Acyclic systems of representatives and acyclic colorings of digraphs

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Abstract

A natural digraph-analogue of the graph concept of "independent set" is that of an "acyclic set of vertices", namely a set not spanning a directed cycle. By this token, an analogue of the notion of "coloring" of a graph is that of decomposition of a digraph into acyclic sets. We extend some known results on independent sets and colorings in graphs to acyclic sets and acyclic colorings of digraphs. In particular, we prove bounds on the topological connectivity of the complex of acyclic sets, and using them we prove sufficient conditions for the existence of acyclic systems of representatives of a system of sets of vertices. We prove a fractional version of a strong-acyclic-coloring conjecture for digraphs. We propose the following conjecture: every simple planar digraph is 2-acyclically colorable (namely, it can be decomposed into two acyclic sets).

1 Introduction

An *acyclic* set of vertices in a digraph is a set not spanning a directed cycle. This is a natural extension of the notion of independent sets in graphs. Consequently, a natural extension of the notion of coloring of a graph is that of a decomposition of a digraph into acyclic sets. Indeed, upon viewing an undirected graph as a digraph in which all edges are arranged in oppositely directed pairs, these notions yield the ordinary notions of an independent set and a coloring.

Let us start with some terminology and notation. For two vertices x and y of D we write $x \to y$ if there is a directed edge from x to y. For two sets A and B of vertices we write $A \succ B$ if there do not exist vertices $a \in A, b \in B$ such that $b \to a$. A vertex $x \in X \subseteq V(D)$ is called *maximal* in X, if $y \not\to x$ for every $y \in X$.

The minimal length of a directed cycle in a digraph D is denoted by $c(D)$. Throughout the paper we assume that the graph D considered contains no loops, namely $c(D) > 1$. We say that D is *simple* if $c(D) > 2$, i.e., if there are no oppositely directed pairs of edges. For a vertex $x \in V(D)$ we write $\tilde{N}^+(x) = \{y \in V(D) : x \to y\}$, and $N^+(x) = \tilde{N}^+(x) \cup \{x\}$. For a set $X \subseteq V(D)$ we write $N^+(X) = \bigcup_{x \in X} N^+(x)$ and $\tilde{N}^+(X) = \bigcup_{x \in X} \tilde{N}^+(x)$. We write $d^+(x) = |\tilde{N}^+(x)|$, and denote $\max_{v \in V(D)} d^+(v)$ by $\Delta^+(D)$. The maximal size of an acyclic set in D is denoted by $\alpha_A(D)$, and $\chi_A(D)$ denotes the minimal number of acyclic sets whose union is $V(D)$.

A digraph D is called *c-partite* if there exists a partition $V(D) = P_0 \dot\cup \ldots \dot\cup P_{c-1}$ such that $x \to y$ only if $x \in P_i$, $y \in P_j$ and $j - i \equiv_{\text{mod } c} 1$. If all pairs of this form are edges in D then it is said to be *complete c-partite*.

To exemplify the analogy between independent sets in graphs and acyclic sets in digraphs let us mention two facts, parallels of well known facts about graphs:

Lemma 1.1 $\chi_A(D) \le \Delta^+(D) + 1$

Proof: Color the vertices, one at a time, by $\Delta^+(D)+1$ colors. When the turn arrives for a vertex x to be colored, assign to it a color not appearing among those vertices of $N^+(x)$ colored so far.

□

The second fact is the analogue of Turán's theorem:

Lemma 1.2 *Write $V = V(D), E = E(D)$. Then:* $\alpha_A(D) \ge \frac{|V|^2}{|E|+|V|}$

Proof: The proof mimics a well known proof of Turán's theorem. In a random permutation of $V(D)$, for a given vertex x, the probability that there is no edge from x to its predecessors is $\frac{1}{d^+(x)+1}$. Hence the expectation of the size of a set of vertices not sending an arrow to any of their predecessors is $\sum_{x\in V}\frac{1}{d^+(x)+1}$, which by the harmonic average inequality is at least $\frac{|V|^2}{|E|+|V|}$. Since such a set is acyclic, this concludes the proof.

$\square$

Conjecture 1.3 *In a simple digraph* $\alpha_A \geq \frac{|V|^2}{|E|}\log_2\frac{|E|}{|V|}$.

The case that the graph is a tournament is a known conjecture (see [7], Chapter 1). In this case, up to a factor of 2 the conjecture is easy - a greedy algorithm yields an acyclic set of size $log_2|V|$. For general graphs the conjecture is not known even up to a constant factor.

Given a system of (not necessarily disjoint) sets $V_1, \dots, V_m$, a subset X of $V = \bigcup_{i\leq m} V_i$ is called a *system of representatives (SR)* if $X \cap V_i \neq \emptyset$ for every $1 \leq i \leq m$. If a graph G is defined on V, then X is called an *independent system of representatives (ISR)* if it is also independent. If a digraph D is defined on V and X is acyclic in D, we say that X is an *acyclic system of representatives (ASR)*. Given any hypergraph H on V, we say that X is a H-SR if, besides being an SR, it belongs to H.

The first result on ISR's appeared (implicitly) in [8], where a sufficient condition for the existence of ISR's was proved in terms of the domination number of subgraphs of the given graph. Later, in [6], a topological approach was introduced, yielding further sufficient conditions. This approach was developed further in [11, 12], where homological methods were introduced, and in [12, 1, 2]. In the second section we shall extend both methods to the case of ASR's. The results obtained are in terms of the domination number and the shortest length of a cycle in the graph.

For an integer m, we write $[m]$ for $\{1, \dots, m\}$, and $\langle m\rangle$ for $\{0, \dots, m-1\}$.. We shall use the following notation: if $X_1, X_2, \dots, X_m$ are sets and $I \subseteq [m]$, then X_I denotes the union $\bigcup_{i\in I} X_i$. For a graph G and a set of vertices X we denote by $G[X]$ the graph induced by G on X.

2 ISR's, topological connectivity, and domination

Let us recall the basic definitions concerning simplicial complexes. A simplicial complex (or just a *complex*) $\mathcal{C}$ is a hereditary hypergraph, namely satisfying the condition that if $\sigma \in \mathcal{C}$ and $\tau \subset \sigma$ then $\tau \in \mathcal{C}$. The edges of a complex are called *simplices*. Every simplicial complex has a unique (up to homeomorphism) geometric realization in some $\mathbb{R}^n$, in which every simplex σ corresponds to a geometric simplex $\phi(\sigma)$, and $\phi(\sigma) \cap \phi(\sigma') = \phi(\sigma \cap \sigma')$. The complex is called a *triangulation* of its realization. We shall often identify a complex with its realization.

Given a complex $\mathcal{C}$ and a subset X of its ground set we write $\mathcal{C} \upharpoonright X$ for the complex $\{\sigma : \ \sigma \in \mathcal{C}, \ \sigma \subseteq X$ on the ground set X. For a vertex v we denote by $\mathcal{C} - v$ the complex $\mathcal{C} \upharpoonright (V(\mathcal{C} \setminus \{v\}$.

A simplicial complex $\mathcal{C}$ is said to be *m-connected* if for every $k \leq m$, every continuous function from the k-dimensional sphere S^k into the geometric realization $\|\mathcal{C}\|$ of $\mathcal{C}$ is extendable to a continuous function from the$(k+1)$-dimensional ball B^{k+1} into $\|\mathcal{C}\|$. The *topological connectivity* of $\mathcal{C}$ is the largest m for which it is m-connected. If $\mathcal{C}$ is m-connected for every m then it is said to be ∞-connected, and its connectivity is defined as ∞ . Adding 2 to the topological connectivity we obtain $\eta(\mathcal{C})$, which is a more convenient parameter to work with.

Remark: It is more common to work with the homological connectivity of complexes, which is greater than or equal to the topological connectivity. Indeed, all our discussion could be carried out using this notion. Our choice to use topological connectivity was guided by the wish to keep the paper as elementary as possible.

The following property of the connectivity function can be proved as a corollary of the Meyer-Vietoris and Van-Kampen theorems. (It is not very hard to prove it in more elementary methods as well.)

Lemma 2.1 *For any pair $\mathcal{A}, \mathcal{B}$ of complexes,* $\eta(\mathcal{A}) \geq \min(\eta(\mathcal{A} \cap \mathcal{B}), \eta(\mathcal{A} \cup \mathcal{B}))$

If $\mathcal{A}$ and $\mathcal{B}$ are two simplicial complexes with disjoint ground sets we write $\mathcal{A} * \mathcal{B} = \{\sigma \cup \tau : \ \sigma \in \mathcal{A}, \ \tau \in \mathcal{B}\}$. It is not hard to show the following:

Lemma 2.2 *For every pair of complexes $\mathcal{A}, \mathcal{B}$*

$$\eta(\mathcal{A} * \mathcal{B}) \geq \eta(\mathcal{A}) + \eta(\mathcal{B})$$

Given a graph G, we write $\mathcal{I}(G)$ for the complex of independent sets of G. For a digraph D we denote by $\mathcal{A}(D)$ the complex of acyclic sets in D. Generalizing the notion of ISR's and ASR's, we define the following: given

In [6] the following was implicitly shown:

Theorem 2.3 *Let $\mathcal{C}$ be a simplicial complex, and let $V_1, \ldots, V_m$ be a partition of its vertex set. If $\eta(\mathcal{C} \restriction V_I) \geq |I|$ for every $I \subseteq [m]$ then there exists a $\mathcal{C}$-SR.*

In particular, we have:

Theorem 2.4 *Let $V_1, \ldots, V_m$ be sets, and let D be a digraph whose vertex set is $V = \bigcup_{i \in [m]} V_i$. If $\eta(\mathcal{A}(D[V_I])) \geq |I|$ for every $I \subseteq [m]$ then there exist an ASR.*

To obtain combinatorial criteria for the existence of ASR's, we need lower bounds on the connectivity of $\mathcal{A}(D)$ for a digraph D, in terms of some combinatorial parameters. In the undirected case, almost all known bounds on η are given in terms of domination parameters of the graph (the only exception so far is a criterion proved in [2], in which a bound is given in terms of the first eigenvalue of the Laplacian of the graph). Similarly, the bounds we shall prove in the directed case will use domination parameters.

Definition 2.5 *Let X be a subset of $V(D)$. A set $Y \subseteq V(D)$ is called X-*dominating *if $X \subseteq \tilde{N}^+(Y)$. A $V(D)$-dominating set is plainly called* dominating. *A set Y is called* totally dominating *if $V(D) = \tilde{N}^+(Y)$.*

Using this terminology we define the following parameters:

- The *domination number*, $\gamma(D)$, of D is the smallest size of a dominating set.
- The *total domination number*, $\tilde{\gamma}(D)$, of D is the minimal size of a totally dominating set.
- The *acyclic domination number*, $a\gamma(D)$, of D is the maximum, over all acyclic sets A in D, of the minimal size of a set dominating A.

These are generalizations of parameters in the undirected case, denoted respectively by γ, $\tilde{\gamma}$ and $i\gamma$.

Straightforward generalizations of two results in the undirected case [6, 11, 5] are the following:

Theorem 2.6 $\eta(\mathcal{A}(D)) \geq a\gamma(D)$.

Theorem 2.7 $\eta(\mathcal{A}(D)) \geq \frac{\tilde{\gamma}(D)}{2}$.

The proofs of these results follow the same outline as the proofs in [6] and [5]. They are based on results on extensions of triangulations from the sphere to the ball. The first of these was proved in [6]: any triangulation $\mathcal{D}$ of the sphere S^{k-1} can be extended to a triangulation $\mathcal{C}$ of the ball B^k, such that the neighbors of every point of $V(\mathcal{C})$ belonging to $V(\mathcal{D})$ form a simplex in $\mathcal{D}$ (still another proof was given in [5]).

Here is a short outline of the proof of Theorem 2.6. Let $n = a\gamma(D)$, and let A be an acyclic set in D needing at least n points to dominate it. Let ϕ be a simplicial map from $\mathcal{A}(D)$ into a triangulation $\mathcal{D}$ of S^{n-2}. Let $\mathcal{C}$ be a triangulation of B^n extending $\mathcal{C}$, such that the neighborhood in $\mathcal{D}$ of any vertex x in $\mathcal{C}$ is a simplex $\sigma(x)$. Since $|V(\sigma(x))| < n$ there exists a vertex $v(x) \in A$ not dominated by $\phi[V(\sigma(x))$. Extend ϕ to $\mathcal{C}$ by defining $\phi(x) = v(x)$ for every $x \in V(\mathcal{C}) \setminus V(\mathcal{D})$. The image $\phi[V(\tau)]$ of every simplex $\tau \in \mathcal{C}$ is acyclic, since its intersection with $\phi[V(\tau)] \cap A$ is acyclic, $\phi[V(\tau)] \setminus A$ is acyclic, and all edges between these two sets go in one direction - from the first to the second.

Theorem 2.7 is proved in a similar way, using the following result from [5]: any triangulation $\mathcal{D}$ of the sphere S^{k-1} can be extended to a triangulation $\mathcal{C}$ of the ball B^k, by adding a point at a time, which is connected in the 1-skeleton of $\mathcal{C}$ to at most $2k$ points added so far.

Theorem 2.7 can be strengthened, as follows:

Theorem 2.8 $\eta_A(D) \geq (1 - \frac{1}{c(D)})\tilde{\gamma}^+(D)$

In fact, we can prove an even stronger theorem, using the following notations. We say that a set $S \subseteq V(D)$ is r-*granular* if no strongly connected component of $D[S]$ has more than r vertices. Thus S is acyclic if and only if

S is 1-granular. In fact, for every number $r < c(D)$, the set S is acyclic if and only if S is r-granular. We write $\mathcal{G}_r(D)$ for the simplicial complex of all r-granular sets and note that as discussed above, $\mathcal{A}(D) = \mathcal{G}_1(D) = \mathcal{G}_{c(D)-1}(D)$. We will now prove the following.

Theorem 2.9 $\eta(\mathcal{G}_r(D)) \geq \frac{r}{r+1}\tilde{\gamma}^+(D)$

The outline of the proof, in particular the use of Lemma 2.1 is borrowed from [12], where a very general bound is proved on $\eta(\mathcal{I}(G))$ for an undirected graph G.

Write $\mathcal{C} = \mathcal{A}(D)$. Choose an arbitrary vertex v of D, and define $\mathcal{C}' = v * (\mathcal{C} - v)$. Then $\mathcal{C}'$ is contractible, hence:

$$\eta(\mathcal{C}') = \infty \tag{1}$$

For a simplex $\sigma \in \mathcal{C} - v$ such that $\sigma + v \notin \mathcal{C}$ denote by $t(\sigma)$ the set consisting of v together with all vertices u in σ for which there exists a directed path within $\sigma + v$ from v to u. Order the set of distinct $t(\sigma)$'s as $t_1, \ldots, t_s$, in a way compatible with the order of inclusion, namely if $t_i \subset t_j$ then $i < j$. Note that each t_i satisfies

$$|t_i| \geq r + 1 \tag{2}$$

For a set t we denote by Σ_t the complete simplicial complex on t, namely $\mathcal{P}(t)$, and by $\hat{\Sigma}_t$ the complex $\Sigma_t \setminus \{t\}$.

Define inductively complexes $\mathcal{C}_i$ as follows. Let $\mathcal{C}_0 = \mathcal{C}$, and for $i > 0$ define $\mathcal{C}_i = \mathcal{C}_{i-1} \cup \mathcal{B}_i$, where $\mathcal{B}_i = \Sigma_{t_i} * (\mathcal{C} - N^+(t_i))$. Note that if $\sigma \in \mathcal{C} - v$ is such that $t(\sigma) = t_i$ then σ does not contain any vertex $x \in N^+(t_i) \setminus t_i$ (or else $x \in t(\sigma)$), and thus $\sigma \in \mathcal{B}_i$. Also note that for each simplex $\sigma \in \mathcal{B}_i$ every strongly connected component must be contained either in t_i or in $V \setminus N^+(t_i)$. Therefore $\sigma - v \in \mathcal{C}$ and $\mathcal{B}_i \subseteq \mathcal{C}'$.

Let $\tau \in \mathcal{C}' \setminus \mathcal{C}$. Then $v \in \tau$, and $\tau - v \in \mathcal{C}$, and hence $t(\tau - v) = t_i$ for some i. From the definition of $\mathcal{B}_i$ it follows then that $\tau \in \mathcal{B}_i$. This shows that in the last step, s, of the construction, we have:

$$\mathcal{C}_s = \mathcal{C}' \tag{3}$$

Next we show:

$$\mathcal{C}_{i-1} \cap \mathcal{B}_i = \hat{\Sigma}_{t_i} * (\mathcal{C} - N^+(t_i)) \tag{4}$$

For that end we need to show

$$\mathcal{C}_{i-1} \cap \mathcal{B}_i \subseteq \hat{\Sigma}_{t_i} * (\mathcal{C} - N^+(t_i)) \tag{5}$$

$$\mathcal{C}_{i-1} \supseteq \hat{\Sigma}_{t_i} * (\mathcal{C} - N^+(t_i)) \tag{6}$$

$$\mathcal{B}_i \supseteq \hat{\Sigma}_{t_i} * (\mathcal{C} - N^+(t_i)) \tag{7}$$

Equation (7) follows directly from the definition of $\mathcal{B}_i$.

We now want to prove (6). Let $\sigma \in \hat{\Sigma}_{t_i} * (\mathcal{C} - N^+(t_i))$ and we need to show $\sigma \in \mathcal{C}_{i-1}$. We may assume $v \in \sigma$ (or else $\sigma \in \mathcal{C} \subseteq \mathcal{C}_{i-1}$). We note that $t(\sigma - v) \subsetneqq t_i$, meaning that $t(\sigma - v) = t_j$ for some $j < i$. Since $N^+(t_j) \subseteq N^+(t_i)$ it follows that $\sigma \in \mathcal{B}_j$, and hence $\sigma \in \mathcal{C}_{i-1}$. This completes the proof of (6).

In order to prove (5), it suffices to show that no set in $\mathcal{C}_{i-1}$ contains t_i. Equivalently, we have to show this for any set in $\mathcal{B}_j$, $j < i$. Consider any $j < i$. Choose a vertex x in $t_i \backslash t_j$, take a path in t_i from v to x, and let z be the first vertex on this path not belonging to t_j. Then $z \in t_i \cap N^+(t_j) \setminus t_j$, hence $z \notin \tau$ for any $\tau \in \mathcal{B}_j$, proving that $\tau \not\supseteq t_i$, which is the desired conclusion.

By (4) and Lemma 2.1 we have:

$$\eta(\mathcal{C}_{i-1}) \geq \min(\eta(\mathcal{C}_i), \eta(\hat{\Sigma}_{t_i} * (\mathcal{C} - N^+(t_i)))) \tag{8}$$

We shall now show, by induction on $s - i$, that $\eta(\mathcal{C}_i) \geq \frac{r}{r+1}\tilde{\gamma}^+(D)$. For $i = s$, this is given by (1). Assume that the assertion is true for i. Invoking an inductive assumption on the size of D, we have

$$\eta(\mathcal{C} - N^+(t_i)) \geq \frac{r}{r+1}\tilde{\gamma}^+(D - N^+(t_i)) \tag{9}$$

For a set X dominating $D - N^+(t_i)$ the set $X \cup t_i$ dominates D, and hence:

$$\tilde{\gamma}^+(D - N^+(t_i)) \geq \tilde{\gamma}^+(D) - |t_i| \tag{10}$$

This, together with (9), implies:

$$\eta(\mathcal{C} - N^+(t_i)) \geq \frac{r}{r+1}(\tilde{\gamma}^+(D) - |t_i|) \tag{11}$$

Since $\eta(\hat{\Sigma}_{t_i}) = |t_i| - 1$ we can use Lemma 2.2 and get

$$\begin{aligned} \eta(\mathcal{C}_{i-1} \cap \mathcal{B}_i) &\geq \eta(\hat{\Sigma}_{t_i}) + \eta(\mathcal{C} - N^+(t_i)) \\ &\geq |t_i| - 1 + \tfrac{r}{r+1}(\tilde{\gamma}^+(D) - |t_i|) = \tfrac{r}{r+1}\tilde{\gamma}^+(D) + \tfrac{|t_i|}{r+1} - 1 \end{aligned} \tag{12}$$

which, in view of (2) yields

$$\eta(\mathcal{C}_{i-1} \cap \mathcal{B}_i) \geq \frac{r}{r+1}\tilde{\gamma}^+(D) \tag{13}$$

By (8) this proves the theorem.

□

Applying this theorem to undirected graphs we get the following theorem of Tardos and Szabo [**?**].

Corollary 2.10 *Let G be a graph, let $r > 0$ be an integer and let $\mathcal{C}$ be the simplicial complex of all sets of vertices not containing a subset of size more than r inducing a connected graph. Then $\eta(\mathcal{C}) \geq \frac{r}{r+1}\tilde{\gamma}(G)$.*

Combining Theorems 2.8 and 2.4 yields:

Corollary 2.11 *If $\tilde{\gamma}(D[V_I]) > (1 + \frac{1}{c(D[V_I])-1})(|I| - 1)$ for every $I \subseteq [m]$ then there exist an ASR in D.*

This is in fact sharp. To see this, consider a digraph D which is a disjoint union of t directed cycles of length c. Let V_1 contain t vertices, one of each cycle and let $V_i,\ 2 \leq i \leq c$ each consist of the choice of one vertex from every cycle. For every set I of indices $\tilde{\gamma}(D[V_I])$ is either ∞ or $\frac{c}{c-1}(|I| - 1)$, and thus the assumption of the lemma is valid, and yet the system does not possess an ASR.

Corollary 2.11 yields:

Corollary 2.12 *If $|V_i| \geq \frac{c(D)}{c(D)-1}\Delta^+(D)$ for all $i \in [m]$ then there exists an ASR. Moreover, every vertex x is contained in some ASR.*

proof Without loss of generality $x \in V_1$. Define $V_1' = \{x\}$ and $V_i' = V_i$ for every $i \geq 2$. Every k sets of this system contain at least $1 + \frac{c}{c-1}\Delta^+(D)(k-1)$ vertices and hence is totally dominated by at least $(1 + \frac{c}{c-1}\Delta^+(D)(k-1))/\Delta^+(D) > \frac{c}{c-1}(k-1)$ vertices and so Corollary 2.11 ensures the existence of an ASR.

$\square$

This result is sharp: for any integers $c,\ k$ we shall construct a digraph $H = H_{c,k}$ with $c(H) = c,\ \Delta^+(H) = c^k$, and a partition of $V(H)$ as $V_1\dot{\cup}\ldots\dot{\cup}V_m$, such that $|V_i| = \lfloor\frac{c}{c-1}\Delta^+(H)\rfloor$ and H does not possess an ASR. We shall have $|V(H)| = c^{k+1} \times \sum_{j=0}^{k} c^j$, and $V(H)$ is partitioned into $m = c^{k+1}$ sets V_α, indexed by $\alpha \in \langle c\rangle^{k+1}$. Each set V_α is of size $\sum_{j=0}^{k} c^j$, and its vertices $v_\alpha^{j,\beta}$ are indexed by $j \in \{0,\ldots,k\},\ \beta \in \langle c\rangle^j$.

The graph H will consist of $\sum_{j=0}^{k} c^j$ connected components, each being complete c-partite and having size c^k. We shall index these components D_ψ^j, where $j \in \{0,\ldots,k\}, \psi \in \langle c\rangle^j$.

The first such component, $D_\emptyset^0$, has $\{v_\alpha^{0,\emptyset},\ \alpha \in \langle c\rangle^{k+1}\}$ as its vertex set, and its parts P_i are $\{v_\alpha^{0,\emptyset} : \alpha(0) = i\}$ for every $i \in \langle c\rangle$. In general, for each $j \in \{0,\ldots,k\}$ define c^j complete c-partite graphs $D_\psi^j,\ \psi \in \langle c\rangle^j$. The vertex set of D_ψ^j is $\{v_\alpha^{j,\beta} : \ \alpha \restriction \{0,\ldots,j-1\} = \psi\}$, and its parts are $P_i = \{v_\alpha^{j,\beta} \in V(D_\psi^j) :\ \alpha(j) = i\}$.

Clearly, $c(H) = c,\ \Delta^+(H) = c^k$ and $|V_\alpha| = \sum_{j=0}^{k} c^j = \lfloor\frac{c}{c-1}\Delta^+(H)\rfloor$. Assume that there exist an ASR A. Let us examine $D_\emptyset^0$. Obviously, not all of the sides P_i of $D_\emptyset^0$ can intersect A, so assume $P_0 \cap A = \emptyset$. Hence if $v_\alpha^{j,\beta} \in A$ and $\alpha(0) = 0$, then $j \geq 1$. Let us now examine D_ψ^1, where $\psi(0) = 0$. Again we may assume that $P_0 \cap A = \emptyset$ and so if $v_\alpha^{j,\beta} \in A$ and $\alpha(i) = 0$ for $i \in \{0,1\}$, then $j \geq 2$. Proceed this way, examining at each stage l the component D_ψ^l, where $\psi(i) = 0$ for $i \in \{0,\ldots,l-1\}$. As before, we may assume $P_0 \cap A = \emptyset$ and conclude that if $v_\alpha^{j,\beta} \in A$ and $\alpha(i) = 0$ for $i = \{0,\ldots,l\}$, then $j \geq l$. Eventually, we deduce that V_α is not represented by A for $\alpha(i) = 0,\ i = \{0,1,\ldots,k\}$.

This example generalizes an example of Yuster [13] for graphs. In fact, in the case $c = 2$, the two examples coincide.

Note that by using granularity we can formulate and prove an extension of Corollary 2.11.

Corollary 2.13 *If $\tilde{\gamma}(D[V_I]) > (\frac{r+1}{r})(|I|-1)$ for every $I \subseteq [m]$ then there exist an r-granular set in D with a representative from each V_i.*

The undirected case of this corollary was proved By Haxell, Tardos and Szabo [**?**].

3 A combinatorial approach

In this section we re-prove Corollary 2.11 in a combinatorial way, using an alternating paths technique similar to that used in [8] (see also [3]). The reason is that the combinatorial proof provides some more information, which will be needed for applications.

As before, let $\mathcal{P}$ be a partition $V_1, V_2, \ldots, V_m$ of $V(D)$.

Theorem 3.1 *Suppose that the partition $V_1, \ldots, V_m$ of $V(D)$ does not possess an ASR in D. Then there exists $I \subseteq [m]$ such that, writing $c = c(D[V_I])$, there exists a subset W of V_I of size at most $\frac{c}{c-1}(|I|-1)$ that totally dominates $D[V_I]$. Furthermore, such a set W can be chosen in such a way that $W = X \dot{\cup} Y$, where*
(i) X is acyclic and Y is a partial ASR.
(ii) $|Y| \geq (c-1)|X|$

proof Let $J \subseteq [m]$ be a maximal set of indices such that $D[V_J]$ has an ASR. Write $\mathcal{A}$ for the collection of all ASR's of $(V_j : j \in J)$ in $D[V_J]$. By assumption, $J \neq [m]$, so without loss of generality we may assume that $1 \notin J$. Pick any vertex $x_1 \in V_1$. We say that (Y, Z) is a *good decomposition* of an ASR $R \in \mathcal{A}$ if $R = Y \cup Z$, and $Y \cup \{x_1\} < Z$. For example, $(R, \emptyset)$ is a good decomposition. Among all good decompositions of all ASR's in $\mathcal{A}$ choose a good decomposition (Y_1, Z_1) of an ASR R_1 in which Y_1 is minimal.

The set $Y_1 \cup \{x_1\}$ is cyclic, or else $R_1 \cup \{x_1\}$ would be an ASR, contradicting the maximality of J. In particular, $Y_1 \neq \emptyset$. Let $W_1 = Y_1 \cup \{x_1\}$ and $I_1 = I(W_1)$. If W_1 totally dominates $D[V_{I_1}]$ then the theorem is proved, with $W = W_1,\ X = \{x_1\},\ Y = Y_1$ and $I = I(W_1)$.
So, assume that there is some $x_2 \in V_{I_1}$ which is not dominated by W_1.

Let $\mathcal{A}_1$ be the set of sets R of vertices such that $Y_1\dot{\cup}R \in \mathcal{A}$ and $W_1 < R$. We have $\mathcal{A}_1 \neq \emptyset$ since $Z_1 \in \mathcal{A}_1$. Let $R \in \mathcal{A}_1$ and assume that the set $R\cup\{x_2\}$ is acyclic. If $x_2 \in V_1$ then $\{x_2\}\cup Y_1 \cup R$ is an ASR in $D[V_{J\cup\{1\}}]$, contradicting the maximality of J. If $x_2 \in V_j$, where $j = i(y)$ for some $y \in Y_1$, then the pair (Y', Z'), where $Y' = Y_1 - \{y\}$ and $Z' = R \cup \{x_2\}$, is a good decomposition of $Y' \cup Z'$, contradicting the minimality property of Y_1. As before, R can be written as $R = Y\dot{\cup}Z$ such that $Y \cup \{x_2\} < Z$, a type of decomposition which we call *good*. Take $R_2 = Y_2\dot{\cup}Z_2$ to be a member of $\mathcal{A}_1$ for which there exists a good decomposition in which Y_2 is minimal, for all choices of R and all good decompositions. As before, $Y_2 \cup \{x_2\}$ is cyclic.
Let $W_2 = Y_1 \cup Y_2 \cup \{x_1, x_2\}$ and $I_2 = I(W_2)$. If W_2 totally dominates $D[V_{I_2}]$ then the theorem is proved, with $X = \{x_1, x_2\}$, $Y = Y_1 \cup Y_2$ and $I = I_2$.
Thus we may assume that there exists $x_3 \in V_{I_2}$ not dominated by W_2 and proceed with the same argument.

Since our setting is finite, at some finite stage t, upon denoting $X = \{x_1, \ldots, x_t\}$, $Y = Y_1 \cup \cdots \cup Y_t$, $W = Y \cup X$ and $I = I(W)$, the set W dominates $D[V_I]$. By our construction Y is a partial ASR and X is acyclic. For every $k \leq t$ the set $x_k \cup Y_k$ is cyclic, implying that $|Y_k| \geq (c-1)$, where $c = c(D[V_I])$, and thus $|Y| \geq (c-1)|X|$. Since $|I| = |Y| + 1$ we get $|W| = |Y| + |X| \leq \frac{c}{c-1}|Y| = \frac{c}{c-1}(|I| - 1)$.

$\square$

4 Weighted Domination and Weighted ASR's

To our usual setting of a digraph D and a partition $\mathcal{P}$ of its vertex set we add another element - a weight function $w : V(D) \to \mathbb{R}$. We denote by ν_w be the maximum of $w[A]$, over all partial ASR's A of $(D, \mathcal{P})$. Note that if $w(v) < 0$ for some ertex v, this vertex will not appear in any partial ASR of maximum wight. Thus we may assume w is nonnegative. A pair of functions $(f : V(D) \to \mathbb{R}^+, g : [m] \to \mathbb{R}^+)$ is said to be *w-dominating* if $g(i(v)) + f[\tilde{N}^+(v)] \geq w(v)$ for every $v \in V$ (where $i(v)$ is the $i \in [m]$ for which $v \in V_i$). For such a pair, we write $|(g, f)| = |g| + \frac{|f|}{c(D)}$. Let τ_w be the minimum of $|(g, f)|$, over all w-dominating pairs (g, f).

Theorem 4.1 $\tau_w \leq \nu_w$. *Furthermore, it is possible to find a dominating pair of functions* (g, f) *of weight at most* ν_w *such that*
(*) $f(x) + g(i(x)) \leq \max_{y \in V_{i(x)}} w(y)$ *for every vertex* x.
If all weights $w(v)$ *are integral, then* g, f *can be assumed to be integral as well.*

proof In outline, the proof follows closely the proof of the analogous undirected result, in [4]. Hence we shall give it rather briefly.

The theorem is easily reducible to the integral case, hence we assume integrality of the weight function w.

The proof is by induction on $w[V]$. The basis of the induction, $w[V] = 0$, is trivial. Assuming that the theorem is true for $w[V] < s$, we prove it for $w[V] = s$. For each $i \leq m$ let t_i be the maximal weight of a vertex in V_i, and let M_i be the set $\{x \in V_i : \; w(x) = t_i$. If the sets M_i have an ISR, then $\nu_w = \sum t_i$, while the pair of functions defined by $f \equiv 0, \; g(i) = t_i$ is dominating and satisfies $|(g, f)| = \sum t_i$, proving the theorem. Thus we may assume that the sets M_i do not have an ISR in the subgraph of G they induce. Let $I \subseteq [m]$, X and Y be as in Theorem 3.1. Let $w'(v) = w(v) - |\tilde{N}^+(v) \cap D|$. By the induction hypothesis there exists a dominating pair (g', f') for w' such that $|(g', f')| \leq nu_{w'}$. Then $(g', f' + \chi_D)$ is dominating for w, where χ_D is the characteristic function of D. Since $|(g', f' + \chi_D)| = |(g', f')| + |D|$, in order to complete the proof it suffices to show that $\nu_w \geq \nu_{w'} + |D|/c(D)$. Since $|D| \leq c(D)|Y|$, it suffices to show $\nu_w \geq \nu_{w'} + |Y|$.

Choose an ASR T with $w'[T] = \nu_{w'}$, having the additional property that its intersection with Y is maximal. Then every vertex $y \in Y \setminus T$ is connected to some vertex in T, or else it could replace, in T, the representative of $i(y)$, contradicting the maximality property of T. By the definition of w' it follows then that $w'[T] \leq w[T] - |Y|$, implying $\nu_w \geq \nu_{w'} + |Y|$, as required.

□

5 Strong Acyclic Colorability

Definition 5.1 *A digraph* D *is called* strongly k-acyclically colorable *if for every partition* $V_1, \ldots, V_m$ *of* $V(D)$ *such that* $|V_i| \leq k$ *for every* $i = 1, \ldots, m$ *there exists an acyclic coloring of* D *such that every color class intersects each* V_i *in at most one vertex.*

This is an analogue of the notion of k-strong colorability in undirected graphs (see [9, 10]). The main conjecture regarding the latter, is as follows:

Conjecture 5.2 *Every graph G is $2\Delta(G)$-strongly colored.*

We extend this conjecture to digraphs:

Conjecture 5.3 *Every digraph D is strongly $\frac{c(D)}{c(D)-1}\Delta^+(D)$-acyclically colorable.*

This conjecture, if true is sharp, as is shown in the construction that follows Corollary 2.12. In this paper we bring two slightly weaker versions of Conjecture 5.3, which extends former results [4, 10]:

Theorem 5.4 *Every digraph D is strongly $(2 + \frac{1}{c(D)-1})\Delta^+(D)$-acyclically colorable*

and

Theorem 5.5 *Every digraph D is fractionally strongly $\frac{c(D)}{c(D)-1})\Delta^+(D)$-acyclically colorable. In other words, for every partition of $V(D)$ to sets of size $\frac{c(D)}{c(D)-1})\Delta^+(D)$ one can assign weights to partial ASR's so that the sum of all weights is $\frac{c(D)}{c(D)-1}\Delta^+(D)$ and for every vertex the sum of weights of the ASR's containing it is 1.*

The proof of Theorem 5.4 uses Corollary 2.12. The proof of Theorem 5.5 uses Theorem 4.1. Both proofs are similar to the original proofs in the undirected case (see [4, 10]) and hence are omitted.

6 Acyclic Coloring of Planar Digraphs

It is easy to show that every planar simple digraph D is 3-acyclically colorable: take any vertex v of degree 5 or less, and using an inductive hypothesis, color $D - v$ acyclically by 3 colors. In at least one of the three color classes v has degree smaller than or equal to 1, and adding v to this color class yields an acyclic 3-coloring of D.

We conjecture that the following is true:

Conjecture 6.1 *Every simple digraph is 2-acyclic colorable.*

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