PERIODIC REPORT

Project No. 276919 of Marie-Curie actions FP7-PEOPLE-2010-RG,

01/03/2013 – 30/05/2014

Extrinsic Geometric Flows on Foliated Manifolds

Project coordinator: ***Vladimir Rovenski*** (University of Haifa)

PUBLISHABLE SUMMARY.

The period was devoted to organization of the International workshop, see [7], training of researcher (participation in seminars, conferences and reading literature) in areas of the project, and to proving and publishing results (due to the project objectives). The researcher continued investigation of extrinsic geometry of foliations, developing **Extrinsic Geometric Flows** (EGF) and applications of EGF and **Integral Formulas** (IF) to minimizing certain functions defined for plane fields on Riemannian manifolds, prescribing mean curvatures of a foliation or its orthogonal distribution, existence of foliations with certain extrinsic geometry (see problems by Gluck-Ziller, Walczak, and Toponogov). There was large progress of the work in line of Annex I.

In [1], researcher developed EGF, depending on the second fundamental form of a codimension-one foliations, which are associated with parabolic PDEs. The work begins with formulae for deformations of geometric quantities as the Riemannian metric varies along the leaves of a foliation. Under suitable assumptions, the existence/uniqueness and convergence of a solution have been shown. Applications to prescribing the higher mean curvatures were given.

In [2], authors developed the D-conformal EGF, related to parabolic PDEs, for arbitrary codimension foliations (D is the orthogonal distribution, and the speed of the flow is proportional to the divergence of the mean curvature vector H). The EGF is equivalent to the heat flow of 1-form dual to H. Under suitable assumptions, they show that the flow has a solution converging to a metric with $H = 0$; actually under some topological assumptions they can prescribe the mean curvature.

In [4, 12], authors studied the **mixed Einstein-Hilbert action** on foliations (imitative of Einstein-Hilbert one, except that the scalar curvature is replaced by the mixed scalar curvature, Smix). They used integral formulas ([9] surveys recent results for integral formulae in the foliation theory) and developed variational formulas for the extrinsic geometric quantities on foliations of arbitrary codimension, aiming to calculate the Euler-Lagrange equations (of the action) and present them in two equivalent forms: in terms of extrinsic geometry and intrinsically using the **partial Ricci tensor** (which was first defined in [6]). Certainly, these equations admit amount of solutions (e.g., twisted products). For a globally hyperbolic spacetime, they embodying the smooth Geroch infinitesimal splitting, they determined an exact solution to mixed field equations in empty space-time provided the foliation is isoparametric and totally umbilical.

In [5, 11], authors developed D-conformal EGF on foliations of arbitrary codimension, which yield parabolic PDEs. The EGF, whose velocity along the orthogonal distribution is proportional to Smix, has been used to examine the question: Which foliations admit a metric with a given property of Smix (e.g., positive/negative or constant)? The flow preserves harmonicity of foliations and yields the leaf-wise Burgers type equation for the mean curvature vector H of orthogonal distribution. When H is leaf-wise conservative, its potential obeys the non-linear heat equation. Under certain conditions (in terms of spectral parameters of Schrodinger operator), the flow admits a unique global solution, converging exponentially to a metrics, whose Smix is leaf-wise constant. Hence, in certain cases, there exists D-conformal metrics with Smix positive/negative or constant.

In [13], authors study the **mixed Yamabe problem** (i.e., the constancy of Smix by a biconformal change of metrics) using non-linear heat equation and spectral parameters of leaf-wise Schrodinger operator.

In [3], authors developed the **partial Ricci flow** for a codimension one foliation, these results were generalized in [10] for arbitrary foliations. We expect that the flow will be used to examine the question: When a foliation admits a metric with constant mixed sectional curvature?

In [8], authors generalized result for hypersurfaces in [J. of Evolution Equations, 10 (2010) 413-423].

In [14], authors classified hypersurfaces in Euclidean space with codimension one totally geodesic foliations.

Project website: http://math.haifa.ac.il/ROVENSKI/rovenski_ERG.pdf

PROJECT OBJECTIVES FOR THE PERIOD

The specific objectives for the period are
- Developing theory of EGF for a codimension-1 foliation and for arbitrary codimension foliation (existence/uniqueness theorems, converging as *t* runs to infinity, asymptotic behavior of curvature, singularities), foliated surfaces, variations of related total quantities and geometry of stable critical metrics, etc.
- Applications of EGF to problems of extrinsic geometry of foliations: minimizing functions (like volume and energy) defined for plane fields on Riemannian manifolds, foliated Riemannian submanifolds, etc.

WORK PROGRESS AND ACHIEVEMENTS DURING THE PERIOD

Significant results in the reporting period:
a) Theory of parabolic EGF for codimension-one foliations, see [1].
b) Theory of *D*-conformal EGF for arbitrary codimension foliations, with application to the mixed Yamabe problem, see [2, 5, 11, 13].
c) Theory of the partial Ricci flow (first steps), see [3, 10].
d) The mixed Einstein –Hilbert action on foliations with applications to theoretical physics, see [4, 6, 12].
e) Classification of Euclidean hypersurfaces with codimension one totally geodesic foliations, see [14].

There were no deviations from Annex I and no corrective actions.

ADDITIONAL INFORMATION

List of research publications in the reporting period:

1. *Extrinsic geometric flows on codimension-one foliations,* J. of Geometric Analysis, 23 (3), 1530–1558**, 2013**, see Zbl 1278.53030.
2. *Deforming metrics of foliations*. Cent. Eur. J. Math., 11 (6), 1039–1055, **2013** (with R. Wolak), see ArXiv:1109.1868.
3. *The partial Ricci flow on one-dimensional foliations*, preprint, **2013**, ArXiv:1203.6361 (with V. Sharafutdinov).
4. *Mixed gravitational field equations on globally hyperbolic spacetimes,* Classical and Quantum Gravity, 30 (8), **2013**, 085015, 26 pp. (with E. Barletta, S. Dragomir and M. Soret).
5. *Prescribing the positive mixed scalar curvature flow on totally geodesic foliations*, Proceedings of International Conference "Foliations-2012", 163–203, **2013** (with L. Zelenko), see ArXiv:1203.6361.
6. *On solutions to equations with partial Ricci curvature*, accepted J. Geom. & Physics, **2014**, see ArXiv:1010.2986.
7. Editor (with P. Walczak) of ``*Geometry and its Applications*". Selected papers based on the presentations at the 2nd international workshop on geometry and symbolic computation. Proceedings in Mathematics & Statistics, 72, Berlin: Springer, **2014**, 243 pp. see Zbl 06265683.
8. *Gaussian mean curvature flow for submanifolds in space forms*. In [**7**], 39 – 55 (with A. Borisenko).
9. *Integral formulas in foliations theory*. In [**7**], 73–82 (with K. Andrzejewski and P. Walczak).
10. *The partial Ricci flow for foliations*. In [**7**], 125–155.
11. *Prescribing the mixed scalar curvature of a foliation*. In [**7**], 83–123 (with L. Zelenko), see ArXiv:1303.0548.
12. *The mixed Einstein-Hilbert action and extrinsic geometry of foliated manifolds*, **2014**, see ArXiv:1405.6011 (with E. Barletta, S. Dragomir), submitted to Annales Polonici Mathematici.
13. *The mixed Yamabe problem for harmonic foliations*. Proc. ``Geometry, Dynamics, and Foliations 2013" in the series ASPM from the Math. Society of Japan, **2014** (with L. Zelenko), see ArXiv:1405.3809.
14. *Euclidean hypersurfaces with a totally geodesic foliation of codimension one*, Geometria Dedicata, **2014** (with M. Dajczer and R.Tojeiro), see ArXiv:1306.4570.

Training of students
1. Sergey Liflyandsky: "EGF on foliated manifolds" (M.S. 2011-2014).
2. Julian Tenenhauzer: "Nonlinear dynamics on a surface with a time-dependent metric" (M.S. 2011-2014).

DISSEMINATION ACTIVITIES

Organization of the 2nd International Workshop "Geometry & Symbolic Computations", Haifa, Israel (May, 2013), see http://www.cri.haifa.ac.il/crievents/2013/ . Editing (with P. Walczak) and preparing for publication of [**7**].

Lectures on International Conferences and seminar presentations in the reporting period

1. Report "*The mixed scalar curvature flow on foliations*", 10 of March, 2013. Special Geometry and Topology Seminar, University of Haifa.

2. Reports: "*Flows of metrics on foliations with visualizing for surfaces*", and "*Euclidean hypersurfaces with a totally geodesic foliation of codimension one*", 16 – 17 of May, 2013. The 2nd International Workshop "Geometry and Symbolic Computations", Haifa.

3. Report ``Mixed scalar curvature flow and harmonic foliations", Int. Conference ``Geometry & Foliations", Sept. 2013, Tokyo.

4. Reports ``The mixed scalar curvature flow on a foliated manifold" and ``Total mixed scalar curvature of foliated Riemannian manifolds", 2nd Joint International Meeting of the AMS and the IMU, June, 2014, Tel-Aviv.

5. Report ``The mixed scalar curvature of a harmonic foliation", ICM-2014, Geometry section, Aug. 2014, Seoul.

6. Report ``The mixed Yamabe problem for harmonic foliations", Int. Workshop ``Geometry of Foliations", 1–5 Sept. 2014, Madrid.

PROJECT MANAGEMENT

No management problems were occured.
No change of the legal status of the beneficiary occured.

2014–06–30 Prof. Vladimir Rovenski

MID-TERM REPORT
FP7-PEOPLE-2010-RG, No. 276919 of Marie-Curie action,
01/06/2011 – 30/02/2013

Extrinsic Geometric Flows on Foliated Manifolds

Vladimir Rovenski (University of Haifa)

The objectives of project are to develop the Extrinsic Geometric Flows (**EGF**), recently introduced by V. Rovenski and P. Walczak for codimension-one foliations, as a new research tool for studying the Riemannian geometry of foliations. We also use the methods of geometric analysis and PDE's, theory of Riemannian submanifolds, integral and variation formulae for foliations, topology and dynamics of foliations, and computer simulations.

The results concern

- Developing of EGF for a codimension-1 foliation/distribution and for a foliation of arbitrary codimension: existence/uniqueness theorems, converging as $t \to T$, behavior of curvature, singularities, Extrinsic Geometric solitons for totally umbilical metrics, foliated surfaces and 3-manifolds, truncated variations of related total quantities and geometry of stable critical metrics etc.
- Applications of EGF to problems of extrinsic geometry of foliations: minimizing functions (like volume and energy) defined for plane fields on Riemannian manifolds, foliated Riemannian submanifolds, combining integral and variation formulae for real and complex foliations with the approach of EGF etc.

1. In [1] authors develop EGF, integral and variation formulae on codimension-one foliations, these are related to hyperbolic PDEs (objectives 1, 2) and complete our previous project FP7-PEOPLE-2007-2-1-IEF, N° 219696.

In [2, 3, 5, 6] we develop EGF on codimension-one foliations, these are related to parabolic PDEs (objective 1). We begin with formulae for deformations of geometric quantities as the Riemannian metric varies along the leaves of a foliation. Then the EGF depending on the second fundamental form of the foliation is introduced. Under suitable assumptions, this evolution yields the second-order parabolic PDEs, for which the existence/uniqueness and in some cases convergence of a solution are shown. Applications to the problem of prescribing the mean curvature function of a codimension-one foliation, and examples with harmonic and umbilical foliations (e.g., foliated surfaces) and with twisted product metrics are given.

In [4, 7] we develop D-conformal EGF on a foliation F of arbitrary codimension, these are related to parabolic PDEs (objective 1). D is the orthogonal distribution and the speed of the flow is proportional to the divergence of the mean curvature vector H. We study the question: When the metrics converge to one for which D enjoys a given geometric property, e.g., is harmonic, or totally geodesic? Our main observation is that EGF is equivalent to the heat flow of 1-form dual to H, provided the initial 1-form is TF-closed. Assuming that F consists of compact and orientable leaves, we use existence results for the heat flow to show that our flow has a solution converging to a metric for which H = 0; actually under some topological assumptions we can prescribe the mean curvature H.

In [13, 14] we develop D-conformal EGF on foliations of any codimension related to the mixed scalar curvature, S_{mix}, that yields parabolic PDEs (objective 1). The mixed curvature of a foliation regulates the deviation of leaves. We study the flow of metrics on a foliation, whose velocity along the orthogonal distribution is proportional to S_{mix}. The flow is used to examine the question: Which foliations admit a metric with a given property of S_{mix} (e.g., positive or negative)? We observe that the flow preserves harmonicity of foliations and yields the Burgers type equation along the leaves for the mean curvature vector H of orthogonal distribution. If H is leaf-wise conservative, then its potential obeys the non-linear Schrödinger heat equation. Under certain conditions the asymptotic behavior of its solutions is the same as of the linear equation, when the Burgers equation for H has a single-point global attractor. The flow admits a unique global solution g(t), whose S_{mix} converges exponentially to a leaf-wise constant. The metrics are smooth on M when all leaves are compact and have finite holonomy group. Using rescaling along D one may provide convergence to metrics with S_{mix} as positive so negative.

In [8, 11, 12] we develop variation and integral formulae for D-conformal variations of metrics related to mixed Einstein-Hilbert action on foliations of arbitrary codimension (objective 2). We study stationary points of the mixed Einstein-Hilbert action (or the total S_{mix}) on a manifold M endowed with a Pfaffian system D and a complementary distribution D'. For a globally hyperbolic spacetime M we derive new mixed gravitational field equations embodying the smooth Geroch infinitesimal splitting. We determine an exact solution to mixed field equations in free space provided F is isoparametric and totally umbilical. We solve linearized mixed field equations for empty space.

2. In 2011, I invited well-known geometers for joint investigations and training related to the project:

a) Prof. A. Borisenko (University of Kharkov) is well-known expert in Riemannian geometry and geometric flows. The purposes of his visit are my training and research in common in the area of GFs on submanifolds. We reviewed and studied the *Mean Curvature Flow* on submanifolds of any codimension (in Riemannian, Euclidean and hyperbolic spaces) and with a density function. We generalized recent result of [Borisenko A. and Miquel V. Gaussian mean curvature flow, J. of Evolution Equations, 10 (2010) 413-423], extended result of [Andrews B., Baker C. Mean curvature flow of pinched submanifolds to spheres. J. Diff. Geom. 85 (2010) 357-395] and prepared a preprint [9]. Due to project objectives, I propose to develop the technique (including concept of density) for EGFs on foliations of any codimension, and to obtain applications for Riemannian submanifolds.

b) Prof. V. Sharafutdinov (Sobolev Institute of Mathematics at Novosibirsk), is well-known expert in geometric analysis, see his homepage http://www.math.nsc.ru/~sharafutdinov/ and the list of publications. The purposes of his scientific visit are my training in the area of modern geometric analysis and spectral theory for elliptic operators on Riemannian manifolds, the investigation in common in the area related to the ERG project (see Toponogov problem), to prepare a preprint [10]. After fruitful discussions with V. Sharafutdinov, I solved the particular case of above deformations of metrics (i.e., leaf-wise conformal deformations), studied the corresponding EGF.

Training of students

1. S. Liflyandski: “EGF on foliated hypersurfaces” (M.S).

2. J. Tenenhazer: “Nonlinear dynamics on a surface with a time-dependent metric” (M.S).

3. O. Kelis: “Multi-weighted systems of parabolic equations” (Ph.D. co-advising with A.Kozghevnikov).

Transfer of knowledge and integration activities:

Lectures on International Conferences and Seminars in Mathematics abroad [1, 2, 3, 9, 10] and [6, 7, 8, 11] in Israel.

I organaze 2nd International Workshop "Geometry and Symbolic Computations", Haifa, Israel (May 15–May 17), see http://www.cri.haifa.ac.il/crievents/2013/details/322-Geometry%20and%20Symbolic%20Computations

3. Significant results:

a) General theory of parabolic EGF for codimension-one foliations, see works [1, 2, 3, 5, 6].

b) General theory for D-conformal EGF of any codimension, see works [4, 7].

c) Theory of the mixed scalar curvature flow, see papers [13, 14, 15].

d) Mixed Einstein –Hilbert action on foliations with applications to general relativity, see works [1, 11, 12].

There were no deviations from Annex I and no corrective actions.

Conferences and seminar presentations:

1. Lecture: “*Variational formulae for a pair of complementary distributions on a Riemannian manifold*”. Co-organizer of the section “Contemporary Geometry”. Israeli-Polish Mathematical meeting; Lodz, 11 – 15 of September 2011, see http://imuptm.math.uni.lodz.pl

2. Lecture “Extrinsic *Geometry of Codimension-One Foliations*”. The International Conference “Geometry Days in Novosibirsk, 2011”, dedicated to the 50th anniversary of the Chair of Geometry and Topology of Novosibirsk State University; Novosibirsk, 1 – 4 of Sept. 2011, http://math.nsc.ru/conference/geomtop2011

3. Lecture “*Variational Formulae for Codimension-One Foliations and Their Applications*”. The International Conference on Applied and Engineering Mathematics (AEM-2011); Oct. 28 – Nov. 3, 2011. Shanghai, China, see http://www.engii.org/cet2011/AEM2011.aspx

4. Participation in the International Workshop “Waves and Quantum Fields on Fractals”, June 26 – 29, 2011, Technion, Haifa, Israel, see http://physics.technion.ac.il/~conf/Waves/

5. Participation in the International Conference “*Recent Advances in Non-Linear Evolutionary Equations and Analysis of Multi-Scale Phenomena*”, July 25 – 29, 2011. Weizmann Institute of Sciences, Rehovot, Israel.

6. Lecture “On Extrinsic *Geometry of Codimension-One Foliations*”, 27 of November, 2011. The Mathematical Colloquium. Holon Institute of Technology.

7. Lecture “*Flows of Riemannian metrics on fiber bundles*”, March, 2012. Special Geometry and Topology Seminar. University of Haifa, Israel.

8. Lecture “*Flows of Riemannian metrics on fiber bundles*”, April, 2012. Analysis Seminar. Univ. of Bar-Ilan, Israel.

9. Visiting professor of Universitaa degli Studi della Basilicata, Dipartimento di Matematica e Informatica, Italy, April 2012. Lecture: “*Flows of Metrics on Fiber Bundles*” on Seminario Interdisciplinare di Matematica.

10. Lecture: "*Flows of Metrics on a Fiber Bundle*". The Fourth Geometry Meeting dedicated to the centenary of A.D. Alexandrov, August 2012, Saint-Petersburg, Russia.

11. Lecture "*Flows of metrics on fiber bundles*". Sixth European Congress of Mathematics satellite conference "Foliations 2012". Lodz, June 2012.

12. Lecture "*The mixed scalar curvature flow on foliations*", 10 of March, 2013. Special Geometry and Topology Seminar. University of Haifa, Israel.

List of research publications:

15. *Topics in Extrinsic Geometry of Codimension-One Foliations*. Series: Springer Briefs in Mathematics, Berlin: Springer, September of **2011**, 114 pp. (with P. Walczak), see Zbl 1228.53002.
16. *Extrinsic geometric flows on foliated manifolds*, III, ArXiv:1108.5071, preprint, 20 pp. **2011**.
17. *Extrinsic geometric flows on codimension-one foliations,* J. of Geom. Analysis, 29 pp., **2012**, DOI 10.1007/s12220-012-9297-1.
18. *Deforming metrics of foliations*, ArXiv:1109.1868, preprint, 17 pp., **2011** (with R.Wolak).
19. *Variational Formulae for Codimension-One Foliations and Their Applications,* 4 pages. Proc. of the IEEE World Congress on Engineering and Technology (CET2011); Int. Conf. on Applied and Engineering Math. (AEM **2011**), Shanghai, China.
20. *On extrinsic geometry of codimension-one foliations*, 34 - 44. Proc. of the International Conference "The Geometry Days in Novosibirsk, **2011**" (with P.Walczak)
21. *Deforming metrics of foliations*. Cent. Eur. J. Math. Vol. 11 - Issue 6 - June **2013** (with R.Wolak).
22. *Integral formulae on foliated symmetric spaces*, Math. Ann. 352, No. 1, 223-237 (**2012**) (with P.Walczak).
23. *Gaussian mean curvature flow for submanifolds*, preprint, 15 pp. **2012** (with A.Borisenko).
24. *Deforming metrics of geodesic foliations*, preprint, preprint, 13 pp. **2012** (with V.Sharafutdinov).
25. *Mixed gravitational field equations on globally hyperbolic spacetimes,* preprint, 32 pp. **2012** (with E.Barletta, S.Dragomir and M.Soret).
26. *Mixed Einstein-Hilbert actions on foliated manifolds*, preprint, 14 pp. **2012** (with E.Barletta, S.Dragomir).
27. *The mixed scalar curvature flow on a fiber bundle,* ArXiv:1203.6361, preprint, 15 pp. **2012** (with L.Zelenko).
28. *The mixed scalar curvature flow and harmonic foliations,* ArXiv:1303.0548, preprint, 20 pp. **2013** (with L.Zelenko)
29. *The mixed scalar curvature flow on totally geodesic foliations*, Proceedings of International Conference "Foliations 2012", 25 pp. **2013** (with L.Zelenko).
30. *Integral formulas in foliations theory*. Proceedings of International Conference "Foliations 2012", 11 pp. **2013** (with K.Andrzejewski and P.Walczak).

2013–02–30 Prof. Vladimir Rovenski