$, while $\pi(\mathcal{M}, \mathcal{N}) = 1$.

But we can prove:

Theorem 11.4. $\pi(\mathcal{M},\mathcal{N}) \geq \min\{\lfloor \frac{1}{2}\delta(\mathcal{M})\rfloor, \lfloor \frac{1}{2}\delta(\mathcal{N})\rfloor\}$

Proof. It suffices to show that if $\delta(\mathcal{M}) \geq 2k$ and $\delta(\mathcal{N}) \geq 2k$ then $\pi(\mathcal{M},\mathcal{N}) \geq k$. Recall that a set is spanning in $\mathcal{M}$ if and only if its complement belongs to $\mathcal{M}^*$. Hence, what we have to show is equivalent to the existence of k sets $C_1,\dots,C_k$ in $\mathcal{C} \triangleq \mathcal{M}^* \cap \mathcal{N}^*$ whose union covers every element at least $k-1$ times - then $V \setminus C_i$ are k disjoint sets, each spanning in both $\mathcal{M}$ and $\mathcal{N}$. By Theorem 10.1 it suffices to show that $\Delta(\mathcal{C}) \leq \frac{k}{k-1}$. Namely, we have to show that $\eta(\mathcal{C} \restriction X) \geq |X|\frac{k-1}{k}$ for every subset X of V. By assumption, $\rho(\mathcal{M}.X) \leq \frac{|X|}{2k}$, meaning that $\rho(\mathcal{M}^* \restriction X) \geq |X| - \frac{|X|}{2k}$. Similarly, $\rho(\mathcal{N}^* \restriction X) \geq |X| - \frac{|X|}{2k}$. Hence, by Lemma 6.9 $\eta(\mathcal{C}) \geq |X| - \frac{|X|}{k}$, as required. □

The packing number π of two matroids has a fractional version too. Let $\mathcal{M},\mathcal{N}$ be two matroids and let $\mathcal{H}$ be the set of sets spanning in both matroids. In other words, $\mathcal{H}$ is the set of the complements of the sets in $\mathcal{M}^* \cap \mathcal{N}^*$. A function $f: \mathcal{H} \to \mathbb{R}^+$ is a *fractional packing* of $\mathcal{M}$ and $\mathcal{N}$ if $\sum_{X \ni v} f(X) \leq 1$ for every vertex v. We write $\pi^*(\mathcal{M},\mathcal{N})$ for the maximum of $\sum_{X\in\mathcal{H}} f(X)$ over all fractional packings f of $\mathcal{M}$ and $\mathcal{N}$. By LP-duality $\pi^*(\mathcal{M},\mathcal{N})$ is equal to the the minimum over all functions $w: V \to \mathbb{R}^+$ ($w \not\equiv 0$) of

$$\frac{|w|}{\min_{X\in\mathcal{H}} w[X]} .$$

Theorem 11.5. *For any two matroids $\mathcal{M},\mathcal{N}$ it is true that $\pi^*(\mathcal{M},\mathcal{N}) = \min(\delta(\mathcal{M}),\delta(\mathcal{N}))$.*

Proof. Let $\mathcal{H}$ be as above and let $k = \min(\delta(\mathcal{M}),\delta(\mathcal{N}))$. By Observation 11.2 we have $\Delta(\mathcal{M}^*) \leq \frac{k}{k-1}$ and $\Delta(\mathcal{N}^*) \leq \frac{k}{k-1}$ The inequality $\pi^*(\mathcal{M},\mathcal{N}) \geq k$ can be shown in a way similar to the proof of Corollary 10.3 . Assume for contradiction that $\pi^*(\mathcal{M},\mathcal{N}) > k$. Then for some function $w: V \to \mathbb{R}^+$ ($w \not\equiv 0$) we have

$$\frac{|w|}{\min_{X\in\mathcal{H}} w[X]} > k .$$

But

$$\min_{X\in\mathcal{H}} w[X] = \min_{I\in\mathcal{M}^*\cap\mathcal{N}^*}(|w| - w[I]) = |w| - \nu_w(\mathcal{M}^*,\mathcal{N}^*) = |w| - \tau_w(\mathcal{M}^*,\mathcal{N}^*) .$$

Thus there exist two functions $f,g: V \to \mathbb{R}^+$ with $f^{\mathcal{M}^*} + g^{\mathcal{N}^*} \geq w$ and $\frac{|w|}{|w|-(|f|+|g|)} < k$. In other words, $(|f|+|g|)\frac{k}{k-1} < |w|$. By Lemma 9.2 we have $|w| \leq |f^{\mathcal{M}^*}| + |g^{\mathcal{N}^*}| \leq |f|\Delta(\mathcal{M}^*) + |g|\Delta(\mathcal{N}^*) \leq (|f|+|g|)\frac{k}{k-1} < |w|$. A contradiction. □

12. DeVos' Stable Base Problem

DeVos [9] proved the following:

Theorem 12.1. *Let G be a graph with maximal degree Δ and let $\mathcal{M}$ be a matroid with $2^{\Delta+1}$ disjoint bases. Then there exists a base in $\mathcal{M}$ which is an independent set in G.*

DeVos asked for the minimal number $f(\Delta)$ that can be used in the above theorem instead of $2^{\Delta+1}$.

Using the methods of this chapter we can easily solve this problem.

Theorem 12.2. *Let G be a graph with maximal degree Δ and let $\mathcal{M}$ be a matroid with 2Δ disjoint bases. Then there exists a base in $\mathcal{M}$ which is an independent set in G.*

Proof. We have

$$\delta(\mathcal{M}) \geq \pi(\mathcal{M}) \geq 2\Delta \geq \Delta(\mathcal{I}(G)) .$$

The theorem now follows from Theorem 11.3. □

Note that $\pi(\mathcal{M}) \geq 2\Delta - 1$ deos not suffice, as is shown by an example of Yuster in [25].

13. Matroidal colorability

A graph is called *strongly k-colorable* if for every partition of its vertex set into sets $V_1, V_2, \ldots, V_m$ of size at most k there exists a coloring of G with k colors, such that every color class meets each V_i in at most one point. A conjecture that has been hanging around for a few years now (it first appeared in writing in [3], but was known before to many who worked in the subject) is that every graph G is strongly $2\Delta(G)$-colorable. (For partial results see [18, 3].)

In our terminology, the conjecture states that for any partition matroid $\mathcal{P}$ on $V = V(G)$ with $\Delta(\mathcal{P}) \leq 2\Delta(G)$ it is true that $\chi(\mathcal{P} \cap \mathcal{I}(G)) \leq 2\Delta(G)$. This can be extended to any matroid. Call a complex $\mathcal{C}$ *matroidally k-colorable* if for every matroid $\mathcal{M}$ on its vertex set with $\Delta(\mathcal{M}) \leq k$ it is true that $\chi(\mathcal{M} \cap \mathcal{C}) \leq k$. We say that a graph G is matroidally k-colorable if $\mathcal{I}(G)$ is matroidally k-colorable.

Conjecture 13.1. *Every graph G is matroidally $2\Delta(G)$-colorable.*

Following [3], in which it was proved that every graph G is fractionally strongly $2\Delta(G)$-colorable, we prove in this section the fractional version of Conjecture 13.1.

For this purpose we shall need some terminology. Consider a graph G and a matroid $\mathcal{M}$ on the same ground set V. Call sets in $\mathcal{M} \cap \mathcal{I}(G)$ *bi-independent*. A pair (D, I) of subsets of V is said to *cover* a vertex $v \in V$ if $v \in sp_{\mathcal{M}}(I) \cup \tilde{N}(D)$. We say that (D, I) covers V if it covers all vertices in V.

¿From Theorems 4.6 and 6.1 there follows:

Corollary 13.2. *Let $G = (V, E)$ be a graph and let $\mathcal{M}$ be a matroid of rank r on V. Suppose that $[\mathcal{M}, \mathcal{C}]$ is not matchable. Then there exist a pair (D, I) covering V such that $\frac{|D|}{2} + |I| < r$*

In fact, using combinatorial methods we can prove a stronger result:

Theorem 13.3. *Let $G = (V, E)$ be a graph and let $\mathcal{M}$ be a matroid on V. Suppose that $[\mathcal{M}, \mathcal{C}]$ is not matchable. Then there exist sets $X, Y, I \subseteq V$ such that*
(1) $(X \cup Y, I)$ covers V
(2) $Y \cup I$ is independent in $\mathcal{M}$
(3) $G[X \cup Y]$ is a union of disjoint stars, where X is the set of centers of the stars and Y is the set of rays. (In particular, $|X| \leq |Y|$)

Proof. Let J be a bi-independent set of maximal cardinality. By assumption, J does not span V. Choose any vertex x_1 not spanned by J. Among all bi-independent sets spanning $sp_{\mathcal{M}}(J)$ choose one, say R_1, such that x_1 has a minimal number of neighbors in R_1. Let Y_1 be $\tilde{N}(x) \cup R$. Clearly, Y_1 is non-empty, or else $R_1 \cup \{x\}$ would be bi-independent, contradicting the maximality property of J.

Let $D_1 = \{x_1\} \cup Y_1$. If $(D_1, R_1 - Y_1)$ covers V then we are done. Thus we may assume that there exists a vertex x_2 not covered by $(D_1, R_1 - Y_1)$. Since $G[D_1]$ does not contain isolated vertices, $x_2 \notin D_1$.

Among all bi-independent sets R_2 such that $Y_1 \subseteq R_2$ and $sp_{\mathcal{M}}(R_2 - Y_1) = sp_{\mathcal{M}}(R_1 - Y_1)$ pick one in which x_2 has a minimal number of neighbors. Denote by Y_2 the set of neighbors of x_2 in R_2. Suppose that $Y_2 = \emptyset$. If $x_2 \notin sp_{\mathcal{M}}(R_2)$ then $R_2 \cup \{x_2\}$ is bi-independent, contradicting the maximality of J. If $x_2 \in sp_{\mathcal{M}}(R_2)$ then the circuit containing x_2 in $R_2 + x_2$ must contain some $y \in Y_1$. Then the set $R_2 - x_2 + y$ is bi-independent and x_1 has fewer neighbors in it than in R_1, contradicting the minimality property of R_1. Thus Y_2 can be assumed to be non-empty.

Let $D_2 = \{x_1, x_2\} \cup Y_1 \cup Y_2$. If $(D_2, R_2 - Y_1 - Y_2)$ covers V, then the theorem is proved.

Thus we may assume that there exists a vertex x_3 not covered by $(D_2, R_2 - Y_1 - Y_2)$. The argument now goes as before. Since our setting is finite, this process must terminate at some point, yielding the theorem. $\square$

Another element needed for the proof of the fractional version of Conjecture 13.1 is a weighted version of Theorem 13.3, namely a version in which the elements of V carry weights. Let $G = (V, E)$ be a graph, let $\mathcal{M}$ be a matroid on V and let g, f, w be three functions from V to $\mathbb{R}^+$. We say that the pair (g, f) *dominates* w if $g^{\mathcal{M}}(v) + f[\tilde{N}(v)] \geq w(v)$ for every $v \in V$. For such a pair of functions (not necessarily dominating) we write

$$|(g, f)| = |g| + |f|/2.$$

By ν_w we denote the maximal value of $w[B]$ over all bi-independent bases B, and by τ_w the minimal value of $|(g, f)|$, over all pairs (g, f) dominating w.

Theorem 13.4. *$\tau_w \leq \nu_w$. Furthermore, it is possible to find a dominating pair of functions (g, f) of weight at most ν_w such that*
(*) $f(x) + g(x) \leq w(x)$ for every vertex x.
If all weights $w(v)$ are integral, then g, f can be assumed to be integral as well.

Proof. The case of rational weights, and hence by approximation arguments also that of general real-valued weights, can be reduced to the case of integral weights. Hence we shall assume integrality of the weights.

The proof is by induction on $w[V]$. When this sum is 0, there is nothing to prove. So, we assume that the theorem is true whenever $w[V] < s$, and prove it when $w[V] = s$.

We consider the matroid $\mathcal{M}_w$. If there exists a base B of $\mathcal{M}_w$ which is independent in G, then we can define $g : V \to \mathbb{R}^+$ by $g(v) = w(v)$ for $v \in B$ and $g(v) = 0$ for $v \notin B$, and define f to be indentically zero. The pair (g, f) is then dominating and satisfies $|(g, f)| = w[B] = \nu_w$, proving the theorem. Thus we may assume that the pair $[\mathcal{M}_w, \mathcal{I}(G)]$ is not matchable. Let X, Y, I be as in Theorem 13.3, and let $D = X \cup Y$.

Define a new weight function w' by $w'(v) = w(v) - |\tilde{N}(v) \cap D|$. If (g', f') is a dominating pair for w', then $(g', f' + \chi_D)$ is a dominating pair for w, where χ_D is the characteristic function of D. Clearly, also, if (g', f') satisfies (*) for w', then $(g', f' + \chi_D)$ satisfies (*) for w. By the induction hypothesis there exists a dominating pair (g', f') for w' satisfying (*) and the property $|(g', f')| \leq \nu_{w'}$. Since

$|(g', f' + \chi_D) = |(g', f')| + |D|$, in order to complete the proof it suffices to show that $\nu_w \geq \nu_{w'} + |D|/2$. Since $|D| \leq 2|Y|$, it suffices to show $\nu_w \geq \nu_{w'} + |Y|$.

Choose $T \in \mathcal{I}(G) \cap \mathcal{M}$ with $w'[T] = \nu_{w'}$, having the additional property that its intersection with Y is maximal.

Suppose that there exists $y \in Y \setminus T$, and let C be the $\mathcal{M}$-circuit containing y in $T + y$. We claim that there exists $x \in C \setminus Y$ such that $w'(x) \leq w'(y)$.

To prove this, Let k be the minimal value of w in C. Consider first the case $w(y) > k$. By Lemma 9.3, there exists a circuit D of $\mathcal{M}_w$ such that $D \subseteq C$ and $w(v) = k$ for all $v \in D$. Since Y is independent in $\mathcal{M}_w$ there exists some $x \in D \setminus Y$. Now we have

$$w'(x) \leq w(x) = k \leq w(y) - 1 = w'(y) .$$

In the case that $w(y) = k$ we can demand that $y \in D$ and thus there exists $x \in D \setminus Y \setminus sp_{\mathcal{M}_w}(I)$. Then

$$w'(x) \leq w(x) - 1 = k - 1 = w(y) - 1 = w'(y) .$$

This proves the claim. If y had no neighbors in T then $T - x + y$ would contradict the maximality of T. Thus every vertex $y \in Y \setminus T$ is connected to some vertex in T. By the definition of w' it follows then that $w'[T] \leq w[T] - |Y|$, implying $\nu_w \geq \nu_{w'} + |Y|$, as required. □

We suspect that the same result is true replacing $\mathcal{I}(G)$ by any complex. To define the general conjecture, we first have to extend the notion of the "span" of a function from matroids to general simplicial complexes. Given a function $f : V(\mathcal{C}) \to \mathbb{R}^+$ on the vertex set of a simplicial complex $\mathcal{C}$, we define $f^{\mathcal{C}}(x)$ to be $\max\{\min_{y \in Y} f(y) : Y \subseteq V,\ x \in sp_{\mathcal{C}}(Y)\}$. Let $\mathcal{C}, \mathcal{D}$ be two complexes on the same ground set V, and let w be a non-negative real-valued function on V. We define $\nu_w(\mathcal{C}, \mathcal{D})$ as for matroids, namely as $\max\{w[\sigma] : \sigma \in \mathcal{C} \cap \mathcal{D}\}$.

Conjecture 13.5. *Let $\mathcal{M}, \mathcal{C}$ be a matroid and a simplicial complex on the same ground set V, and let $w : V \to \mathbb{R}^+$. Then $\nu_w(\mathcal{M}, \mathcal{C}) \geq \min\{|g| + \frac{1}{2}|f| : g^{\mathcal{M}} + f^{\mathcal{C}} \geq w\}$*

(We thus conjecture that there is a price of a factor of $\frac{1}{2}$ in τ_w for the fact that $\mathcal{C}$ is a general complex, and not a matroid. For $w \equiv 1$ the conjecture is akin to Theorem 6.3, although unlike in the case of graphic complexes it does not follow from it directly.)

As a corollary of Theorem 13.4 we get the fractional version of Conjecture 13.1.

Theorem 13.6. *Every graph $G = (V, E)$ is fractionally matroidally $2\Delta(G)$-colorable. Namely, for every matroid $\mathcal{M}$ on V, if $\chi(\mathcal{M}) \leq 2\Delta(G)$ then $\chi^*(\mathcal{M} \cap \mathcal{I}(G)) \leq 2\Delta(G)$.*

Proof. Suppose that $\chi^*(\mathcal{M} \cap \mathcal{I}(G)) > 2\Delta(G)$. Then there exists a function $w : V \to \mathbb{R}^+$ with $w[V] > 2\Delta(G)$ and $w[A] \leq 1$ for every bi-independent set. This means that $\nu_w \leq 1$, which by Theorem 13.4 implies that $\tau_w \leq 1$. Thus there exist functions $g, f : V \to \mathbb{R}^+$ such that $|(g, f)| \leq 1$ and the pair (g, f) dominates w. But this is easily seen to imply that $w[V] \leq 2\Delta(G)$, yielding a contradiction. □

Let us end with three conjectures on matroidal colorability.

Conjecture 13.7. *Every complex $\mathcal{C}$ is matroidally $2\Delta(\mathcal{C})$-colorable.*

Conjecture 13.8. *For every k, the cycle C_{3k} is matroidally 3-colorable.*

This is a generalization of the theorem of Fleischner and Stiebitz [12] that C_{3k} is strongly 3-colorable. The graph $G = C_{3k}$ has the property that $\eta(\mathcal{I}(G[S])) \geq \frac{|S|}{3}$ for every set S of vertices (see, e.g., [3]), namely $\Delta(\mathcal{I}(G)) = 3$. By Theorem 8.3 it follows that for every matroid $\mathcal{M}$ with $\Delta(\mathcal{M}) \leq 3$ there exists a set of size k in $\mathcal{M} \cap \mathcal{I}(G)$. The conjecture is that $V(G)$ can be partitioned into 3 such sets.

The following conjecture has similar motivation, stemming from the results of [2]:

Conjecture 13.9. *Every chordal graph G is matroidally $\Delta(G)+1$-colorable.*

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References

1. R. Aharoni, Ryser's conjecture for 3-partite 3-graphs, *Combinatorica*, **21**(2001), 1-4.
2. R. Aharoni, E. Berger and R. Ziv, A tree version of Kőnig's theorem, *Combinatorica*, **22**(2002), 335-343.
3. R. Aharoni, E. Berger and R. Ziv, Independent systems of representatives in weighted graphs, *submitted for publication*
4. R. Aharoni, M. Chudnovsky and A. Kotlov, Triangulated spheres and colored cliques, *Disc. Comp. Geometry* **28**(2002), 223-229.
5. R. Aharoni and R. Ziv, The intersection of two infinite matroids, *J. London Math. Soc.* **58**(1998), 513-525.
6. R. Aharoni and P. Haxell, Hall's theorem for hypergraphs, *J. of Graph Theory* **35**(2000), 83-88.
7. A. Björner, B.Korte and L. Lovász, Homotopy properties of greedoids, *Adv. in Appl. Math.* **6**, 447-494.
8. J. Davies and C. McDiarmid, Disjoint common transversals and exchange structures, *J. London Math. Soc.* **14**(1976), 55-62.
9. M. DeVos, Stable bases and circuit decompositions, preprint.
10. J. Edmonds, Lehman's switching game and a theorem of Tutte and Nash-Williams, *J. Res. Nat. Bur. Stand.* **69B**(1965), 73-77.
11. J. Edmonds, Matroid intersection, *Discrete optimization* Ann. Discrete Math. **4**(1979), 39-49.
12. H. Fleischner and M. Stiebitz, A solution to a colouring problem of P. Erdös, *Discrete Math.* **101**(1992), 39–48.
13. A. Frank, A weighted matroid intersection algorithm, *J. Algorithms* **2**(1981), 328–336.
14. B. Knaster, C. Kuratowski, and C. Mazurkiewicz, Ein Beweis des Fixpunktsatzes fuer n-dimensionale Simplexe, Fundamenta Mathematicae **14** (1929), 132-137.
15. D. Kőnig, Über Graphen und ihre Andwendung auf Determinantentheorie und Mengenlehre, *Math Ann.* **77** (1916), Jbuch. 46.146-147.
16. P. Hall, On representation of subsets, *J. London Math. Soc.* **10**(1935), 26-30.
17. P. E. Haxell, A condition for matchability in hypergraphs, *Graphs and Combinatorics* **11** (1995), 245-248.
18. P. E. Haxell, On the strong chromatic number, to appear in *Comb. Prob. and Computing*
19. R. Meshulam, The clique complex and hypergraph matching, *Combinatorica* **21**(2001) 89-94.
20. R. Meshulam, Domination numbers and homology, *Jour. Combin. Th., Ser. A*, **102**(2003), 321-330.
21. A. Schrijver, Combinatorial Optimization, Springer-Verlag, 2003.
22. D. J. A. Welsh, Matroid Theory, Academic Press, 1976.
23. D. J. A. Welsh, On matroid theorems of Edmonds and Rado, *J. London Math. Soc.* **2**(1970), 251–256.
24. M. Wild, On Rota's problem about n bases in a rank n matroid. *Adv. Math.* **108**(1994), 336–345.
25. R. Yuster, Independent transversals in r-partite graphs, *Discrete Math.* **176**(1997), 255-261.

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