abstract

Let $G = (V, E)$ be a simple undirected graph and W_0 a subset of V. Consider the following repetitive polling game. At round 0 the vertices of W_0 are colored white and the other vertices are colored black. At each round, each vertex v is colored according to the following rule. If at round r the vertex v has more than half of its neighbors colored c, then at round $r+1$ the vertex v will be colored c. If at round r the vertex v has exactly half of its neighbors colored white and half of its neighbors colored black, then we say there is a tie. In this case v is colored at round $r+1$ by the same color it had at round r. Peleg considered other models for dealing with ties. We will refer to these models in section 5. Additional models and further study of this game may be found at [2], [3], [4], [5] and [6].

Note the famous result of Goles and Olivos (see [2]) that for finite graphs the behavior is eventually 2-periodic. That is, there exist a finite r_0 so that for every $r > r_0$ all vertices are colored at round $r+2$ the same as they are colored at round r.

If there exists a finite r so that at round r all vertices in V are white, then we say that W_0 is a *dynamic monopoly*, abbreviated *dynamo*. If $r = 1$, that is, all vertices in V are white at round 1, we say that W_0 is a *static monopoly*. Peleg obtained many results regarding static monopolies (see for example [1], [5], [24] [25]). The work done so far on dynamic monopolies is less comprehensive. The main result so far is ([1] Corollary 2.5)

Theorem 1 *A dynamo is called* monotone *if every vertex which is white at some round will stay white from this round on. For every n-vertex graph G, every monotone dynamo is of size $\Omega(\sqrt{n})$*

In this thesis we prove that the above result is no longer valid when the monotonicity requirement is omitted.

Theorem 2 *For every natural number n there exists a graph with more than n vertices and with a dynamic monopoly of 18 vertices.*

The Structure of the Thesis

Section 1 is an introduction. It contains some definitions, and gives some motivation for the game. Section 2 gives the proof of Theorem 2. In section 3 we deal with the problem of finding the maximal size of the smallest monopoly over all connected graphs with n vertices. In Section 4 we deal with infinite graphs. In Section 5 we deal with other models of the game:

models with a different ways for dealing with ties, and models with handicaps.

List of Symbols

....

$[n]$	the set $\{1 \ldots n\}$
$\|A\|$	the cardinality of the set A
(V, E)	the graph with the set V of vertices and the set E of edges
$N(v)$	the neighborhood of the vertex v (when using this notation we assume the graph to be known)
$d(v)$	the degree of the vertex v ($d(v) = \|N(v)\|$)
$\mathcal{W}$	the white color
$\mathcal{B}$	the black color
$C_r(v)$	the color of the vertex v at round r
W_r	the set of white vertices at round r
B_r	the set of black vertices at round r
T_r	the set of 2-in-a-row vertices ($T_r = W_r \cap W_{r-1} (r > 0)$)
S_r	the set of historically 2-in-a-row vertices ($S_r = T_1 \cup \ldots \cup T_r$)
$\sigma(n)$	the maximal size of the smallest static monopoly over all connected graphs with n vertices
$\delta(n)$	the maximal size of the smallest dynamic monopoly over all connected graphs with n vertices
$\overline{\delta}_\infty$	the upper limit $\overline{\lim} \frac{\delta(n)}{n}$
$\underline{\delta}_\infty$	the lower limit $\underline{\lim} \frac{\delta(n)}{n}$
σ_∞	the limit $\lim \frac{\sigma(n)}{n}$
D	the set of duplicated vertices
U	the set of unduplicated vertices
J_n	the graph obtained from the graph J by duplicating n times the vertices of D. (when using this notation we assume D is known.)
$\hat{G}$	the bipartate graph obtained from the graph G by introducing two copies of every vertex and connecting them so that every vertex has an odd degree
$(S, \overline{S})$	the cut of set S (the set of all edges from the graph G with one vertex in S and one not in S. When using this notation We assume G is known)

1 introduction

1.1 definitions

A *simple undirected graph* is a pair $G = (V, E)$, where V is a set, and E is a set of subsets of V of size 2. The elements of V are called *vertices* and the elements of E are called *edges*. If $v \in V$ then $N(v)$ denotes the set of neighbors of v (not including v itself), that is $N(v) = \{u : \{v, u\} \in E\}$. We call $d(v) = |N(v)|$ the *degree* of v.

For every $r = 0, 1 \ldots$ we define C_r as a function from V to $\{\mathcal{B}, \mathcal{W}\}$. If $C_r(v) = \mathcal{W}$ we say that v is white at round r and if $C_r(v) = \mathcal{B}$ we say that v is black at this round. The definition of C_r is recursive. The function C_0 is chosen arbitrarily. For $r > 0$, and for every $v \in V$ if $|C_{r-1}^{-1}(\mathcal{B}) \cap N(v)| > |C_{r-1}^{-1}(\mathcal{W}) \cap N(v)|$ then $C_r(v) = \mathcal{B}$. If $|C_{r-1}^{-1}(\mathcal{W}) \cap N(v)| > |C_{r-1}^{-1}(\mathcal{B}) \cap N(v)|$ then $C_r(v) = \mathcal{W}$. If $|C_{r-1}^{-1}(\mathcal{B}) \cap N(v)| = |C_{r-1}^{-1}(\mathcal{W}) \cap N(v)|$ then $C_r(v) = C_{r-1}(v)$.

We also define $W_r = C_r^{-1}(\mathcal{W})$, $B_r = C_r^{-1}(\mathcal{B})$, $T_r = W_r \cap W_{r-1} (r > 0)$ and $S_r = T_1 \cup \ldots \cup T_r$.

Note that once W_0 is known, the functions $C_0, C_1, C_2 \ldots$ are uniquely defined. If $W_r = V$ for some finite r then we say that W_0 is a *dynamic monopoly* abbreviated *dynamo*. If $W_1 = V$ we say that W_0 is a *static monopoly*.

1.2 Motivation

One of the central problems in distributed computing is the overcoming of failures. The applicability of the polling game described above to this purpose is well established by Peleg in the introduction to [5] (see also [24]). A common method for overcoming failures is the *majority ruling*. The assumption is that the number of failures is small, and therefore by performing a *voting* process one can eliminate, or at least restrict, the damage caused by failures.

Peleg brings many examples of this method used as a component of fault-tolerant algorithms in a wide variety of contexts, such as consensus and agreement problems ([7], [8], [9]), mutual exclusion ([10], [11]) and other quorum system applications ([12], [13], [14], [15], [16]), system level diagnosis ([17], [18], [19]), distributed database management algorithms ([20], [21]), and fault local mending ([22], [23]).

Let us elaborate on a few of the above problems:

consensus

A typical consensus problem is the following (see [8]):

Consider n processes $p_1 \dots p_n$ out of which, at most t are malfunctioning (or controlled by hackers) and the other are correct. Each process p_i holds a value v_i. A *consensus protocol* is a protocol (usually probabilistic) with the following properties:

Decidability: If all correct processes obey the protocol, and assuming the worst about the malfunctioning processes, with probability 1, a finite time after the beginning of the protocol, every correct process makes an irreversible *decision* on some value.

Validity: If all correct processes start with the same initial value v, they all decide on v.

agreement: All correct processes decide on the same value. (Even if initially they have different values).

Note that we must assume the possibility that the malfunctioning processes deviate from the protocol. They may fail to send any message when required and may send different and contradicting messages to different processes.

If $n \leq 3t$ then it is easy to prove such a protocol does not exist. If $n > 3t$ then a protocol exists. Actually, Bracha [8] introduced an $O(\log n)$ expected rounds consensus protocol when $n \geq (3+\epsilon)t$ for some constant ϵ.

A simple algorithm can be found at [29]. The main idea is repetitive voting, when action is determined according to the size of the majority. A big majority causes a decision, an intermediate majority determines the next vote and a small majority (or a tie) will be followed by a coin toss. Bracha's algorithm is more complicated, but the basic building block is still repetitive voting.

A famous very related problem is the *Byzantine Generals Problem.* Consider a commanding general who wants to send an order to $n-1$ lieutenant generals. Out of the n generals, at most t are traitors, the other are loyal. note that the commanding general himself may be a traitor. The problem is to bring all the loyal lieutenant general to obey the order. If the commanding general is himself loyal, we may assume he gives the same order to all his $n-1$ lieutenant generals, and we want all loyal lieutenant generals to obey this order. If the commanding general is a traitor, he may send contradicting orders. In that case the loyal lieutenants should find a way to perform the same order. (Anything is good as long as they do the same thing.)

fault local mending

Consider a graph $G = (V, E)$. Each vertex represents an FSM (*finite state machine*) that can be in any state of the set S. Let S^V be the state space of the graph, that is, the set of functions from V to S. Let $L \subseteq S^V$ be a set of states called *legal* states.

For example (see [22]), we may take $S = \{0, 1\}$ and we may take L to be the set of all maximal matchings. That is, every $F \in L$ assigns 1 and 0 to vertices, so that no two connected vertices u, v have $F(u) = F(v) = 1$, and if a vertex v has $F(v) = 0$ then v must have a neighbor u with $F(u) = 1$.

The problem of *fault local mending* is the following: Given $F \in S^V$ and assuming there exists a legal state $F' \in L$ obtained from F by changing the value in at most t of the vertices, find a legal state (not necessarily F') in a complexity which depends only on t. Of course, f' is not given. In one model of this problem, neither t is given.

In the algorithm given at [22], every vertex holds a record about every other vertex up to distance 2. The records are used to vote in polls done by these vertices.

quorum systems

A *quorum system* (see e.g. [15]) is a collection of pair-wise intersecting sets in a finite universe. For example, $\{\{a, b\}, \{b, c, d\}, \{a, d, e\}, \{a, c, e\}\}$ is a quorum in $\{a, b, c, d, e\}$. If we force that every change in a network will be on all members of one set at the quorum system, then we may be sure every change will override earlier changes. This property allows us to archive features such as *mutual exclusion* (see [10]), that is, preventing two processes from using the same resource at the same time, and *replication control*, namely, forcing the system to use the most up-to-date data, when the data is stored in several locations.

Note that if the universe size is $2n - 1$ or $2n - 2$ for some $n = 2, 3, \ldots$ then the collection of subsets of size n is a quorum system. Thus we may recognize this quorum simply by voting. Indeed, there exist other types of easy-to-use quorum systems, but voting systems are most favored due to their simplicity.

In most of the above algorithms, optimal results are archived, if all components of the system participate in all voting polls. However, in today's networks, performing repetitive global polls may be very expensive. The hardware usually forces a certain graph, such that in t time units a vertex v (representing a processor) can collect data only from vertices with distance at most t from v. Therefore, in many cases we have to compromise on *local* voting, when a vertex performs a poll only among its neighbors.

The research made so far on static monopolies is a good model when voting is performed only once (simultaneously for all vertices). However, this is rarely the case. In most of the algorithms, voting are performed again and again, allowing a vertex to indirectly influence another vertex far away. This highly motivates the research on dynamic monopolies.

The game discussed above may also be used as a model of many social, financial and linguistic phenomena, such as the propagation of slang words, the pattern of use of new technologies, the creation of religions and so on. Of course, this may be a very rough model. Many factors in reality are ignored in the model, the continuous time is simplified to discrete time, and so on. However, the main theme of local majority repeats in a wide variety of phenomena. Still, much more research must be done before the results on dynamic monopolies can be used outside computer science, but the results of the models studied in this thesis may give a good hint on the behavior of other models, more complicated and more faithful to specific real-life phenomena.

2 Proof of Theorem 2

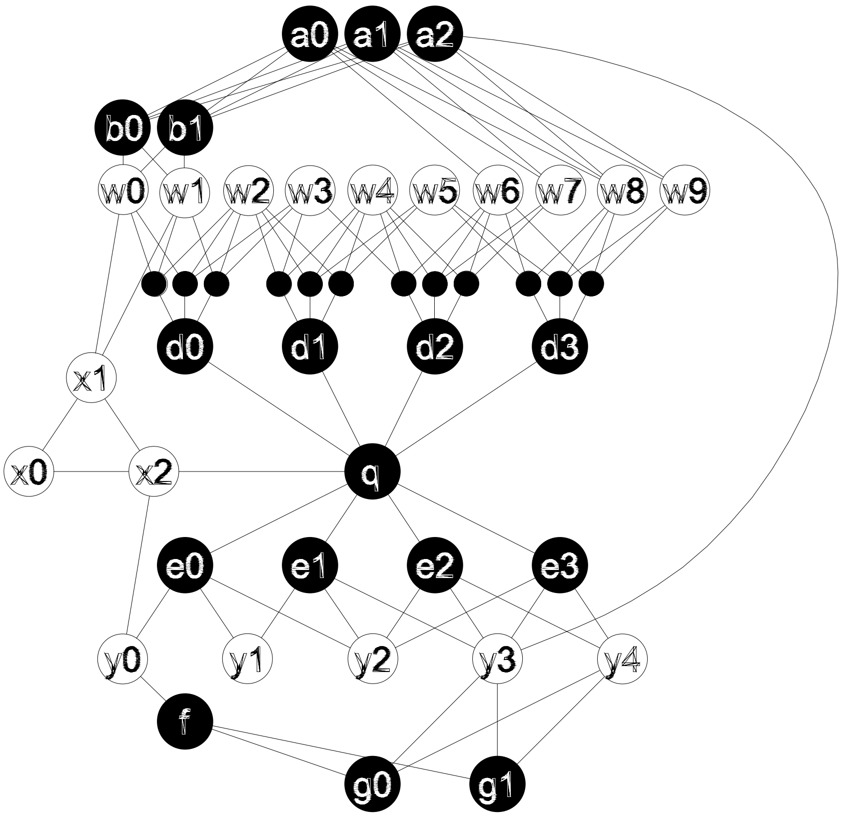

Figure 1: The graph J. (The small black circles are the vertices $c_0 \dots c_{11}$.)

Let $J = (V_J, E_J)$ be the graph in Figure 1. Let

$$W_0 = \{w_0, \dots, w_9, x_0, \dots, x_2, y_0, \dots, y_4\}$$

and let $U = W_0 \cup \{q\}$ and $D = V_J - U$. We construct a graph J_n by duplicating n times the vertices in D. That is,

$$J_n = (V_n, E_n)$$

where

$$V_n = U \cup [n] \times D$$

and

$$E_n = \{\{u,v\} \in J : u,v \in U\} \cup \{\{u,(i,v)\} : \{u,v\} \in J, u \in U, v \in D, i \in [n]\}$$
$$\cup \{\{(i,u),(i,v)\} : \{u,v\} \in J,\ u,v \in D, i \in [n]\}$$

(Here, as usual, $[n]$ denotes the set $\{1 \ldots n\}$).

Note that for reasons of symmetry, at a given round, all copies of a vertex in J have the same color. Thus we may write "e_0 is white at round 4" instead of "(i, e_0) is white at round 4 for every $i \in [n]$" etc.

The following table describes the evolution of J_n. The symbol 1 stands for white and 0 stands for black. Note that the table does *not* depend on n. (This property is peculiar to the graph J. In general graphs duplication of vertices may change the pattern of evolution of the graph).

Table 1

r	a_{012}	b_{01}	$c_0 \ldots c_{11}$	d_{0123}	e_{0123}	f	g_{01}	q	$w_0 \ldots w_9$	y_{01234}
0	000	00	000000000000	0000	0000	0	00	0	1111111111	11111
1	111	00	111111111111	0000	1111	0	11	0	0000000000	00000
2	000	11	000000000000	1111	0000	1	00	1	1111111111	11111
3	111	00	111111111111	0000	1111	0	11	1	1100000000	10000
4	000	11	100000000000	1111	1000	1	00	1	1111111111	11111
5	111	00	111111111111	1000	1111	0	11	1	1100000000	11000
6	000	11	111000000000	1111	1100	1	00	1	1111111111	11111
7	111	00	111111111111	1000	1111	0	11	1	1111000000	11100
8	000	11	111100000000	1111	1111	1	00	1	1111111111	11111
9	111	00	111111111111	1100	1111	0	11	1	1111000000	11111
10	000	11	111111000000	1111	1111	1	11	1	1111111111	11111
11	111	00	111111111111	1100	1111	1	11	1	1111110000	11111
12	000	11	111111100000	1111	1111	1	11	1	1111111111	11111
13	111	00	111111111111	1110	1111	1	11	1	1111110000	11111
14	000	11	111111111000	1111	1111	1	11	1	1111111111	11111
15	111	00	111111111111	1110	1111	1	11	1	1111111100	11111
16	000	11	111111111100	1111	1111	1	11	1	1111111111	11111
17	111	00	111111111111	1111	1111	1	11	1	1111111100	11111
18	000	11	111111111111	1111	1111	1	11	1	1111111111	11111
19	111	00	111111111111	1111	1111	1	11	1	1111111111	11111
20	111	11	111111111111	1111	1111	1	11	1	1111111111	11111
21	111	11	111111111111	1111	1111	1	11	1	1111111111	11111

The table shows that at round 20 the entire system is white and therefore W_0 is a dynamo. The reader may go through the table by himself, but in order to facilitate the understanding of what happens in the table let us add some explanations as to the mechanism of "conquest" used in this graph.

We say that round j *dominates* round i if $W_i \subseteq W_j$.

We shall make use of the following obvious fact:

Observation 1 *If round j dominates round i $(i, j = 0, 1 \ldots)$ then round $j+1$ dominates round $i+1$.*

By applying this observation k times, we find that if round j dominates round i then round $j+k$ dominates round $i+k$ $(i, j, k = 0, 1 \ldots)$. By looking at the table one can see that in the graph J_n round 2 dominates round 0 and thus we have

Corollary 1 *Round $k+2$ dominates round k in J_n for every $k = 0, 1 \ldots$*

We say that a vertex v *blinks* at round r if $C_{r+2i}(v) = \mathcal{W}$ for every $i = 0, 1 \ldots$. We say that a vertex v is *conquered* at round r if $C_{r+i}(v) = \mathcal{W}$ for every $i = 0, 1 \ldots$. Examining rounds 0 to 3 in the table and using Corollary 1 one can see that x_0, x_1 and x_2 are conquered at round 0, and in addition q, w_0, w_1 and y_0 are conquered at round 2. Furthermore, every vertex in J_n blinks either at round 1 or at round 2.

Finally, we have

Lemma 1 *If at round r a vertex v in J_n has at least half of its neighbors conquered then v is conquered at round $r+2$.*

Proof: Every vertex in J_n blinks either at round 1 or at round 2, and hence v is white either at round $r+1$ or at round $r+2$. From this round on, at least half of the neighbors of v are white, so v will stay white.

$\square$

Now the vertices will be conquered in the following order:

$x_0, x_1, x_2, q, w_0, w_1, y_0, c_0, e_0, d_0, y_1, c_1, c_2, e_1, w_2 w_3, y_2, c_3, e_2, e_3, d_1, y_3, y_4,$
$c_4, c_5, g_0, g_1, f, w_4, w_5, c_6, d_2, c_7, c_8, w_6, w_7, c_9, d_3, c_{10}, c_{11}, w_8, w_9,$
$a_0, a_1, a_2, b_0, b_1.$

Eventually, the entire graph is colored white. J_n is a graph with $19 + 27n > n$ vertices and W_0 is a dynamo of size 18, proving Theorem 2.

3 Other Problems Related to the Game

For every natural number n, let $\delta(n)$ be the maximum of the minimum of the size of a dynamic monopoly, and let $\sigma(n)$ be the maximum of the minimum of the size of a static monopoly, when in both cases, the maximum is taken over all the connected graphs with n vertices, and the minimum is taken over all the (dynamic or static) monopolies in those graphs.

Let $\overline{\delta}_\infty = \overline{\lim}\frac{\delta(n)}{n}$, $\underline{\delta}_\infty = \underline{\lim}\frac{\delta(n)}{n}$, $\sigma_\infty = \lim \frac{\sigma(n)}{n}$. The last limit exists because $\sigma(n+m) \geq \sigma(n)+\sigma(m)-2$ for every natural numbers m, n. Obviously $\underline{\delta}_\infty \leq \overline{\delta}_\infty \leq \sigma_\infty$. The following theorem gives a lower bound for these three:

Theorem 3 *Every natural number $k > 1$ and every $i = 0, 1, 2, 3$ satisfy $\delta(4k+1-i) \geq 3k-i$.*

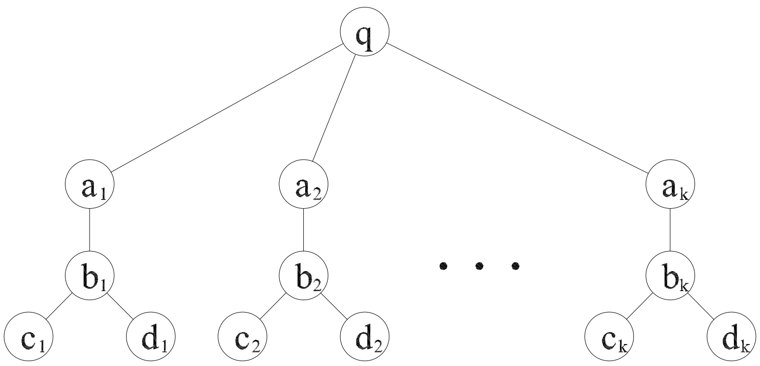

Figure 2: The graph used to proove $\underline{\delta}_\infty \geq 0.75$. (Compare this graph with the graph in Figure 5 of [24])

Proof: Consider the graph in Figure 2. For every $j = 1, \ldots, k$, if at any round b_j is black then b_j can never be conquered. Thus every dynamo must contain b_j and at least two of the three a_j, c_j and d_j. This proves $\delta(4k+1) \geq 3k$. By eliminating d_k we have $\delta(4k) \geq 3k-1$. By eliminating also c_k we have $\delta(4k-1) \geq 3k-2$. And finally, by eliminating also b_k we have $\delta(4k-2) \geq 3k-3$. $\Box$

Corollary 2 $\underline{\delta}_\infty \geq 0.75$

The following theorem shows that for trees the bound is tight:

Theorem 4 *Let T be a tree with n vertices ($n > 2$), then T has a static monopoly with at most $0.75n$ vertices.*

Proof: Choose one vertex r with $d(r) > 1$ to be the *root*. Every vertex $v \neq r$ has exactly one neighbor $f(v)$ closer than v to r. The vertex $f(v)$ will be called the *father* of v, and v will be called the *child* of $f(v)$. Note that the neighbors of a vertex are exactly its father and its children.

Now apply the following algorithm:

1. If all vertices are colored, halt.
2. Let v be the vertex furthest from r which is not colored yet. Color it black.
3. If $v = r$ goto stage 1. Otherwise, let $u = f(v)$. If u is not colored yet, color it white.
4. If $u \neq r$ let $w = f(u)$. If $u = r$ and u has children not already colored, let w be one of these children. If there are no such children goto stage 1. If w is not colored yet, color it white.
5. If u has children not already colored, choose one of them and color it white.
6. Goto stage 1

Every time the algorithm goes through stage 2 a vertex is colored black, and at each of the next 3 stages, at most one vertex is colored white. Thus the set W of the vertices colored white when the algorithm halts satisfies $|W| \leq 0.75n$.

Note that the black vertices have 3 generations distance. That is, if v is black then $f(v)$ and $f(f(v))$ if exist can not be black. Consider a vertex $u \neq r$. We will show that u becomes white in one round. If $f(u)$ is black then u is white and u has children all of which are white and we are done. If u has black children then u and $f(u)$ are white and the number of black children is at most 1 more than the number of white children. Again this is enough to keep u white. If $u = r$ then u has at most half of its children black, and if u has any black children then u is white and will stay white.

Therefore W is a static monopoly. $\square$

My conjecture is that $\underline{\delta}_\infty = \overline{\delta}_\infty = 0.75$. However, σ_∞ does not have this value.

Theorem 5

i. $\sigma(14) \geq 12$

ii. $\sigma_\infty \geq 0.8$

Proof:

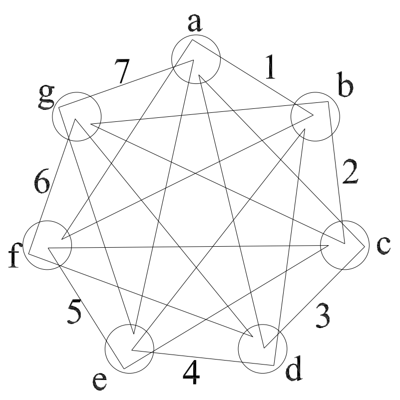

Figure 3: The complete graph on 7 vertices divided to 7 triangles

i. It is known that the complete graph on 7 vertices can be divided to 7 triangles (see Figure 3). Form a bipartate graph by introducing vertices instead of the triangles. The obtain graph has 7 vertices on each side, each with degree 3, where every two vertices at the same side have exactly one common neighbor (see Figure 4). Every set with less that 6 vertices on each side can not be a static monopoly, and thus $\sigma(14) \geq 12$.

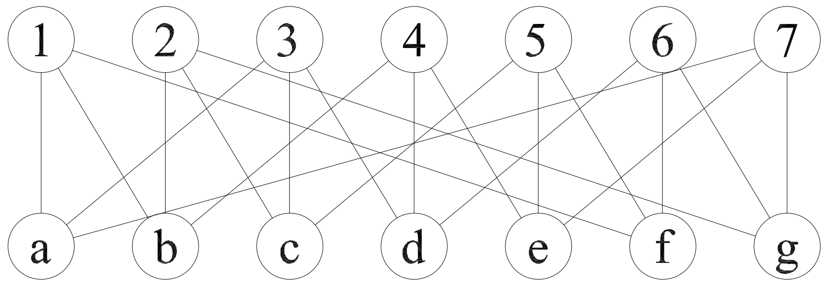

Figure 4: The bipartate graph used to proove $\sigma(14) \geq 12$. (This graph belongs to a class of graphs described at [24]. According to the notations there, this graph is $G(\mathcal{B}(7,3,1))$.)

ii. For every $k > 1$ take k copies of the bipartite graph described above and connect them as in Figure 5. The obtained graph has $15k$ vertices and for reasons similar to the above, it has no static monopoly of size less than $12k$. Thus, $\sigma_\infty \geq 0.8$.

$\square$

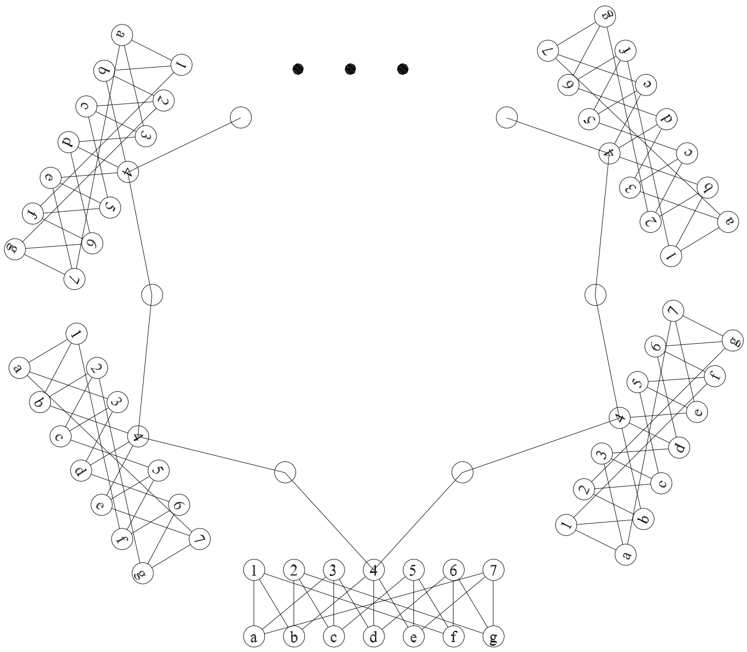

Figure 5: The graph used to proove $\sigma_\infty \geq 0.8$

4 Infinite Graphs

4.1 The cardinal model

The game described at the introduction may be applied to infinite graph as well. The result of Goles and Olivos regarding periodic behavior is no longer valid for infinite graphs. Moran (see [26], [27], [28]) established sufficient conditions to ensure the periodicity. However, the definition of a dynamo stays as in finite graphs.

The result of Section 2 gives rise to the following question:

Knowing that a constant number may be the size of

Question 1 *Does there exist an infinite graph with a finite dynamo?*

The answer is *no*. This follows from the following theorem:

Theorem 6 *If W_0 is finite then T_r is finite for all $r = 1, 2 \ldots$. Moreover, every vertex in T_r has a finite degree.*

Proof: The proof is by induction on r.

For $r = 1$ note that W_0 is finite, and therefore $T_1 = W_0 \cap W_1$ is also finite. Furthermore, every vertex $v \in W_0$ with an infinite degree becomes black at round 1, and thus $v \notin W_1$, in other words, a vertex with an infinite degree can not be both in W_0 and W_1 and therefore can not be in T_1.

For $r > 1$, let v be any vertex.

First, suppose v has an infinite degree λ. We will show that $v \notin T_r$. If $C_{r-1}(v) = \mathcal{B}$ we are done. If $C_{r-1}(v) = \mathcal{W}$ note that by the induction hypotheses $v \notin T_{r-1}$, and therefore $C_{r-2}(v) = \mathcal{B}$. In order to make v change color to white at round $r-1$ it must have a majority of white neighbors at round $r-2$, that is $|N(v) \cap B_{r-2}| < \lambda$. By the definition of T_{r-1} we have $W_{r-1} \subseteq B_{r-2} \cup T_{r-1}$ and hence $|N(v) \cap W_{r-1}| \leq |N(v) \cap B_{r-2}| + |T_{r-1}| < \lambda$. In other words, v has a majority of black neighbors at round $r-1$. Therefore at round r the vertex v will be colored black and again we have $v \notin T_r$. In any case, T_r does not have a vertex with an infinite degree.

If $v \in T_r$ has a finite degree we will show that either $v \in T_{r-1}$ or v has a neighbor in T_{r-1}. By the induction hypotheses only finitely many vertices have such a neighbor, and this will prove that T_r is finite. If v is in T_r but not in T_{r-1} then by the definition $C_{r-2}(v) = \mathcal{B}$, $C_{r-1}(v) = \mathcal{W}$, $C_r(v) = \mathcal{W}$. In order to have this, v must have more than half of its neighbors white at round $r-2$ and at least half of its neighbors white at round $r-1$. That is, $|N(v) \cap W_{r-2}| > \frac{|N(v)|}{2}$ and $|N(v) \cap W_{r-1}| \geq \frac{|N(v)|}{2}$.

Hence $|N(v) \cap W_{r-2}| + |N(v) \cap W_{r-1}| > |N(v)|$ and $N(v) \cap W_{r-2} \cap W_{r-1} = N(v) \cap T_{r-1} \neq \emptyset$, so v has a neighbor in T_{r-1}. This proves

$$T_r \subseteq T_{r-1} \cup \bigcup_{u \in T_{r-1}} N(u)$$

which is a finite set by the induction hypothesis.
$\square$

4.2 The ordinal model

In the above theorem we assumed there is a tie when a vertex has the same cardinality of black neighbors and white neighbors. There is, however, the possibility to deal with these cases using ordinal number.

Let G be a graph and suppose the set of vertices is well ordered. Actually we may assume that the set of vertices is an ordinal number α. Note that every subset of α is isomorphic (as an ordered set) to some $\beta \leq \alpha$. This allows us to define another model of the game.

For any $v \in \alpha$ and $r = 0, 1, \ldots$, let β be the ordinal number isomorphic to $N(v) \cap B_r$ and let γ be the ordinal number isomorphic to $N(v) \cap W_r$. If $\gamma > \beta$ then we say v has a *white ordinal majority* and at round $r+1$ we color v to white. Respectively, if $\gamma < \beta$ then we say v has a *black ordinal majority* and at round $r+1$ we color v to black. If $\gamma = \beta$ then we say there is a tie and at round $r+1$ the vertex v stays with the color it had at round r.

Theorem 7 *Let λ be an infinite cardinality and let α be the minimal ordinal number of cardinality λ.*

i. If G is a graph on α then G has no dynamo of cardinality less than λ.

ii. If $\beta > \alpha$ is another ordinal number of the same cardinality λ then there exists a graph on β with a finite dynamo.

Proof:

i. We will show by induction on r that if $|W_0| < \lambda$ then $|T_r| < \lambda$ for all $r = 1, 2 \ldots$. Moreover, every vertex in T_r has a degree less than λ.

For $r = 1$ note that $|W_0| < \lambda$, and therefore $|T_1| = |W_0 \cap W_1| < \lambda$. Furthermore, every vertex $v \in W_0$ with degree λ becomes black at round 1, and thus $v \notin W_1$, in other words, a vertex with degree λ can not be both in W_0 and W_1 and therefore can not be in T_1.

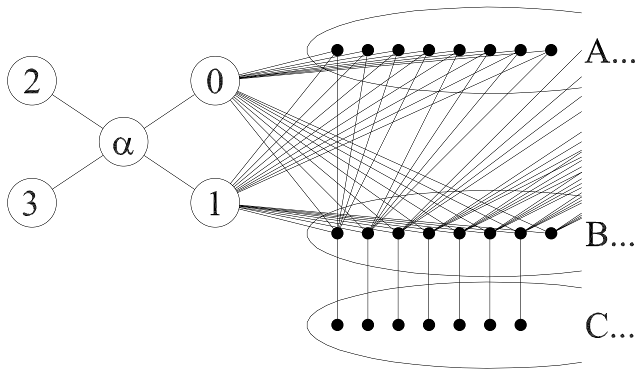

Figure 6: A grph with a dynamo of size 3 in the ordinal model

For $r > 1$, let v be any vertex.

First, suppose $d(v) = \lambda$. We will show that $v \notin T_r$. If $C_{r-1}(v) = \mathcal{B}$ we are done. If $C_{r-1}(v) = \mathcal{W}$ note that by the induction hypotheses $v \notin T_{r-1}$, and therefore $C_{r-2}(v) = \mathcal{B}$. In order to make v change color to white at round $r-1$ it must have an ordinal majority of white neighbors at round $r-2$. By using the well known fact that every subset of α with cardinality λ is isomorphic to α, we have $|N(v) \cap B_{r-2}| < \lambda$. As before, we use the fact $W_{r-1} \subseteq B_{r-2} \cup T_{r-1}$ to show $|N(v) \cap W_{r-1}| \leq |N(v) \cap B_{r-2}| + |T_{r-1}| < \lambda$. In other words, v has a majority (even a cardinal majority) of black neighbors at round $r-1$. Therefore at round r the vertex v will be colored black and again we have $v \notin T_r$. In any case, T_r does not have a vertex with degree λ.

If $v \in T_r$ has a degree less than λ we will show that either $v \in T_{r-1}$ or v has a neighbor in T_{r-1}. By the induction hypotheses only less than λ vertices have such a neighbor, and this will prove that $|T_r| < \lambda$. If v is in T_r but not in T_{r-1} then by the definition $C_{r-2}(v) = \mathcal{B}$, $C_{r-1}(v) = \mathcal{W}$, $C_r(v) = \mathcal{W}$. In order to have this, $N(v) \cap W_{r-2}$ must be isomorphic to an ordinal number greater than the ordinal number isomorphic to $N(v) \cap B_{r-2}$. Suppose $N(v) \cap T_{r-1} = \emptyset$ then $N(v) \cap$

$W_{r-2} \subseteq N(v) \cap B_{r-1}$, $N(v) \cap W_{r-1} \subseteq N(v) \cap B_{r-2}$, and hence $N(v) \cap B_{r-1}$ must be isomorphic to an ordinal number greater than the ordinal number isomorphic to $N(v) \cap W_{r-1}$. This makes v black at round r, in contradiction with the assumption $v \in T_r$. Thus $N(v) \cap T_{r-1} \neq \emptyset$, so v has a neighbor in T_{r-1}. This proves

$$T_r \subseteq T_{r-1} \cup \bigcup_{u \in T_{r-1}} N(u)$$

which is a set of cardinality less than λ by the induction hypothesis.

ii. One way to prove this is by noticing that the proof of Theorem 2 is still valid if we take n to be an ordinal number rather than a natural number. However, the ordinal case is actually much easier than the finite case. In fact, in the ordinal case there exists a dynamo of size 3.

Let $\beta > \alpha$ be an ordinal of cardinality λ. Divide the set $\{v : 3 < v < \alpha\}$ to two disjoint sets A and B, each of cardinality λ, and let $C = \{v : \alpha < v < \beta\}$. Let f be a one-to-one function from C into B, and let g be a four-to-one function from A onto B. (That is, every $v \in B$ satisfies $|g^{-1}(v)| = 4$.) Let $G = (\beta, E)$ where

$$E = \{\{\alpha, 0\}, \{\alpha, 1\}, \{\alpha, 2\}, \{\alpha, 3\}\} \cup \{\{0, v\}, \{1, v\} : v \in A \cup B\}$$

$$\cup\{\{v, f(v)\} : v \in C\} \cup \{\{v, g(v)\} : v \in A\}$$

Note that A and B are both isomorphic to α, $A \cup \{\alpha\}$ and $B \cup \{\alpha\}$ are both isomorphic to $\alpha + 1$, every vertex in A has degree 3, every vertex in B has degree 6 or 7 and the vertex α has degree 4 (see Figure 6).

Now if we take $W_0 = \{\alpha, 0, 1\}$ we have $W_1 = \{\alpha, 2, 3\} \cup A$, $W_2 = \{\alpha, 0, 1, 2, 3\} \cup B$, $W_3 = \{\alpha, 0, 1, 2, 3\} \cup A \cup C$, $W_3 = \{\alpha, 0, 1, 2, 3\} \cup A \cup B$ and finally $W_4 = \beta$.

□

5 Other Models on Finite Graphs

5.1 Dealing with ties

The next question deals with other models considered by Peleg:

Question 2 *Do we still have a dynamo of size O(1) if we change the rules of dealing with ties? (e.g. if a vertex becomes black whenever there is a tie.)*

The answer here is *yes*. This follows from the following lemma:

Lemma 2 *Let $G = (V, E)$ be a graph, introduce a new vertex v' for every $v \in V$ and consider the graph $\hat{G} = (\hat{V}, \hat{E})$ where*

$$\hat{V} = \{v, v' : v \in V\}$$

and

$$\hat{E} = E \cup \{\{u', v'\} : \{u, v\} \in E\} \cup \{\{v, v'\} : 2|d(v)\}$$

If W_0 is a dynamo of G according to the model in Theorem 2, then $\hat{W_0} = \{v, v' : v \in W_0\}$ is a dynamo of $\hat{G}$.

Proof:

Let $W_0, W_1, W_2 \ldots$ denote the sets of white vertices in the evolution of G and respectively, let $\hat{W_0}, \hat{W_1}, \hat{W_2} \ldots$ denote the sets of white vertices in the evolution of $\hat{G}$.

We will show by induction on r that $\{v, v' : v \in W_r\} \subseteq \hat{W_r}$. (Actually there is an equation here.)

For $r = 0$ this is true by definition.

For $r > 0$ let v be any vertex in W_r. We will show that $v' \in \hat{W_r}$

If $d(v)$ is odd then the neighbors of v' in $\hat{G}$ are exactly the neighbors of v in G. More than half of these vertices are in W_{r-1} and hence by the induction hypothesis in $\hat{W_{r-1}}$. This implies $v' \in \hat{W_r}$.

If $d(v)$ is even then the neighbors of v' in $\hat{G}$ are the neighbors of v in G and v itself. Again, more than half of these vertices are in W_{r-1} and hence by the induction hypothesis in $\hat{W_{r-1}}$. Still we have $v' \in \hat{W_r}$.

The proof that $v \in \hat{W_r}$ is analogous.

□

Note now that all vertices of $\hat{G}$ have odd degrees, and thus ties are not possible and $\hat{W_0}$ is a dynamo of $\hat{G}$ according to *any* rule of dealing with ties.

Therefore, for every $n = 1, 2 \ldots$ the graph $\hat{J_n}$ has a dynamo of size 36.

5.2 The ρ model

Let $\rho > 1$ be a real number. Consider the following model, which will henceforth be called *the ρ-model.* At every round, for every vertex v with b neighbors colored black and w neighbors colored white, if $w > \rho b$ then v is colored white at the next round, otherwise it is black. For the sake of simplicity we will assume that ρ is irrational and that there are no isolated vertices, so that $w = \rho b$ is impossible.

The most interesting question regarding this model is whether there exist graphs with O(1) dynamo like in Theorem 2. This question is as yet open. We only have some partial results, which can be summarized as follows:

i. If ρ is big enough then the size of a dynamo is $\Omega(\sqrt{n})$.

ii. If ρ is small enough then there exist graphs in which the size of a dynamo is $O(\log n)$.

iii. If there exist graphs with O(1) dynamo then the number of rounds needed until the entire system becomes white is $\Omega(\log n)$.

More explicitly:

Theorem 8 *Let $\rho > 3$. If a graph with n vertices has a dynamo of size k in the ρ-model then*

$$n < k^2$$

proof:

For every $r = 1, 2, \ldots$, let $(S_r, \overline{S}_r)$ be the set of edges with one vertex in S_r and the other not in S_r. Call $s_r = |S_r| + |(S_r, \overline{S}_r)|$. Note that S_1 is the set of vertices which are white at both round 0 and round 1. Every $v \in S_1$ is connected to at most $k - |S_1|$ vertices in $W_0 \setminus S_1$ and at most $\frac{k-1}{\rho} < k - 1$ vertices outside of W_0. Therefore we have

$$s_1 < |S_1| + |S_1|(k - |S_1| + k - 1) = k^2 - (k - |S_1|)^2 \leq k^2$$

Thus all we need is to show $s_{r+1} \leq s_r$ and we are done.

Let r be fixed. By definition $S_r \subseteq S_{r+1}$. Let $\Delta = S_{r+1} \setminus S_r$, and let $v \in \Delta$. More than $\frac{3}{4}$ of the neighbors of v are white at round r and more than $\frac{3}{4}$ of the neighbors of v are white at round $r - 1$. Thus more than $\frac{1}{2}$ of the neighbors of v belong to S_r. In other words $|\{u : \{u, v\} \in (S_r, \overline{S}_r)\}| - |\{w : \{v, w\} \in (S_{r+1}, \overline{S}_{r+1})\}| \geq 1$.

Summing this inequality for every $v \in \Delta$ yields

$$|(S_r, \overline{S}_r) \setminus (S_{r+1}, \overline{S}_{r+1})| - |(S_{r+1}, \overline{S}_{r+1}) \setminus (S_r, \overline{S}_r)| \geq |\Delta|$$

which implies $s_{r+1} \leq s_r$. By induction $s_r < k^2$ for all r. If we begin with a dynamo then for some finite m we have $S_m = V$ and $n = s_m < k^2$ $\Box$

Theorem 9 *Let $\rho > 1$. If $|W_0| = k$ and $W_m = V$ (the set of all vertices), then the number e of edges in the graph satisfies*

$$e < k^2 (\frac{2\rho}{\rho - 1})^m$$

proof:
Let d_r denote the sum of the degrees of the vertices in S_r. Recall that every $v \in S_1$ is white at both round 0 and round 1, and thus $|N(v) \cap B_0| < k$ and $d(v) < k$. Therefore, $d_1 < 2k^2$. Again, let r be fixed, let Δ be as in the proof of Theorem 8 and let $v \in \Delta$. More than $\frac{\rho}{\rho+1}$ of the neighbors of v are white at round r and more than $\frac{\rho}{\rho+1}$ of the neighbors of v are white at round $r-1$. Thus more than $\frac{\rho-1}{\rho+1}$ of the neighbors of v belong to S_r. Therefore, we have

$$d_{r+1} < d_r + \frac{\rho+1}{\rho-1} d_r = \frac{2\rho}{\rho-1} d_r$$

By induction $d_r < 2k^2 (\frac{2\rho}{\rho-1})^{r-1}$. If the entire system is white at round m then $d_{m+1} = 2e$ and thus we have

$$e < k^2 (\frac{2\rho}{\rho - 1})^m$$

$\Box$

Theorem 10 *Let $1 < \rho < \frac{257}{256}$. For every integer $n > 5$ there exists in the ρ model a graph with more than 2^n vertices and with a dynamo of size $30(n-5) + 36$.*

Outline of proof:
Let $\hat{J}$ be as defined in the answer to Question 2. Construct $\tilde{J}$ by eliminating f from $\hat{J}$ and connecting f' to y_0 and g_1 (but *not* to g_0). Note that in $\tilde{J}$ the vertex g_0 is connected only to y_3 and to y_4.
In figure 7, the upper graph is a part of $\hat{J}$. The lower graph is the corresponding part in $\tilde{J}$. The rest of $\tilde{J}$ is identical to the rest of $\hat{J}$.

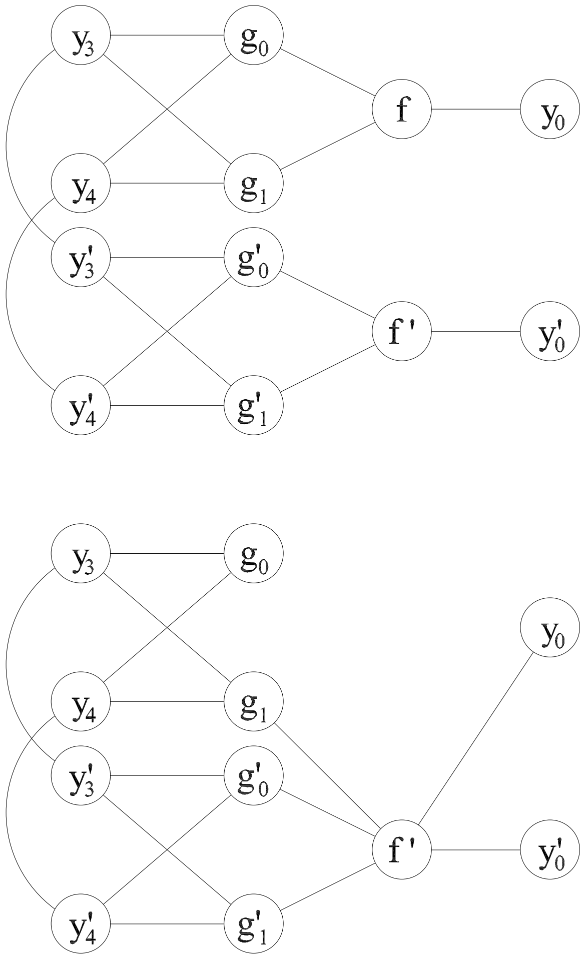

Figure 7:

Construct $\tilde{J}_{32}, \tilde{J}_{64}, \ldots \tilde{J}_{2^n}$ as in the construction of J_n, where the duplicated vertices are all black vertices except for q and q'. Note that the graphs are constructed separately, namely, the sets of vertices of $\tilde{J}_{2^i}$ and $\tilde{J}_{2^j}$ are disjoint for $i \neq j$. In order to avoid ambiguity we will use the following notation: the triplet $(v, i; 2^j)$ will refer to the i-th copy of v in $\tilde{J}_{2^j}$. If v is not duplicated, the pair $(v; 2^j)$ will refer to the single copy of v in $\tilde{J}_{2^j}$.

Now connect the graphs in the following way. First, eliminate the copies of x_0, x_1, x_2 from all graphs except for $\tilde{J}_{32}$. Note that in $\tilde{J}_{2^i}$ there are 2^i copies of g_0 (when $i = 5, \ldots n-1$). Divide them into 32 disjoint sets $P_0, \ldots P_{31}$, of size 2^{i-5} each. Now connect the vertices in P_0 to the copy of q in $\tilde{J}_{2^{i+1}}$, connect P_1 to the copy of q', and connect each one of $P_2 \ldots P_{31}$ to a respective white vertex in $\tilde{J}_{2^{i+1}}$ (see in figure 8).

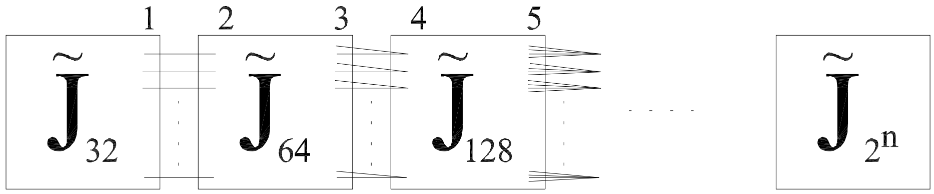

Figure 8: This figure illustrates the graph used in the proof of Theorem 10. The vertices under the numeral 1 are the 32 copies of g_0 in $\tilde{J}_{32}$. Under the numeral 2 are the 32 unduplicated vertices in $\tilde{J}_{64}$ (q, q' and the initially white vertices). Under the numeral 3 are the 64 copies of g_0 in $\tilde{J}_{64}$, under the numeral 4 are the 32 unduplicated vertices in $\tilde{J}_{128}$, under the numeral 5 are the 128 copies of g_0 in $\tilde{J}_{128}$, and so on.

More explicitly, we define the graph $\hat{J} = (\hat{V}, \hat{E})$ where

$$\hat{V} = \{v, v' : v \in V_J\}$$
$$\hat{E} = E_J \cup \{(u', v') : (u, v) \in E\} \cup \{(v, v')) : 2|d(v)\}$$

We define another graph $\tilde{J} = (\tilde{V}, \tilde{E})$ where

$$\tilde{V} = \hat{V} \setminus \{f\}$$

$$\tilde{E} = \{e \in \hat{E} : f \notin e\} \cup \{\{f', y_0\}, \{f', g_1\}\}$$

We divide the set $\tilde{V}$ to three sets

$$X = \{x_0, x_1, x_2, x'_0, x'_1, x'_2\}$$

$$\tilde{U} = \{u, u' : u \in U\} \setminus X$$

$$\tilde{D} = \tilde{V} \setminus \tilde{U} \setminus X$$

Note that $|U| = 32$. In some cases it may be convenient to write $U = \{u_0, \ldots, u_{31}\}$. Finally we construct the graph $K = (V_K, E_K)$ where

$$\begin{aligned} V_K = \{(v, i; 2^j) : v \in \tilde{D}, i \in [2^j], 5 \le j \le n\} \\ \cup\{(u; 2^j) : u \in \tilde{U}, 5 \le j \le n\} \\ \cup\{(u; 32) : u \in X\} \end{aligned}$$

$$\begin{aligned} E_K = \{\{(u; 32), (v; 32)\} : v \in X, u \in U \cup X, \{u, v\} \in \tilde{E}\} \\ \cup\{\{(u; 2^j), (v; 2^j)\} : u, v \in \tilde{U}, \{u, v\} \in \tilde{E}, 5 \le j \le n\} \\ \cup\{\{(u; 2^j), (v, i; 2^j)\} : u \in \tilde{U}, v \in \tilde{D}, \{u, v\} \in \tilde{E}, i \in [2^j], 5 \le j \le n\} \\ \cup\{\{(u, i; 2^j), (v, i; 2^j)\} : u, v \in \tilde{D}, (u, v) \in \tilde{E}, i \in [2^j], 5 \le j \le n\} \\ \cup\{\{(g_0, i; 2^j), (u_{i \bmod 32}; 2^{j+1})\} : i \in [2^j], 5 \le j \le n-1\} \end{aligned}$$

Table 1 is still valid for rounds 0,1,2. (However now one should read it a little differently. For example, the numeral 1 for a_2 at round 1 means that every vertex in V_K of the form $(a_2, i, 2^j)$ or $(a'_2, i, 2^j)$ is white at round 1.)

Moreover, Observation 1 and Corollary 1 are still valid for K with the ρ-model, with similar proofs. Lemma 1 is no longer valid, but the following lemma can fill its place:

Lemma 3 *Let G be a graph colored according to the ρ-model.*

i. If at round r a vertex v has more than $\frac{\rho}{\rho+1}$ of its neighbors conquered then v is conquered at round $r+1$.

ii. *If v and v' are two connected vertices blinking at the same phase (that is, either they both blink at round 1 or they both blink at round 2), with the same odd degree $2k+1 < \frac{\rho+1}{\rho-1}$ and if at round r at least k of the neighbors of v and k of the neighbors of v' are conquered then v and v' are conquered at round $r+2$.*

Proof:

i. This is trivial.

ii. The two vertices blink at the same phase, and hence they are both white either at round $r+1$ or at round $r+2$. From this round on, at least $k+1$ of the neighbors of each vertex are white, so v and v' will stay white.

Now the vertices of $\tilde{J}_{32}$ will be conquered in the following order:

$(x_0;32)$, $(x'_0;32)$, $(x_1;32)$, $(x'_1;32)$, $(x_2;32)$, $(x'_2;32)$, $(q;32)$, $(q';32)$, $(w_0;32)$, $(w'_0;32)$, $(w_1;32)$, $(w'_0;32)$, $(y_0;32)$, $(y'_0;32)$, $(c_0,i;32)$ and $(c'_0,i;32)$ (for every $i\in[32]$ at the same round), $(e_0,i;32)$ and $(e'_0,i;32)$, $(d_0,i;32)$ and $(d'_0,i;32)$, $(y_1;32)$, $(y'1;32)$, $(c_1,i;32)$ and $(c'_1,i;32)$, $(c_2,i;32)$ and $(c'_2,i;32)$, $(e_1,i;32)$ and $(e'_1,i;32)$, $(w_2;32)$, $(w'_2;32)$, $(w_3;32)$, $(w'_3;32)$, $(y_2;32)$, $(y'_2;32)$, $(c_3,i;32)$ and $(c'_3,i;32)$, $(e_2,i;32)$ and $(e'_2,i;32)$, $(e_3,i;32)$ and $(e'_3,i;32)$, $(d_1,i;32)$ and $(d'_1,i;32)$, $(y_3;32)$, $(y'_3;32)$, $(y_4;32)$, $(y'_4;32)$, $(c_4,i;32)$ and $(c'_4,i;32)$, $(c_5,i;32)$ and $(c'_5,i;32)$, $(g_0,i;32)$, $(g'_0,i;32)$, $(g_1,i;32)$, $(g'_1,i;32)$, $(f',i;32)$, $(w_4;32)$, $(w'_4;32)$, $(w_5;32)$, $(w'_5;32)$, $(c_6,i;32)$ and $(c'_6,i;32)$, $(d_2,i;32)$ and $(d'_2,i;32)$, $(c_7,i;32)$ and $(c'_7,i;32)$, $(c_8,i;32)$ and $(c'_8,i;32)$, $(w_6;32)$, $(w'_6;32)$, $(w_7;32)$, $(w'_7;32)$, $(c_9,i;32)$ and $(c'_9,i;32)$, $(d_3,i;32)$ and $(d'_3,i;32)$, $(c_{10},i;32)$ and $(c'_{10},i;32)$, $(c_{11},i;32)$ and $(c'_{11},i;32)$, $(w_8;32)$, $(w'_8;32)$, $(w_9;32)$, $(w'_9;32)$, $(a_0,i;32)$, $(a'_0,i;32)$, $(a_1,i;32)$, $(a'_1,i;32)$, $(a_2,i;32)$, $(a'_2,i;32)$, $(b_0,i;32)$, $(b'_0,i;32)$, $(b_1,i;32)$, $(b'_1,i;32)$.

Once $(g_0,1;2^{j-1}),\ldots,(g_0,2^{j-1};2^{j-1})$ are conquered, the vertices of $\tilde{J}_{2^j}$ will be conquered in the following order:

$(q;2^j)$, $(q';2^j)$ (Note that here the bound $\rho < \frac{257}{256}$ is used.), $(w_0;2^j)$, $(w'_0;2^j)$, $(w_1;2^j)$, $(w'_0;2^j)$, $(y_0;2^j)$, $(y'_0;2^j)$, $(c_0,i;2^j)$ and $(c'_0,i;2^j)$ (for every $i\in[2^j]$ at the same round), $(e_0,i;2^j)$ and $(e'_0,i;2^j)$, $(d_0,i;2^j)$ and $(d'_0,i;2^j)$, $(y_1;2^j)$, $(y'1;2^j)$, $(c_1,i;2^j)$ and $(c'_1,i;2^j)$, $(c_2,i;2^j)$ and $(c'_2,i;2^j)$, $(e_1,i;2^j)$ and $(e'_1,i;2^j)$, $(w_2;2^j)$, $(w'_2;2^j)$, $(w_3;2^j)$, $(w'_3;2^j)$, $(y_2;2^j)$, $(y'_2;2^j)$, $(c_3,i;2^j)$ and $(c'_3,i;2^j)$, $(e_2,i;2^j)$ and $(e'_2,i;2^j)$, $(e_3,i;2^j)$ and $(e'_3,i;2^j)$, $(d_1,i;2^j)$ and $(d'_1,i;2^j)$, $(y_3;2^j)$, $(y'_3;2^j)$, $(y_4;2^j)$, $(y'_4;2^j)$, $(c_4,i;2^j)$ and $(c'_4,i;2^j)$, $(c_5,i;2^j)$ and $(c'_5,i;2^j)$, $(g_0,i;2^j)$, $(g'_0,i;2^j)$, $(g_1,i;2^j)$, $(g'_1,i;2^j)$, $(f',i;2^j)$, $(w_4;2^j)$, $(w'_4;2^j)$, $(w_5;2^j)$, $(w'_5;2^j)$, $(c_6,i;2^j)$ and $(c'_6,i;2^j)$, $(d_2,i;2^j)$ and $(d'_2,i;2^j)$, $(c_7,i;2^j)$

and $(c'_7, i; 2^j)$, $(c_8, i; 2^j)$ and $(c'_8, i; 2^j)$, $(w_6; 2^j)$, $(w'_6; 2^j)$, $(w_7; 2^j)$, $(w'_7; 2^j)$, $(c_9, i; 2^j)$ and $(c'_9, i; 2^j)$, $(d_3, i; 2^j)$ and $(d'_3, i; 2^j)$, $(c_{10}, i; 2^j)$ and $(c'_{10}, i; 2^j)$, $(c_{11}, i; 2^j)$ and $(c'_{11}, i; 2^j)$, $(w_8; 2^j)$, $(w'_8; 2^j)$, $(w_9; 2^j)$, $(w'_9; 2^j)$, $(a_0, i; 2^j)$, $(a'_0, i; 2^j)$, $(a_1, i; 2^j)$, $(a'_1, i; 2^j)$, $(a_2, i; 2^j)$, $(a'_2, i; 2^j)$, $(b_0, i; 2^j)$, $(b'_0, i; 2^j)$, $(b_1, i; 2^j)$, $(b'_1, i; 2^j)$.

Thus all vertices of K are eventually conquered. The theorem follows upon noticing that K has more than 2^n vertices, and the size of the dynamo is $30(n-5)+36$. $\square$

6 Conclusions

In this thesis we introduced several models of a game highly motivated by problem of overcoming failures in distributed systems. The results showed that in many cases it is possible for small sets of processes to affect the whole system. One conclusion is that one must be precocious when using repetitive local polls (especially when using heuristics that have no rigorous proof of validity). However, we believe that, when properly used, the graphs described in this thesis may be applied to systems such as neural networks, due to their ability to propagate information through a small number of channels.

We believe that the results obtained in this thesis can be extended to a wide variety of models, applicable in computer sciences and in social sciences.

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