FINAL REPORT

P-IEF, No. 219696 of Marie-Curie action, period **06/10/2008 to 05/10/2010**

Integral formulae and Extrinsic Geometry of Foliations

Vladimir Rovenski and Pawel Walczak (Uniwersytet Lodzki)

The project objectives are to study extrinsic geometry of foliations (properties expressed by the 2nd fundamental form b of the leaves) and its relations with topology and dynamics. The main results concern the development of research tools: 1) *Extrinsic Geometric Flow* (**EGF**), 2) *Integral Formulae* (**IF**), 3) *Variation Formulae* (**VF**), and applications of them to problems on foliations. The research could be relevant to geometers and young specialists in areas of Riemannian geometry and foliations, in Spain, France, Poland, USA, Japan etc.

1. **EGF** is introduced [1—3] as deformations of Riemannian metrics on a manifold equipped with a codimension-1 foliation subject to quantities expressed in terms of b. It is a tool for studying the question: *Under what conditions on (M, F, g_0) the metrics g_t of corresponding EGF converges to one for which F is totally umbilical, geodesic, or minimal*.

Definition [1]: Let (M, F) be a foliated manifold equipped with a transversal vector field N. Given transformation h acting on F-truncated symmetric (0, 2)-tensors, a 1-parameter family g_t of Riemannian metrics on M making N unit normal to F and satisfying $\partial_t g_t = \mathrm{h}(b^t)$, is called **EGF**. The important example is $\mathrm{h}(b) = \sum_{j=0}^{n-1} f_j \cdot \hat{b}_j$ where $\hat{b}_j$ ($\hat{b}_0 = \hat{g}$) are symmetric (0, 2)-tensors dual to j-th degree of Weingarten operator A, extended on TM by A(N) = 0. Auxiliary functions $f_j \in C^2$ are either (a) $f_j(p,t),\ p \in M$ or (b) $f_j(\vec{\tau},t)$, where $t \in \mathbb{R}_+$, $\vec{\tau} = (\tau_1,\dots,\tau_n)$, and $\tau_i = \mathrm{Tr}\, A^i$ are power sums of the principal curvatures k_i of F.

The terms $\hat{b}_j$ $(j < n)$ are meaningful: for example, the **EGF** corresponding to

– the *extrinsic Ricci* curvature tensor $\mathrm{Ric}^{\mathrm{ex}} = \tau_1 \hat{b}_1 - \hat{b}_2$ depends on $\hat{b}_1,\ \hat{b}_2$;

– the *Newton transformation* $T_i(b) = \sigma_i \hat{g} - \sigma_{i-1}\hat{b}_1 + \dots + (-1)^i \hat{b}_i$ $(i < n)$ depends on all $\hat{b}_j$ $(j \le i)$.

Definition [2]: ***EGS*** structure on a foliated manifold (M, F, N) is a pair (g, X) of a metric g, and a complete vector field X preserving F satisfying the PDE $\mathrm{h}(b) = \varepsilon \hat{g} + L_X \hat{g}$ (with the Lie derivative) for a $\varepsilon \in \mathbb{R}$. Depending on X, **EGS** might be *tangent* (X ∈ TF) and *normal* (X ⊥ TF).

The existing and classifying **EGS** have great interest and are closely related to the basic problem in theory of foliations (first posed by H. Gluck for geodesic foliations):

Problem: *Given* (P), *a property of Riemannian submanifolds expressed in terms of the second fundamental form and its invariants, and a foliated manifold (M, F), decide if there exist* (P)*-metrics on (M, F), that is, Riemannian metrics such that all the leaves enjoy the property* (P)*. If they do exist, study their properties, classify them etc*.

EGF is really original approach which should provide striking new strategies and result in studying the extrinsic geometry of foliations. The approach is somehow similar to that for the Ricci flow which implied solutions of *Poincare Conjecture* and *Thurtston Geometrization Conjecture*. The **EGF** extends the application area of **IF**'s and **VF's** for foliations. The results [1 — 3] are the local existence/uniqueness theorems, estimating the existence time of solutions, the convergeness in a sense to minimal and totally geodesic foliations, geometry of extrinsic Ricci and Newton transformation flows, classifying **EGS** among totally umbilical foliations and foliated surfaces.

2. **IF** for a foliation (of any codimension) means the vanishing of a total (over M) expression composed from quantities depending on b (as higher mean curvatures), the co-nullity, integrability and the curvature tensor. The results [4, 5, 11, 12] generalize the Walczak formula, 1990; Andrzejewski-Walczak, 2010 (for special functions f_j providing Newton transformations of the co-nullity tensor), and Brito-Langevin-Rosenberg, 2001 (with constant mixed curvature) formulae.

3. We obtain **VF** for two cases, when either (3a) foliation or (3b) metric are varied.

3a) The results are **VF** [6] for the *i*-th mean curvature: when M has constant curvature c they do not depend on the choice of *k* orthonormal vector fields (see Brito-Langevin-Rosenberg, 1981, for $k = 1$). We study the problem of prescribing the partial Ricci curvature of a foliation [12]: obtain explicit conditions for a symmetric 2-tensor to admit conformal metrics, solve partial Ricci equations, compute variations of the total mixed curvature. We deduce **VF** for the total mixed scalar curvature, and apply them to minimization of the total energy and bending.

3b) The results [3] are **VF** for total quantities under *F*-truncated variations g_t, developing the method: "*deformation of geometric quantities as the Riemannian metric is deformed*", studying geometry of critical metrics and applications to functionals related to **EGF**. Combining with approach of **IF**, we find **VF** for the total mixed scalar curvature, energy and bending.

Other results of the project concern (i) prescribing projection of one field of principal directions of surfaces in (M^3, g) [10] and extension of theorem by Bryant, 2001; (ii) determination of sets with positive reach by their projection-type images [7, 8] and extension of stability theorem by Groemer, 1997; (iiii) modeling in Mathematics and Mechanics [9, 13-15].

List of research publications in 2008-2010

1. *Extrinsic geometric flows on foliated manifolds, I*, 34 pp., **2010**, preprint arXiv:1003.1607v1 (with P. Walczak)
2. *Extrinsic geometric flows on foliated manifolds, II*, 18 pp., **2010**, preprint arXiv:1009.6066v1 (with P. Walczak)
3. *Variational formulae for a codimension-one foliation*, preprint No: 2010/10, Mathematical Department of Lodz University, 29 pp., 2010 (with P. Walczak)
4. *Integral formulae for a Riemannian manifold with two orthogonal distributions*. 21 pp., Preprint N° 946, May **2010**, CRM, Barcelona, see http://www.crm.cat/
5. *Integral formulae for foliations on Riemannian manifolds*, Proc. of the 10th International Conference on Differential Geometry and Its Applications (DGA2007), Olomouc, August 27–31, 2007, 193–204, World Scientific Publ., **2008** (with P. Walczak)
6. *Variational formulae for the total mean curvatures of a codimension-one distribution*, Proc. of the 8-th International Colloquium on Differential Geometry, Santiago-de-Compostela, Spain, July 7–11, 2008, 83–93, World Sci. Publ., **2009** (with P. Walczak)
7. *Some extensions of the class of convex bodies*, preprint arXiv: 0808.1788v3, **2009**, 29 pp. (with V. Golubyatnikov)
8. *Some extensions of the class of k-convex bodies*, Siberian Math. J., 50 (5), **2009**, 820–829 (with V. Golubyatnikov), see also preprint arXiv:0809.3643v1
9. *Entropy Principle Maximum for Hierarchical Dynamic Systems*. Scientific Israel-Technological Advantages, 10, № 4**, 2008,** 147–158 (with I. Gaissinski)
10. *On surfaces with a prescribed curvilinear projection of one field of principal directions*. December 2009, 32 pp. (with L. Zelenko), see preprint arXiv:1003.2184v1
11. *On the partial Ricci curvature of foliations*, 2010, 20 pp., preprint arXiv:1010.2986v1, December 2009
12. *Integral formulae on foliated symmetric spaces*, preprint No: 2007/13, Mathematical Department of Lodz University, 21 pp., 2009 (with P. Walczak)

Monographs and Textbooks in 2008-2010

13. *Modeling of Curves and Surfaces with MatLab*, Textbook, SUMAT, Springer, **2010**, 452 pp.

14. *Hydrodynamic Instability Analysis. Perturbation Methods.* Verlag Dr. Muller, **2009**, 220 pp. (with O. Kelis and I. Gaissinski)

15. *Non-Linear Models in Mechanics: Instabilitioes and Turbulence. Mathematical Methods and Applications*, Lambert Academic Publishing, Germany, **2010**, 685 pp. (with I. Gaissinski)

Conferences and seminar presentations

1. Report with Abstracts: Some classes of epsilon-convex bodies. International Conference dedicated to the 100-th anniversary of the birthday of S.L. Sobolev. Novosibirsk, October 2008, p. 380 (with V. Golubyatnikov)

2. Report: Extrinsic geometric flows on foliated manifolds. Topology and Geometry Seminar, Department of Mathematics, University of Haifa, November 26, 2009

3. Report: Extrinsic geometric flows on foliated Riemannian manifolds. Analysis seminar, Bar-Ilan University, December 17, 2009

4. Report: Extrinsic geometric flows on foliated manifolds. Geometry seminar, University of Lodz, November 30, 2009

5. Report: Determination of sets with positive reach by their projection type images (with V. P. Golubyatnikov) Conference in honour of Prof. V. Milman's 70th birthday, Tel-Aviv University, Israel, 24-30 June 2009.

6. The conference on Geometry in honour of S.-T. Yau on his 60th birthday, Warsaw, Poland, 6-8 April 2009

7. International Symposium on Differebtial Geometry in honour of M. Dajczer on his 60th birthday, Rio de Janeiro, Brazil, 16-23 August 2009

8. Joint with P. Walczak 3 hours report: Extrinsic geometry of foliations. International Seminar "Geometry and Topology of Foliations", Tokio, Japan, 24 October 2009

9. Joint with P. Walczak 2 hours report: Extrinsic geometry flows on foliatiated Riemannian manifolds. Mathematical Seminar, IM PAN, Warsaw, 20 January 2010

10. Report: Extrinsic geometric flows on foliated manifolds. Research workshop of the ISF on "Nonlinear PDE and Boundary Value Problems with Measure Data". Technion, Haifa, Israel, 1-5 March 2010

11. Visiting Professor of CRM (Barcelona) for 1 month during the scientific program "Foliations", in April-May 2010. Lectures (2 hours) on the theme: "Extrinsic geometry of foliations".

12. Participation with reports, CRM (Barcelona):

a) Workshop on Geometry of Foliations, April 26 to 27, 2010;

b) Workshop on Dynamics of Foliations, April 28 to 30, 2010;

c) Advanced Course on Foliations: Dynamics-Geometry-Topology, 3-7 May;

d) Conference: Geometry and Topology of Foliations. 12-16 July 2010.

13. Poster report in Geometry section. International Congress of Mathematicians, ICM-2010, Hyderabad, India, 19-29 August 2010.