FOLIATIONS, SUBMANIFOLDS, AND MIXED CURVATURE

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INTRODUCTION

This survey is based on the author's results on the Riemannian geometry of foliations with a nonnegative mixed curvature and on the geometry of submanifolds with generators (rulings) in a Riemannian space of nonnegative curvature. The main idea is that such a foliated (sub)manifold can be decomposed into a product if the dimension of its leaves (generators) is large. The methods of study are local in the sense of directions transversal to leaves and mostly synthetic in the case of submanifolds with generators.

The paper is divided into four chapters.

Chapter 1 (introductory) contains facts concerning foliated smooth manifolds.

Chapter 2 deals not only with a foliated Riemannian manifold, but also with the slightly more general case of a pair of complementary orthogonal distributions. The latter is sometimes called a Riemannian "almost-product structure." Starting from the "co-nulity operator" of D. Ferus, P. Dombrowski, and others, in Chap. 2, we introduce a pair of structural tensors that are similar to O'Neill's or Gray's pairs but are more convenient for our purposes; these tensors also satisfy a Riccati-type PDE with mixed curvature. Chapter 2 concludes with a survey of (1) constructions of totally geodesic and umbilic foliations, (2) recent studies of Riemannian foliations, and (3) the relationship between the result by V. Toponogov and that of the author, which asserts that if a complete simply connected Riemannian manifold with a sectional curvature \( \leq 4 \) and injectivity radius \( \geq \pi/2 \) has the extremal diameter \( \pi/2 \), then it is isometric to CROSS and large-spheres foliations, in particular, skew-Hopf fibrations.

Chapter 3 is devoted to a key problem of this work, namely, we discuss the role played by the Riemannian curvature in studies of foliated (sub)manifolds. Chapter 3 begins with the Ferus theorem (1970) on the optimal (largest) dimension of a totally geodesic foliation with constant positive \( K_{\text{mix}} \) (mixed sectional curvature) on a given manifold. It contains a striking relationship between the Riemannian geometry of foliations and the topological concept of the number of vector fields on the \( n \)-sphere. For a nonconstant \( K_{\text{mix}} > 0 \), the dimension of a compact totally geodesic foliation can be easily estimated from above by half of the dimension of the manifold itself: the idea (by T. Frankel) is that two compact totally geodesic submanifolds (for instance, large spheres in a round sphere) in a space of positive curvature necessarily intersect each other if the sum of their dimensions is not less than the dimension of the whole space. In Sec. 3.7, we extend this result to the case of positive \textit{partial mixed curvature}.

The problem by V. Toponogov is to obtain a Ferus-type estimate for the dimension of a foliation over a (compact) manifold with \( K_{\text{mix}} > 0 \). A local counterexample containing a fibration over closed geodesics that presupposes the necessity of additional assumptions concerning a foliation is given in Sec. 3.4. In Sec. 3.5, we introduce a variation procedure based on the concept of the \textit{volume} (in particular, \textit{area}) of an \textit{L-parallel vector field} and the \textit{turbulence of a foliation along a leaf}, which allows us to obtain rigidity and splitting theorems for foliations with \( K_{\text{mix}} \geq 0 \) and under some additional conditions; in particular, we generalize Ferus' result.

In Sec. 3.6, the Riccati-equation technique is extended to foliations. Combining some ideas, we obtain integral formulas and inequalities with mixed scalar curvature along a compact manifold or a complete leaf.

In Sec. 3.8, we introduce some classes of foliations that generalize Riemannian foliations and apply to them the concept of an index of relative nullity.

Systematic studies of the local and global structure of submanifolds in (pseudo)Riemannian spaces include a study of the relationships between their intrinsic and extrinsic geometries, tests for totally geodesic and cylindrical submanifolds, estimates of codimension, etc. The attention given to foliated submanifolds


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has increased due to studies of some special embeddings with degenerate second fundamental form: ruled and tubular submanifolds, (strongly) parabolic, k-saddle submanifolds, and submanifolds having nonpositive extrinsic curvature, small codimension, etc.

In the case of a curvature-invariant strongly parabolic submanifold (for example, when the ambient manifold is a space form), the domain where the index of relative nullity is minimal has a ruled developable structure (with constant $K_{mix}$ in the case of a space form). More general parabolic submanifolds, which were introduced by A. Borisenko in 1972, constitute a wider class and contain a large number of rulings under a certain condition on the curvature tensor of the ambient space.

In Secs. 4.1, 4.3, 4.5, and 4.6, a detailed survey of studies of ruled submanifolds in space forms and CROSS is given. In Sec. 4.2, which continues the studies carried out in Sec. 3.7, some recent results on submanifolds with nonpositive extrinsic sectional curvature (Otsuki's lemma and its corollaries on nonembedding of manifolds, and extremal theorems by A. Borisenko) are extended to a more general case of extrinsic partial curvature. In Sec. 4.4, we continue the study of Toponogov's problem in the case of foliated submanifolds. We introduce a (synthetically defined) class of uniquely projectable submanifolds along generators in a Riemannian space, which is closely connected with the class of ruled submanifolds with $K_{mix} \geq 0$. A key result of Sec. 4.4 is that a ruled submanifold in the sphere or in the complex projective space with ruling of "high" dimension and a "small" norm of its second fundamental form (i.e., in particular, $K_{mix} \geq 0$) is congruent to the Segre embedding; the latter plays the role of a "cylinder" in a space form of positive curvature. Combined with the method for the volume (or area) of an $L$-parallel vector field (from Sec. 3.5), we obtain the test for the Segre-type decomposition of a ruled and parabolic submanifold in a space of positive curvature.

Acknowledgments. The author express his gratitude to Professor Victor Toponogov (Novosibirsk) for his help and support during many years of work on the theme, and to his colleagues and friends.

Chapter 1

FOLIATIONS ON SMOOTH MANIFOLDS

1.1. Definition and Examples of Foliations

Intuitively, a foliation corresponds to a decomposition of a manifold into a set of connected submanifolds of the same dimension, called leaves, which locally look like the pages of a book.

Definition 1.1 [280]. A family $\mathcal{F} = \{L_\alpha\}_{\alpha \in A}$ of connected subsets of the manifold $M^m$ is a $\nu$-dimensional foliation if

1. $\bigcup_{\alpha \in A} L_\alpha = M^m$,
2. $\alpha \neq \beta \Rightarrow L_\alpha \cap L_\beta = \emptyset$,
3. for any point $p \in M$, there exists a $C^\infty$-chart (local coordinate system) $(U_p, \varphi_p)$ such that $p \in U_p$ and if $U_p \cap L_\alpha \neq \emptyset$, then the components of the set $\varphi_p(U_p \cap L_\alpha)$ are the following parts of parallel affine subspaces (see Fig. 1):

$$A_c = \{(x_1, \ldots, x_m) \in \varphi_p(U_p) : x_{\nu+1} = c_{\nu+1}, \ldots, x_m = c_m\}.$$

Condition (3) does not follow from (1) and (2); this is obvious from the example Sunrise on $\mathbb{R}^2$ in Fig. 2. The coordinate system $(U_p, \varphi_p)$ in Definition 1.1 is called a foliated chart. A local picture of a foliated manifold is simply a space of the form $\mathbb{R}^\nu \times N$, where sets of the form $\mathbb{R}^\nu \times \{n\}$ are leaves and $N$ is a transversal, as in Fig. 3. The integer $m - \nu = \dim N$ is called the codimension of a foliation. For an equivalent definition of a foliation by means of a pseudogroup structure or integrable $G$-structure, see [51] and [227].

Every leaf of a foliation is a $\nu$-dimensional connected manifold with a countable base. A leaf of a foliation is a component of the manifold $M$ in the leaf topology, which is generated by all sets of the form $U \cap \varphi_p^{-1}(A_c)$, where $U$ is an open subset of $U_p$. The injection mapping of a leaf into the manifold $M$ with the original topology is continuous, but the image is not necessarily a closed subset. A leaf $L$ is said to be closed if it is a closed subset of $M$ in the original topology. A leaf is proper if its two topologies coincide, as, for example,
when a leaf is compact. On a 2-dimensional torus, we have vector fields that generate line foliations, each of whose leaves is everywhere dense in the whole manifold (see Fig. 4). In this example, the leaf topology on each leaf $L = \mathbb{R}$ is different from that induced by the original manifold topology.

Foliations on smooth manifolds were studied in [51, 280, 131, 226, 227]. In [191], a foliated space that generalizes the notion of a foliated manifold is considered. In this case, $M$ is a separable metrizable space equipped with a family of open sets $\{U_p: p \in M\}$ with $p \in U_p$ and homeomorphisms $\varphi_p: U_p \to \mathbb{R}^\nu \times N_p$. The continuous change of coordinates has the form $t' = \varphi(t, n)$, $n' = \psi(n)$; moreover, a set of the form $\mathbb{R}^\nu \times \{n\}$ in $U_p$ is smoothly mapped into a set of the form $\mathbb{R}^\nu \times \{\psi(n)\}$. The level surfaces coalesce to the form of maximally connected sets called leaves. Every leaf is a smooth $\nu$-dimensional manifold. An example of a foliated space that is not a foliated manifold is a solenoid; in this case, $\nu = 1$ and each $N_p$ is homeomorphic to a subspace of a Cantor set.

In [188, 45, 329, 313, 227], foliations with singularities on Riemannian manifolds are studied. In this case, the dimension of the leaves is not constant. The theory of Riemannian foliations (see Sec. 2.5) can be applied to certain infinite-dimensional situations, where foliations arise as a result of special actions of continuous groups (see [286]).

Foliations arise in connections with such topics as vector fields without singularities (when $\nu = 1$), integrable $\nu$-dimensional distributions, submersions and fibrations, actions of Lie groups, and direct constructions of foliations such as Hopf fibrations, Reeb foliations, etc.

**Definition 1.2.** A $\nu$-dimensional distribution on a manifold $M$ is a smooth field $\{T(p)\} (p \in M)$ of $\nu$-dimensional tangent subspaces, i.e., a function whose value at each point $p \in M$ is a $\nu$-dimensional subspace $T(p)$ of the tangent space $T_pM$. A distribution is integrable if a $\nu$-dimensional integral submanifold that is tangent to the given distribution at each of its points passes through each point $p \in M$. A 1-dimensional distribution is also called a line field.

A family of maximal integral submanifolds of an integrable distribution forms a foliation. By the Frobenius theorem (see [177]), a distribution $T$ on $M$ is integrable if and only if, for any vector fields $x$, $y$ on $M$ that are tangent to the distribution, the vector field $[x, y]$ (a bracket or a commutator, see Fig. 5) is also tangent to the distribution at every point. A line field is always integrable; it is equivalent locally, but not globally, to a vector field; see the example below (a Reeb component on a torus with one compact leaf) and the examples in $\mathbb{R}^2 \setminus \{0\}$ in Fig. 6. The 2-dimensional field $T$ on $\mathbb{R}^2$: $T(x_1, x_2, x_3) = \{\text{a plane generated by } X_1 = (1, 0, 0) \text{ and } X_2 = (0, \exp(-x_1), \exp(x_1))\}$ does not have integral surfaces: $[X_1, X_2] = (0, -\exp(-x_1), \exp(x_1)) \not\in T$.

**Definition 1.3** [51]. A smooth mapping $\pi: M \to B$ of the differentiable manifolds $M$ and $B$ is called a submersion if, for any point $m \in M$, the differential $\pi_{(m)}: T_mM \to T_{\pi(m)}B$ is a surjective map of tangent spaces; $M$ is the total space, $B$ is the base, and $\{L_b = \pi^{-1}(b)\}$ are fibers (leaves). A $C^\infty$-submersion $\pi: M^m \to B^n$ is a (locally trivial) $C^\infty$-fibration if the following conditions hold:
(1) $\forall b \in B$, the set $\pi^{-1}(b)$ is a $\nu$-dimensional submanifold that is $C^r$-diffeomorphic to a fixed manifold $L'$, 
(2) for every $b \in B$, there exist a neighborhood $U_b$ and a $C^r$-diffeomorphism $\psi$: $\pi(U_b) \cong U \times L$ such that $\psi(\pi^{-1}(b')) = \{b'\} \times L$, $b' \in U_b$ ($M$ is the total space of the fibration, $B$ is the base, $L$ is a fiber (leaf), and $\pi$ is the projection). 

The foliation $L \times \{b\}$, where $b \in B$, on the direct product of manifolds $M = L \times B$ is called a product bundle. Thus, locally, a foliation is a product bundle. The simplest nontrivial bundles in dimension two are the Möbius band with fiber $\mathbb{R}$ and base $S^1$, and the Klein bottle with $S^1$ as a fiber and a base. The most well-known bundles are vector bundles, in particular, the tangent bundle $p: TM \to M$ of a manifold and the normal bundle $p: TM -+ M$ of a submanifold $M \subset M$. 

A submersion with compact leaves or with compact total space and connected base is actually a fiber bundle [280]. The question on conditions under which a foliated manifold is actually a fiber bundle, in particular, a product of manifolds, was studied in terms of an Ehresmann connection (see Sec. 1.3).

Definition 1.4 [51]. A $C^k$-differentiable action of a Lie group $G$ (with identity $e$) on a manifold $M$ is a $C^k$-map $\varphi: G \times M \to M$ such that 

$$\varphi(e, x) = x, \quad \varphi(g_1 \circ g_2, x) = \varphi(g_1, \varphi(g_2, x)) \quad (g_1, g_2 \in G, \ x \in M).$$

The orbit of the point $x \in M$ under the action $\varphi$ is a subset $O_\varphi(x) = \{\varphi(g, x): g \in G\}$. We say that $\varphi$: $G \times M \to M$ is a foliated action if, for every $x \in M$, the tangent space to the orbit of $\varphi$ passing through $x$ has a fixed dimension $\nu$; if $\nu$ equals the dimension of $G$, then $\varphi$ is locally free. The action of the group $G = \mathbb{R}$ is called a dynamical system (flow) on the manifold $M$. 

Definition 1.5 (J. Milnor). The rank of a manifold $M$ is the maximum $r$ for which there exists a locally free action of $\mathbb{R}^r$ on $M$. Equivalently, the rank of this manifold $M$ is the maximum number of continuous pointwise linearly independent commuting vector fields that are admitted by the manifold. 

A compact 3-dimensional manifold with finite fundamental group (for example, $S^3$) has rank 1; moreover, a compact 3-dimensional manifold of rank 2 is a fiber bundle over $S^1$ with toroidal fibers $T^2$ (see [51]). 

Another “similar” topological invariant, the maximum number of continuous pointwise linearly independent vector fields on a manifold, also has a close relationship with foliations, in particular, with foliations of nonnegative mixed curvature. 

Theorem 1.1 (see [92] and [138]). (a) The maximum number of continuous pointwise linearly independent vector fields on the sphere $S^{n-1}$ is equal to $\rho(n) - 1$, where 

$$\rho((\text{odd}) \ 2^{k+c}) = 8b + 2c, \quad (b \geq 0, \ 0 \leq c \leq 3).$$

(b) If $\nu \leq \frac{n}{2}$, then a continuous $\nu$-dimensional distribution on $S^{n-1}$ (and moreover, a $\nu$-dimensional foliation) exists if and only if there exist $\nu$ continuous pointwise linearly independent vector fields on $S^{n-1}$.
Note that $\rho(n) \leq 2 \log_2 n + 2 \leq n$. Table 1 contains some values for the function $\rho(n) - 1$.

A solution to the algebraic problem on the existence of orthogonal multiplications on $\mathbb{R}^n$ [138] allows us to construct $\rho(n) - 1$ orthogonal unit vector fields $\{w_i(x) = B_i x\}$ on the sphere $S^{n-1}$: for all $\nu < \rho(n)$ there exist $\nu$ orthogonal matrices $\{B_i\}$ of size $n \times n$ with the properties $B_i^2 = -E$, $B_i B_j + B_j B_i = 0$, $i \neq j$.

They define a nonintegrable geodesic distribution on a sphere, i.e., every large circle that is tangent to this distribution at one point preserves this property at all its points.

An example of the Reeb foliation plays an important role in the development of foliation theory. The Reeb component in the strip $S = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1\}$ is the foliation $\{L_a\}$, $(a \in \mathbb{R} \cup \pm \infty)$ (see Fig. 7)

$$L_a = \{(t, a + f(t)) : |t| < 1\}, \quad L_{\pm \infty} = \{(\pm 1, t) : |t| < \infty\},$$

where, for example, $f(t) = \exp(x^2 \pm t) - 1$ leads us to a $C^\infty$-foliation [280]. Reeb components can also appear on surfaces. By factoring a strip via translation along the $Y$-axis, we obtain the following Reeb components: a Möbius band and its double covering on a cylinder (Reeb annulus) (see Fig. 8). The Reeb annulus with pairwise glued boundary lines forms a foliation on the torus or on the Klein bottle. Each of these foliations has exactly one compact leaf, but on the torus such a foliation cannot be obtained from a nonsingular vector field.

In 1979, H. Gluck posed two basic problems concerning the correlation of the structures of foliations and metrics on manifolds.

FM1. The existence of a geodesic metric for a given foliated manifold.

For a foliation on a 2-dimensional manifold, an obstruction to geodesibility consists of a Reeb component (see Sec. 1.1). Among foliations that are generated by Morse–Smale vector fields without singularities, only suspensions of diffeomorphisms are geodesible [103]. For $n \geq 4$, there exist nongeodesible foliations of $M^n$ even with closed curves (see Appendix A in [23]).

FM2. The existence (and classification) of geodesic foliations on a given Riemannian manifold.

Much is known about geodesic (and Riemannian) foliations on space forms, (see Sec. 2.4.3; resp., Sec. 2.5.3). All geodesic foliations on the round 3-sphere were catalogued in [104] (see also Sec. 2.6), but the same question is open for $S^{2n+1}$ with $n > 2$: for foliations on compact Lie groups, see [230]; for foliations on some standard hypersurfaces in Euclidean n-spheres, see [217].

Rotating a strip about the $Y$-axis, we obtain a Reeb component on a solid cylinder $D^2 \times \mathbb{R}$ (and hence, on a solid torus, see Fig. 9), which can also be defined by some submersion $f: D^2 \times \mathbb{R} \rightarrow \mathbb{R}$. Let $f_1: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a similar submersion defined on $\mathbb{R}^3$ by the formula $f_1(x_1, x_2, x_3) = a(r^2) \exp(x_3)$, where $a: \mathbb{R} \rightarrow \mathbb{R}$ is a $C^\infty$-function such that $a(1) = 0$, $a(0) = 1$, and if $t > 0$, then $a'(t) > 0$. Let $\{L\}$ be a foliation of $\mathbb{R}^3$ whose leaves are connected components of the submanifolds $f_1^{-1}(c)$, $(c \in \mathbb{R})$. All leaves in the interior of the solid cylinder $C = f_1^{-1}((0, 1]) = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 \leq 1\}$ are homeomorphic to the plane $\mathbb{R}^2$. The boundary of $C$, $\partial C = f_1^{-1}(0) = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 = 1\}$, is also a leaf (a cylinder). Outside $C$, all leaves are homeomorphic to the cylinder (see Fig. 10). *This submersion is not a fiber bundle.*

We can factor the Reeb component in a solid cylinder and thereby obtain a Reeb foliation in a solid torus $D^2 \times S^1$ filled in by copies of $\mathbb{R}^3$, whose leaves accumulate only in the neighborhood of a compact leaf, the boundary torus $\partial(D^2 \times S^1) = S^1 \times S^1$. *This foliation cannot be defined by a submersion.*

From two Reeb components in $D^2 \times S^1$, we obtain a $C^\infty$-foliation on the whole 3-sphere, since $S^3 = \{x = \dots\}$.
Fig. 10. Submersion that is not a fiber bundle.

Fig. 11a. Holonomy of foliations on a Möbius band. Fig. 11b. Holonomy of the Reeb foliation.

$(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \sum_{i=1}^4 x_i^2 = 1$ can be represented as the union of two solid tori $S_+^3 = \{x \in S^3 : x_1^2 + x_2^2 \geq \frac{1}{2}\}$ and $S_-^3 = \{x \in S^3 : x_1^2 + x_2^2 \leq \frac{1}{2}\}$ glued together along the boundary $T^2 = \{x \in S^3 : x_1^2 + x_2^2 = \frac{1}{2}\}$. This Reeb foliation on $S^3$ has exactly one compact leaf. Therefore, it is optimal according to the Novikov theorem (see [51, 280]), which states that every 2-dimensional $C^2$-foliation on a compact 3-dimensional manifold $M$ with a finite fundamental group has a compact leaf, which is homeomorphic to a torus in the case $M = S^3$.

1.2. Holonomy

The notion of holonomy is a generalization of the Poincaré first return map from flows to foliations. The mappings of transversal cross-sections associated with curves lying in a leaf $L$ allow us to obtain a representation of $\pi_1(L, x)$ (the fundamental group of the leaf $L$) in a group of germs of diffeomorphisms of the transversal manifold at a point $x$. If the holonomy group of the leaf $L$ is small, then the recurrence of leaves near $L$ is small.

We denote by $\Gamma^k_n$ the group of germs of $C^k$-diffeomorphisms of $\mathbb{R}^n$ leaving fixed the point $0$.

**Lemma 1.1** [227]. Let $L$ be a leaf of a foliation of codimension $n$, $x \in L$, and let $h$ be an embedding of $\mathbb{R}^n$ transversal to the foliation such that $h(0) = x$. If the foliation is of class $C^k$, then $h$ is also assumed to be of class $C^k$. Then there exists a homomorphism $f(L, x, h)$ taking $\pi_1(L, x)$ into $\Gamma^k_n$, which is defined up to conjugacy by the foliation and the leaf.

**Definition 1.6.** The holonomy homomorphism of a leaf is the homomorphism $\text{Hol}(L, x): \pi_1(L, x) \to \Gamma^k_n$ introduced in the statement of Lemma 1.1. The holonomy group of a leaf is the image $\Gamma(L, x)$ of $\pi_1(L, x)$ under the holonomy homomorphism.

The holonomy of the core circle $\gamma$ on a flat Möbius band foliated to circles equidistant to $\gamma$ is the group $\mathbb{Z}_2$ generated by the diffeomorphism $f(x) = -x$ (see Fig. 11 a). For the Reeb foliation of $S^3$, the holonomy group of a unique compact leaf (torus $T^2$, see Fig. 11 b) is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and can be represented by two $C^\infty$-diffeomorphisms $f, g: \mathbb{R} \to \mathbb{R}$ with $f(0) = g(0) = 0$ such that

$$f(x) = \begin{cases} < x & \text{for } x > 0 \\ = x & \text{for } x \leq 0 \end{cases}, \quad g(x) = \begin{cases} = x & \text{for } x \geq 0 \\ < x & \text{for } x < 0 \end{cases}.$$

**Lemma 1.2** (see [227]). The union of all leaves with trivial holonomy is an everywhere dense set that is the intersection of no more than a countable number of open sets.
Example 1.1 [191]. Let $K$ be a Cantor set of positive measure in a unit circle, and let $f: S^1 \to S^1$ be a homeomorphism that has $K$ as its fixed point set. The associated foliation (obtained by means of the suspension of homeomorphism construction) on a torus has closed leaves each of which corresponds to each point of $K$ and each of these leaves has a nontrivial holonomy.

All fibers of a fiber bundle are diffeomorphic, and for a foliation, the sufficient conditions for leaves to be diffeomorphic to each other are studied in stability theorems. Holonomy is a key notion when all leaves of a foliation are diffeomorphic.

Theorem 1.2 (local stability) [51]. Let $M$ be a smooth foliated manifold and let $L$ be a compact leaf with (a) a finite holonomy, (b) a trivial holonomy. Then there exists a saturated neighborhood $U$ of $L$ with one of the following properties:

(a) all leaves in $U$ are compact with finite holonomy groups,
(b) there exists a diffeomorphism $\psi: L \times \mathbb{R}^n \to U$ that preserves the leaves.

The global stability theorem for codim $L = 1$ was obtained by Reeb. From Lemma 1.2 and Theorem 1.2, it follows that for a compact foliation (i.e., a foliation with compact leaves), the union of all leaves with trivial holonomy is an open everywhere dense set of $M$.

Theorem 1.3 (see [227]). The following statements are equivalent for a compact foliation: (a) the holonomy group of the leaf $L$ is finite, (b) the space of leaves is Hausdorff in a neighborhood of $L$, (c) the volumes of leaves in a neighborhood of $L$ are uniformly bounded in some (and hence, every) Riemannian metric on $M$, (d) the leaf $L$ admits a fundamental system of saturated neighborhoods.

For a compact foliation, we have the following: (1) if either codim $L = 1$ or codim $L = 2$ and $M$ is compact, then, according to [226] and [74], all leaves have finite holonomy groups, (2) if either codim $L \geq 3$ or codim $L = 2$ and $M$ is not compact, then the assertion in (1) is violated (see [23]).

There are two groupoids associated with a foliation, namely, the homotopy groupoid and the holonomy groupoid, which is sometimes called the graph of a foliation.

Definition 1.7. The graph (holonomy groupoid) $\Gamma(M, L)$ of a foliated manifold $M$ is the set of all triples $(x, y, [\alpha])$, where $x, y$ belong to the same leaf $L$, $\alpha$ is a piecewise-smooth path going from $x$ to $y$ in $L$, and $[\alpha]$ is the holonomy equivalence class of $\alpha$. Two triples $(x, y, [\alpha])$ and $(x', y', [\beta])$ are equivalent iff $x = x'$, $y = y'$, and the holonomy of the curve $\alpha\beta^{-1}$ is trivial. The homotopy groupoid of a foliation is defined in a similar way (as the holonomy groupoid). We identify leaf curves that are homotopic relative to their fixed ends in the corresponding leaf.

The graph $\Gamma(M, L)$ has a locally Euclidean topology of dimension $\dim M + \dim L$, but, in general, this topology is not Hausdorff. The concept of the holonomy groupoid was introduced in [76]; it was later studied in more detail by Winkelkemper in [310], who was primarily interested in its application to Riemannian foliations (the graph is Hausdorff in this case). The study of the analytic properties of the graph of foliation was begun in [66]. Likewise, the homotopy groupoid is a manifold, but not necessarily Hausdorff. There is a natural submersion from the homotopy groupoid onto the holonomy groupoid of a given foliation. In [130] (see also [69]), one can find an example of a foliation for which the holonomy groupoid is Hausdorff, but this is not the case for the homotopy groupoid.

A (homotopy) vanishing cycle for a foliation $\{L\}$ is a mapping $c: S^1 \times [0, 1]$ such that for any $t \in [0, 1]$, $c_t = c_tS^1 \times \{t\}$ is a loop in a leaf of $\{L\}$ and $c_0$ is not homotopic to a constant loop in the leaf, but, for $t > 0$, all loops $c_t$ are homotopic to a trivial loop. Moreover, the curves $c_t$ can be chosen to be tangent to the horizontal subbundle. A holonomy vanishing cycle for a foliation $\{L\}$ is a mapping $c: S^1 \times [0, 1]$ such that for any $t \in [0, 1]$, $c_t = c_tS^1 \times \{t\}$ is a loop in a leaf of $\{L\}$ and $c_0$ has a nontrivial holonomy, but for $t > 0$, the loops $c_t$ have no holonomy [310]. It is demonstrated in [70] that the nonexistence of a vanishing cycle (of a homotopy) is equivalent to the Hausdorff property of the homotopy groupoid of a foliation. The graph of the foliation $\{L\}$ is Hausdorff if and only if $\{L\}$ has no holonomy vanishing cycles [311].
To any holonomy group we can associate the family of \textit{infinitesimal holonomy groups}, which is obtained by taking the $j$-jets ($j \leq k$) of the elements of the holonomy group (or groupoid). In the case $j = 1$, we have the \textit{linear holonomy}.

For a foliated Riemannian manifold $M$, $\{L\}$ the embeddings $h : \mathbb{R}^n \to M$ (see Lemma 1.1) can be chosen to be orthogonal to a given leaf $L$ at all points of the path $\alpha$. Then the linear holonomy acts on the orthogonal subspaces $T_\alpha L^\perp$ along the path $\alpha$. For a Riemannian foliation, the holonomy acts by local isometric transformations on (the germs of) transversal manifolds, and hence, is completely defined by its 1-jets at each point, i.e., by the linear holonomy, namely, by orthogonal transformations of $T_\alpha L^\perp$.

The trajectory of a vector $y \in T_\alpha L^\perp$ under the linear holonomy displacement is called an $L$-parallel (or \textit{basic}) \textit{vector field} along the curve $\alpha \subset L$ (see also Sec. 3.3). These vector fields are defined globally on leaves with a trivial holonomy. An $L$-parallel vector field can be characterized by using the \textit{Bott connection} $\nabla$ in $TL^\perp$, which is defined by

$$\nabla_x s = P_2[x, Y_s], \quad (x \subset TL, s \subset TL^\perp),$$

where $Y_s \subset TM$ is any vector field that is projectable onto $s$ under the action of the orthogonal projection $P_2: TM \to TL^\perp$. The curvature $R(x, u)$ is zero for $x, u \in TL$ by the Jacobi identity for the bracket of vector fields. This means that $(TL^\perp, \nabla)$, restricted to each leaf, is a flat vector bundle. The $\nabla$-parallel displacement in $TL^\perp$ along a path in $L$ is a linearized version of holonomy, and hence, depends only on the homotopy class of a path in a leaf.

1.3. Ehresmann Foliations

Ehresmann [76] defined a \textit{connection in a fiber bundle} as a distribution complementary (transversal) to fibers with the following completeness condition: every curve in the base admits a horizontal lift to the total space. The concept of an \textit{Ehresmann connection} is defined in [28-30] for a foliated manifold without boundary as a complementary (transversal) distribution to the foliation satisfying a certain condition. These studies were continued, in particular, for a foliated manifold with boundary in [158-161]. The graph of a foliation with Ehresmann connection is studied in [327] and [313, 314].

Consider a foliation $\{L\}$ on a smooth manifold $M$, together with a complementary distribution $D$. This distribution is transversal, but in the case of a Riemannian manifold, it may not be orthogonal to $TL$. We call a piecewise-smooth curve in $M$ whose velocity vector field lies in $TL$ (resp., $D$) an $L$-curve (resp., $D$-curve).

**Definition 1.8** [28]. Every $D$-curve $\beta : I \to M$ uniquely defines a family of local diffeomorphisms $g_t : V_0 \to V_t$, ($t \in I$) from one leaf onto another one (an \textit{element of holonomy along $\beta$}) such that (1) $V_t$ is a neighborhood of $\beta(t)$ in the leaf passing through $\beta(t)$, (2) $g_t(\beta(0)) = \beta(t)$ for all $t$, (3) for each $x \in V_0$, the curve $g_t(x)$ is tangent to $D$, (4) $g_0$ is the identity map of $V_0$. When the leaves of a foliation have a geometric structure (a measure, linear connection, and Riemannian metric), we say that $D$ \textit{preserves the geometry of the leaves} if the elements of holonomy along $D$-curves are local isomorphisms of this geometric structure.

For example, when $\{L\}$ is a totally geodesic, umbilic, or minimal foliation on a Riemannian manifold and $D = TL^\perp$, then an element of the holonomy along each horizontal curve is a local isometry, conformal mapping, or volume-preserving map of the induced metrics on the leaves, respectively.

**Definition 1.9** [38]. A piecewise-smooth mapping $\delta : [0, 1] \times [0, 1] \to M$ such that for every fixed $s_0$, the curve $\delta(\cdot, s_0)$ is a $D$-curve, and for every fixed $t_0$, the curve $\delta(t_0, \cdot)$ is an $L$-curve is said to be a \textit{rectangle} in the foliated manifold $M$. Also, the curves $\delta(0, \cdot)$, $\delta(1, \cdot)$, $\delta(\cdot, 0)$, and $\delta(\cdot, 1)$ are called the \textit{initial $L$-edge}, \textit{terminal $L$-edge}, \textit{initial $D$-edge}, and \textit{terminal $D$-edge} of $\delta$, respectively. The rectangle whose initial edges are an $L$-curve $\alpha$ and a $D$-curve $\beta$ is called a \textit{rectangle associated with $\alpha$ and $\beta$} and is denoted by $\delta_{\alpha, \beta}$. (It is easy to show that two rectangles with the same initial edges coincide, so the last notion is well defined.)

It is easy to show that two rectangles with similar initial edges coincide, and therefore, Definition 1.9 holds. For a sufficiently small positive $\varepsilon$, the rectangles $\delta_{\alpha|\alpha, \varepsilon, \varepsilon}$ and $\delta_{\alpha, \beta|\alpha, \varepsilon}$ always exist.
**Definition 1.10** [28]. A complementary distribution \( D \) to the foliation \( \{ L \} \) is called an *Ehresmann connection* for \( \{ L \} \) if, for every \( L \)-curve \( \alpha \) and every \( D \)-curve \( \beta \) with the same initial point, i.e., \( \alpha(0) = \beta(0) \), the rectangle \( \delta_{\alpha, \beta} \) exists. \( \{ L \} \) is called an *Ehresmann foliation* whenever such a distribution \( D \) exists.

The basic property of the Ehresmann connection is that for any rectangle \( \delta_{\alpha, \beta} \), the curve \( \delta(1, \cdot) \) depends only on the homotopy class of the curve \( \alpha \), and the curve \( \delta(\cdot, 1) \) depends only on the homotopy class of the curve \( \beta \). Moreover, for an Ehresmann foliation: (a) any two leaves can be joined by a \( D \)-curve, (b) the universal coverings of any two leaves are diffeomorphic.

**Theorem 1.4** (topological product) [28]. Let \( D \) be an integrable Ehresmann connection (with leaves \( L^\perp \)) for the foliation \( \{ L \} \) on \( M \). Then the universal covering \( \tilde{M} \) is topologically the product \( \tilde{L} \times \tilde{L}^\perp \) of universal coverings of the leaves \( L \) and \( L^\perp \).

**Corollary 1.1** [28]. A compact manifold \( M \) with a finite fundamental group does not admit an Ehresmann foliation of codimension 1. In particular, the Reeb foliation on \( S^3 \) is not geodesible.

An Ehresmann foliation on \( M \) with \( \dim M > 2 \) can be not geodesible and may not admit a bundle-like metric [28]. For a submersion with compact connected leaves, every distribution complementary to the leaves is the Ehresmann connection.

Let \( \{ L \} \) be a foliation on a Riemannian manifold, and let \( \alpha \) (resp. \( \beta \)) be an \( L \)-curve (resp. a \( D \)-curve). We denote by \( \text{Rec}(\alpha, \cdot) \) the set of all rectangles whose initial \( L \)-edge is \( \alpha \) and whose initial \( D \)-edge is a regular curve, and by \( \text{Rec}(\cdot, \beta) \) the set of all rectangles whose initial \( D \)-edge is \( \beta \) and whose initial \( L \)-edge is a regular curve. Let us define the function \( G^L_\alpha \) on \( \text{Rec}(\alpha, \cdot) \) by \( G^L_\alpha(\delta) = \frac{l(\delta, \alpha)}{l(\delta, \cdot)} \) for \( \delta \in \text{Rec}(\alpha, \cdot) \) and the function \( G^D_\beta \) on \( \text{Rec}(\cdot, \beta) \) by \( G^D_\beta(\delta) = \frac{l(\delta, \beta)}{l(\delta, \cdot)} \) for \( \delta \in \text{Rec}(\cdot, \beta) \), where \( l(\cdot) \) is the curve length. Note that \( G^L_\alpha \equiv 1 \) for every \( D \)-curve \( \beta \) in the case of a totally geodesic foliation, and \( G^L_\alpha \equiv 1 \) for every \( L \)-curve \( \alpha \) in the case of a Riemannian foliation (i.e., bundle-like metric). The following theorem generalizes the results for Riemannian foliations and for totally geodesic foliations.

**Theorem 1.5** [161]. Let \( \{ L \} \) be a foliation on a Riemannian manifold \( M \) satisfying one of the following conditions: (1) the induced metrics on the leaves are complete and \( \sup \{ G^L_\alpha \} < \infty \) for every \( TL^\perp \)-curve \( \beta \), (2) \( M \) is complete and \( \sup \{ G^L_\alpha \mid [0, s] : s \in [0, 1) \} < \infty \) for every \( L \)-curve \( \alpha \) without a terminal point. Then \( TL^\perp \) is an Ehresmann connection.

Ehresmann foliations also appear among totally umbilic foliations [28] and foliations with the following transversal structures: the Cartan structure, \( G \)-structure of finite type, system of differential equations of arbitrary order (see [327]), Lagrange foliations under certain constraints, [312], foliations on a manifold over a local algebra [272], and in field theories [171].

**Theorem 1.6** [224]. If the surjective submersion \( \pi: M \to B \) admits an Ehresmann connection \( D \), then it is a fiber bundle.

Theorem 1.6 generalizes the result from [134] for the orthogonal distribution \( D \) of Riemannian submersions and the result from [224] for submersions with totally geodesic fibers.

An Ehresmann foliation has no vanishing cycles; as a consequence, the homotopy groupoid of such a foliation is a Hausdorff manifold [311].

For an Ehresmann foliation, a more special holonomy and graph can be defined in the following way.

**Definition 1.11** [28]. Let \( \{ L \} \) be a foliation on a manifold \( M \) with the Ehresmann connection \( D \), and let \( \Omega_x \) be the set of all horizontal curves with origin \( x \). Then the fundamental group \( \pi_1(L, x) \) of a leaf \( L \) acts on \( \Omega_x \) in the following natural way:

\[
\text{for all } \alpha \in \Omega_x, [\psi] \in \pi_1(L, x) : [\psi] \alpha = \delta(1, \cdot),
\]
where $\delta$ is a rectangle associated with the curves $\psi, \alpha$. Let $K_D(L, x) = \{[\psi] \in \pi_1(L, x): [\psi]\alpha = \alpha \forall \alpha \in \Omega_2\}$ be the kernel of this action. The quotient group $\Gamma(D, L, x) = \pi_1(L, x)/K_D(L, x)$ is called the $D$-holonomy group of the leaf $L$ (compare with Definition 1.6).

$H_D(L, x)$ does not depend on the specific Ehresmann connection, and hence, is an invariant of the foliation. In some sense, $H_D(L, x)$ is a globalization of the germinal holonomy group $\Gamma(L, x)$ and admits a natural surjection onto it.

**Definition 1.12** [327]. Let $\{L\}$ be a foliation on a manifold $M$ with the Ehresmann connection $D$. The $D$-graph ($D$-holonomy groupoid) $\Gamma_D(M, L)$ is the collection of all triples $(x, y, [a])$, where $x, y$ lie in the same leaf $L$, $a$ is a (piecewise) smooth path going from $x$ to $y$ in $L$, and $[a]$ is the equivalence class of $a$ (two paths $a, b$ with similar endpoints $x, y$ are equivalent if $a^{-1}b \in K_D(L, x)$).

The graph $\Gamma_D(M, L)$ is always a Hausdorff $(\dim M + \dim L)$-dimensional manifold, and there exists a local isomorphism $\Gamma_D(M, L) \rightarrow \Gamma(M, L)$. Moreover, the graph $\Gamma(M, L)$ is Hausdorff if and only if it is isomorphic (in the canonical way) to $\Gamma_D(M, L)$ [327].

The graph $\Gamma_D(M, L)$ is useful for studying stability problems.

**Theorem 1.7** (global stability) [28, 327, 311]. Let $M$ be a foliated manifold with the Ehresmann connection $D$, and let $L$ be a compact leaf or a leaf with finite volume; moreover, assume that $L$ has a finite $D$-holonomy group. Then all leaves have finite $D$-holonomy groups and are compact or have finite volume, respectively.
Chapter 2
FOLIATION ON RIEMANNIAN MANIFOLDS

2.1. Basic Facts from the Theory of Submanifolds

Recall some necessary facts and formulas from the theory of submanifolds (see [60, 61] and [155]). If $M$ is a submanifold of a Riemannian space $\tilde{M}$, then

$$\tilde{\nabla}_{xy} \xi = \tilde{\nabla}_x y + h(x, y) \quad \text{(Gauss formula)},$$

$$\tilde{\nabla}_x \xi = -A_\xi x + \nabla^* \xi \quad \text{(Weingarten formula)},$$

where $x, y$ are vector fields tangent to $M$, $\xi$ is a vector field orthogonal to $M$, $h: TM \times TM \to TM^\perp$ is the second fundamental form of embedding, which is symmetric with respect to both arguments: $h(x, y) = h(y, x)$, $\nabla^*$ is a (metric) normal connection in the normal vector bundle $TM^\perp$, and $A_\xi$ is the operator of the second quadratic form for a normal vector $\xi$. From the formulas above it follows that

$$(A_\xi x, y) = (h(x, y), \xi).$$

(2.2)

We denote by $\tilde{R}$, $R$, and $R^\perp$ the curvature tensors of the connections $\tilde{\nabla}$, $\nabla$, and $\nabla^\perp$, respectively. The key role in the theory of submanifolds is played by the following equations of Gauss, Codazzi, and Ricci:

$$(\tilde{R}(x, y)z, u) = (R(x, y)z, u) + (h(x, u), h(y, z)) - (h(x, z), h(y, u)),$$

$$\tilde{\nabla}_{x} h(y, z) = (\tilde{\nabla}_y h)(x, z) - (h(x, z), h(y, u)).$$

(2.3)

where the vectors $x, y, z, u$ are tangent to $M$, the vectors $\xi$ and $\eta$ are orthogonal to $M$, and the derivative $\tilde{\nabla} h$ is defined by

$$(\tilde{\nabla}_x h)(y, z) = \nabla^* _x (h(y, z)) - h(\nabla^*_x y, z) - h(y, \nabla^*_x z).$$

For a submanifold in a space form $\tilde{M}(k)$ (and some submanifolds in $\text{CROSS}$, compact symmetric spaces of rank one), we have the relation

$$\tilde{R}(x, y)z^\perp = 0 \quad (x, y, z \in TM),$$

(2.4)

which is equivalent to $$(\tilde{\nabla}_x h)(y, z) = (\tilde{\nabla}_y h)(x, z).$$ A submanifold with property (2.4) is called a curvature-invariant submanifold. A totally geodesic submanifold is always curvature-invariant; for recent results, see [141].

**Definition 2.1.** For a submanifold $M^n \subset \tilde{M}$, the mean-curvature vector field $H \subset TM$ is defined by the formula $H = \sum_{i=1}^n h(e_i, e_i)$ (note that we have deleted the usual factor $1/n$), where $\{e_i\}$ is a local orthonormal basis in $TM$. A submanifold $M^n \subset \tilde{M}$ having one of the conditions

$$h = 0, \quad h(x, y) = \frac{1}{n} H(x, y), \quad H = 0$$

is totally geodesic, totally umbilic, or minimal, respectively.

**Definition 2.2** [60]. A vector bundle $E$ with metric over a Riemannian manifold $M$ and with the second fundamental form, i.e., a smooth section $h$ in $\text{Hom}(TM \oplus TM, E)$ satisfying the symmetry condition $h(y, z) = h(z, y)$ for all $y, z \in TM$, is called a Riemannian vector bundle.

**Theorem 2.1** [60]. (a) Let $M^n$ be a simply connected Riemannian manifold with a Riemannian $m$-dimensional vector bundle $E$ that satisfies Eqs. (5.3a), (5.3c) and (5.4). Then $M^n$ can isometrically be embedded into a space form $\tilde{M}^{n+m}(k)$ with normal bundle $E$. (b) Let $f_1, f_2: M \to \tilde{M}(k)$ be two isometric embeddings with normal bundles $E_1$ and $E_2$, and let there exist an isometry $\varphi: M \to M$ such that $\varphi$ can be covered by a bundle map $\tilde{\varphi}: E_1 \to E_2$ which preserves the metrics and the second fundamental forms of the bundles $E_1$ and $E_2$. Then there exists a rigid motion $g: \tilde{M}(k) \to \tilde{M}(k)$ with the property $g \cdot f_1 = f_2 \circ \varphi$. 1709
2.2. Main Tensors of a Foliation

We denote by $T_1$ and $T_2$ two complementary orthogonal distributions on a Riemannian manifold $M$. We define the structural tensors $B_1: T_2 \times T_1 \to T_1$ and $B_2: T_1 \times T_2 \to T_2$ by the formulas

$$B_1(y, x) = P_1(\nabla_x y), \quad B_2(x, y) = P_2(\nabla_y x), \tag{2.5}$$

where $P_i: TM \to T_i$ ($i = 1, 2$) are orthogonal projections, $\bar{x} \subset T_1$ and $\bar{y} \subset T_2$ are local vector fields containing the vectors $x$ and $y$ respectively. The equation $B_i = 0$ means that the distribution $T_i$ is involutive and tangent to a totally geodesic foliation. The relation $B_1 = B_2 = 0$ indicate that $M$ splits along $T_1$ and $T_2$, i.e., $M$ is locally a Riemannian product $L_1 \times L_2$, where the leaves $\{L_1\}$ and $\{L_2\}$ are tangent to $T_1$ and $T_2$, respectively. In the special case of a totally geodesic foliation $\{L\}$ (i.e., $B_1 = 0$), the tensor corresponding to $B_2$ is $B: TL \times TL^\perp \to TL^\perp$, which is defined in [86] with the opposite sign and in [78] is as follows:

$$B(x, y) = (\nabla_x y)^\perp. \tag{2.6}$$

This tensor (the conullity operator) is introduced in [234, 117, 202], and [206] for a (relative) nullity foliation.

Historically, the first to be defined (in [118] and [204]) were configuration tensors, $T$ (vertical) and $O$ (horizontal), by formulas different from those in (2.5):

$$T_{uv} = P_2(\nabla_{p_1} P_2 v) + P_1(\nabla_{p_2} P_2 u), \quad O_{uv} = P_1(\nabla_{p_2} P_1 v) + P_2(\nabla_{p_1} P_1 u). \tag{2.7}$$

The tensor $T$ is defined by its values $T_{x_1 x_2}$ or by its values $T_{x_2 y_2}$, and similarly for the tensor $O$, in view of identities

$$(T_{x_1 x_2 y_2} + T_{x_2 y_1 x_2}) = -T_{x_2 y_1 x_2}, \quad (O_{x_1 y_2 x_2} + O_{x_2 y_1 x_2}) = -O_{x_2 y_1 x_2}, \tag{2.8}$$

where $x, x_1, x_2 \in T_1$ and $y, y_1, y_2 \in T_2$. These classical tensors can be expressed in terms of the structural tensors $B_1$ and $B_2$ as follows:

$$T_{xy} = B_1(y, x), \quad O_{xy} = B_2(x, y). \tag{2.9}$$

In the case of the foliation $\{L\}$ on a Riemannian manifold $M$, the configuration tensor $T$ is the second fundamental form of the leaves, and, therefore, the identity $T = 0$ means that the leaves are totally geodesic submanifolds. In the sequel, vectors and vector fields on $TM$ tangent (respectively, orthogonal) to the leaves of a foliation are said to be vertical (resp., horizontal).

The second fundamental forms (symmetric tensors) $h_1: T_1 \times T_1 \to T_2$ and $h_2: T_2 \times T_2 \to T_1$ and the integrability tensors (skew-symmetric) $A_1: T_1 \times T_1 \to T_2$ and $A_2: T_2 \times T_2 \to T_1$ are defined by the formulas

$$h_1(x, u) = \frac{P_2(\nabla_{p_1} \bar{u} + \nabla_{\bar{u}} x)/2, \quad A_1(x, u) = \frac{P_2(\nabla_{\bar{u}} \bar{x} - \nabla_{\bar{u}} \bar{x})/2, \quad h_2(y, z) = \frac{P_1(\nabla_{\bar{x}} \bar{y} + \nabla_{\bar{y}} \bar{x})/2, \quad A_2(y, z) = \frac{P_1(\nabla_{\bar{y}} \bar{x} - \nabla_{\bar{y}} \bar{x})/2, \tag{2.10}$$

where $\bar{u}, \bar{x} \subset T_1$ and $\bar{y}, \bar{z} \subset T_2$ are local vector fields containing the vectors $x, y, z$, respectively. Their geometrical meaning is the following: $h_i$ is the second fundamental form at a point $p \in M$ of the submanifold consisting of geodesics in $M$ that are tangent to the subspace $T_i(p)$. In view of the identities $A_1(x, u) = \frac{1}{2}P_2[x, u]$ and $A_2(y, z) = \frac{1}{2}P_1[y, z]$, the relation $A_i = 0$ means that $T_i$ is tangent to the foliation $\{L_i\}$.

The mean curvature vector fields $H_2 = \text{tr} h_2 \subset T_1$ and $H_1 = \text{tr} h_1 \subset T_2$ (of the distributions $T_2$ and $T_1$, resp.) can be calculated by the formulas

$$(H_2, x) = -\text{tr} B_2(x, \cdot), \quad (H_1, y) = -\text{tr} B_1(y, \cdot). \tag{2.11}$$

We denote by $B^+_i, B^-_i$ ($i = 1, 2$) the symmetric and skew-symmetric components with respect to $T_i^\perp$ of the structural tensors $B_i$.

Lemma 2.1. $B^+_i$ ($i = 1, 2$) plays the role of self-adjoint Weingarten tensors associated with $h_i$, and $B^-_i$ ($i = 1, 2$) serves as an equivalent to integrability tensors of distributions $T_i$ by virtue of the identities

$$(B^+_i(y, x), u) = -(h_i(x, u), y), \quad (B^-_i(x, y), z) = -(h_2(y, z), x), \quad (B^+_1(y, x), u) = -(A_1(x, u), y), \quad (B^-_2(x, y), z) = -(A_2(y, z), x). \tag{2.12}$$
Indeed, $B^+_t$ is the shape operator at a point $p \in M$ of the locally defined transversal submanifold consisting of geodesics in $M$ that are tangent to the subspace $T_t(p)$.

**Definition 2.3.** A distribution $T_t$ on a Riemannian manifold $(M, g)$ is (1) geodesic, (2) minimal, or (3) umbilic if its second fundamental form has the property

\[
(1) \ h_t = 0, \quad (2) \ H_t = 0, \quad \text{or} \quad (3) \ h_t(x, y) = (H_t/\dim T_t)(x, y).
\]

The geometrical meaning of geodesic distribution is that *every geodesic in $M$ that is tangent to $T_t$ at one point is tangent to $T_t$ at each of its points.*

Using Definition 2.3, some well-known classes of foliations can be introduced in terms of their tangent or transversal geometry.

**Definition 2.4.** A foliation $\{L\}$ on a Riemannian manifold $M$ is said to be (1) totally geodesic, (2) minimal, or (3) totally umbilic if its leaves are totally geodesic, minimal, or totally umbilic submanifolds, respectively. A foliation $\{L\}$ on a Riemannian manifold $M$ is said to be *Riemannian* or *conformal* if the orthogonal distribution $TL^\perp$ is geodesic or umbilic, respectively. A totally umbilic foliation is *spherical* if its mean-curvature vector field is parallel in the normal bundle along leaves (i.e., the leaves are *extrinsic spheres* in the total space).

Some pairs of foliations are dual in a certain sense: totally geodesic and Riemannian (see [187]), and totally umbilic and conformal foliations. Also, a foliation whose orthogonal distribution is minimal (i.e., with a holonomically invariant transversal volume form; see Proposition 2.1 below) is dual to a minimal foliation.

Classes of foliations on a smooth manifold can also be defined in terms of the existence of certain adapted Riemannian metrics. A foliation is said to be

*Riemannian* if there exists a Riemannian metric for which the leaves are locally equidistant submanifolds (an example is a *homogeneous* foliation obtained by an isometric action of a Lie group on a Riemannian manifold),

*umbilicalizable* (resp. *geodesible*) if there exists a Riemannian metric for which the leaves are totally umbilic [49] (resp. totally geodesic) submanifolds,

*taut* if there exists a Riemannian metric for which the leaves are minimal submanifolds,

*tense* if there exists a Riemannian metric for which the differential mean-curvature form of the leaves is parallel along the leaves [146],

*mean-curvature invariant* (MCI) if there exists a Riemannian metric for which the mean curvature vector field is a basic vector field along the leaves (i.e., the foliation is invariant under the local flow generated by the mean curvature vector field) [307]:

\[
([H, x], y) = 0, \quad (x \in TL, \ y \in TL^\perp).
\]

**Example 2.1.** Assume that $f: M \to N$ is a smooth map of Riemannian manifolds with constant rank $r(f) < \dim M$ (in particular, a submersion when $r(f) = \dim N < \dim M$), $f_*: TM \to TN$ is the differential of $f$, $f^{-1}(TN)$ is the induced bundle over $M$, and $\nabla$ is a sum of connections of $N$ and $M$ in $f^{-1}(TN) \oplus TM$. Then $TM = T_1 \oplus T_2$, where the distribution $T_1 = \ker f = \{x \in TM: f_*x = 0\}$ is always integrable and $T_2$ is the orthogonal distribution. The *second fundamental form* $h_f: TM \times TM \to f^{-1}(TN)$ of $f$ (a symmetric bilinear map) is defined by the formula

\[
h_f(x, y) = \nabla_x f_*(y) - f_*(\nabla x y).
\]

$H_f = \text{tr} h_f$ is the *mean-curvature vector of $f$. The map $f$ is*

*harmonic* if $H_f = 0$; in this case, $\ker f$ defines a minimal foliation,

*umbilic* if $h_f(x, y) = \frac{1}{m}H_f(x, y)$, where $m = \dim \ker f$; in this case, $\ker f$ defines an umbilic foliation and $T_2$ is tangent to a totally geodesic foliation,
geodesic (or projective) if it preserves the geodesics, i.e., there exists a 1-form ω on M such that

\[ h_f(x, y) = \omega(x)f_y + \omega(y)f_x \]  

[197]; in this case, ker f defines a totally geodesic foliation and \( T_2 \) is tangent to an umbilic foliation,

affine if it is a geodesic (or projective) mapping and preserves the natural parametrization of geodesics; in this case, \( h_f(x, y) = 0 \) [274].

The notion of the second fundamental form of a mapping between manifolds endowed with a connection, constructed for the study of harmonic mappings (see [78]), generalizes the second fundamental form of a submanifold isometrically immersed in a Riemannian manifold; it was used in [297] to study totally geodesic mappings and Riemannian submersions, in [128] to study projective mappings, in [319] to study harmonic and affine mappings, in [197] to study projective and umbilic mappings, and in [274–276] to study mappings of this kind with a nonconstant rank.

An interesting interpretation of the second fundamental form \( h \) of foliation \( \{L\} \) follows from the formula

\[ \Theta(y)g(x_1, x_2) = -2g(h(x_1, x_2), y), \quad (y \in TL, x_1, x_2 \in TL), \]  

(2.14)
i.e., for a vector field \( y \) orthogonal to \( \{L\} \); the \( y \)-component of \( h \) is the Lie derivative \( \Theta \) with respect to \( y \) of the metric \( g \) along the leaves. Hence a totally geodesic (or totally umbilic) foliation \( \{L\} \) on \( (M, g) \) is characterized by the condition that the induced metric \( g_{TL} \) along the leaves is invariant (resp., conformally invariant) under the flows of vector fields orthogonal to the foliation, i.e., for \( y \in TL \), the relation \( \Theta(y)g_{TL} = 0 \) (resp., \( \Theta(y)g_{TL} = -2(H(y), g_{TL}) \) holds [285]. Similarly, a Riemannian foliation is characterized by the condition that the induced metric on \( TL \) is holonomically invariant, that is, \( \Theta(x)g_{TL} = 0, \quad (x \in TL) \) [285].

If all leaves \( L \) are orientable, then the foliation is said to be tangentially orientable. The characteristic form \( \chi \) of the \( ν \)-dimensional tangentially oriented foliation \( \{L\} \) on \( (M, g) \) is a \( ν \)-form on \( TM \) that takes the value 1 when it is evaluated on a locally oriented orthonormal frame \( \{e_j\} \) of \( \{L\} \). For arbitrary vectors \( u_1, \ldots, u_ν \in TM \), the form \( \chi \) is given by

\[ \chi(u_1, \ldots, u_ν) = \det\{g(u_i, e_j)\}_{ij}. \]  

(2.15)
From formulas (2.14) and (2.15), we obtain

\[ \Theta(y)\chi_{|L} = -H(y)\chi_{|L} \]  

(2.16)
(see [285]). In other words, the mean curvature is the first-order variation of the volume of the leaves for variations in the direction of the mean curvature vector field. The transversal volume form \( \chi^\perp \), which is similarly defined, satisfies the following equation, which is dual to (2.16) [285]:

\[ \Theta(x)\chi^\perp_{|TL} = -H^\perp(x)\chi^\perp_{|TL}, \]  

(2.17)
Hence (a) minimal foliations are characterized by the condition that the restriction \( \chi_{|L} \) of the volume form to the leaves is invariant under the flows of vector fields orthogonal to the foliation, i.e., \( \Theta(y)\chi_{|L} = 0 \) \( (y \in TL) \), (b) transversally orientable foliations with a holonomically invariant transversal volume form \( \chi^\perp \) are characterized by the condition \( H^\perp = 0 \) [285].

2.3. A Riemannian Almost-Product Structure

Two complementary orthogonal distributions \( T_1 \) and \( T_2 \) on a Riemannian manifold \( M \) (see Sec. 2.2) can be considered in analytic terms.

**Definition 2.5** [227]. A Riemannian manifold \( M \) with metric \( (\ , \ ) \) and a tensor field \( P \) of type \((1,1)\) on \( TM \) such that

\[ P^2 = \text{Id}, \quad (Pu, Pv) = (u, v) \]  

(2.18)
is called a **Riemannian almost-product structure**. The eigenspaces of \( P \) for the eigenvalues +1 and −1 are called the **vertical distribution** \( T_1 \) and the **horizontal distribution** \( T_2 \); they are orthogonally complementary to
each other, i.e., $TM = T_1 \oplus T_2$. Indeed, $P_1 = (\text{Id} + P)/2$ and $P_2 = (\text{Id} - P)/2$ are orthogonal projections of $TM$ onto $T_1$ and $T_2$, respectively.

The integrability tensors $A_1, A_2$ (cf. (2.10)) are given by

$$A_1(u, v) = \frac{1}{2} P_2[ P_1 u, P_1 v], \quad A_2(u, v) = \frac{1}{2} P_1[ P_2 u, P_2 v].$$  \hspace{1cm} (2.19)

It is easy to see that $A_i$ vanishes if and only if $T_i$ is tangent to the foliation. In view of formulas (2.18) and (2.19), the Nijenhuis tensor (or torsion) $N = 8(A_1 + A_2)$ can be written as

$$N(u, v) = [P_1 P_2](u, v) = [u, v] + [P_2 u, P_2 v] - P[P_2 u, v] - P[u, P_2 v].$$  \hspace{1cm} (2.20)

Both distributions $T_1$ and $T_2$ are integrable whenever $N = 0$ holds. The second fundamental forms $h_1$ and $h_2$ (cf. (2.10)) are given by

$$h_1(u, v) = \frac{1}{2} P_2[ P_1 u, P_1 v], \quad h_2(u, v) = \frac{1}{2} P_1[ P_2 u, P_2 v],$$  \hspace{1cm} (2.21)

where $\{u, v\} = (\nabla_u v + \nabla_v u)/2$ is the Jordan bracket of two vector fields $u$ and $v$. In view of formulas (2.18) and (2.21), the Jordan tensor $L = 8(h_1 + h_2)$ can be written as

$$L(u, v) = \{P_1 P_2\}(u, v) = \{u, v\} + \{P_2 u, P_2 v\} - P\{P_2 u, v\} - P\{u, P_2 v\}.$$  \hspace{1cm} (2.22)

Both distributions $T_1$ and $T_2$ are geodesic whenever $N = 0$ holds. The vector field $H = 8(H_1 + H_2)$, where $H_i = \text{tr} \ h_i \ (i = 1, 2)$, can be written as

$$H = \text{tr}\{P_1 P_2\}. \hspace{1cm} (2.23)$$

The covariant derivative $\nabla P$ can be decomposed into the sum of eight pointwise irreducible components with respect to the action of the group $O(T_1) \oplus O(T_2)$. For the distribution $D$ ($T_1$ or $T_2$), we consider the corresponding eight conditions [101]:

1. $(\nabla_u P)u = 0$, \hspace{1cm} GD is a geodesic distribution,
2. $\sum_{e_i \in D}(\nabla_{e_i} P)e_i = 0$, \hspace{1cm} MD is a minimal distribution,
3. $(\dim D)(\nabla_u P)v = (u, v) \sum_{e_i \in D}(\nabla_{e_i} P)e_i$, \hspace{1cm} UD is an umbilic distribution,
4. $\varnothing \hspace{1cm} \Delta$, no conditions
5. $\nabla_u P = 0$, \hspace{1cm} GF is a totally geodesic foliation,
6. $(\nabla_u P)v = (\nabla_e P)u, \sum_{e_i \in D}(\nabla_{e_i} P)e_i = 0$, \hspace{1cm} MF is a minimal foliation,
7. $(\nabla_u P)v = (\nabla_u P)u, \ (\dim D)(\nabla_u P)u = (u, v) \sum_{e_i \in D}(\nabla_{e_i} P)e_i$, \hspace{1cm} UF is a totally umbilic foliation,
8. $(\nabla_u P)v = (\nabla_u P)u$, \hspace{1cm} $F$ is a foliation.

Properties (5)–(8) are similar to (1)–(4), but with an integrability condition.

Combining these eight conditions and eliminating the dual situations, we obtain 36 different classes of almost-product structures, each of which is characterized by some algebraic condition on $\nabla P$ [194].

For example, a $(GD, \Delta)$ almost-product structure (i.e., the distribution $T_1$ is $GD$ and $T_2$ is $\Delta$; a similar notation is used for the other 35 classes of almost-product structures) is called an almost foliated metric in [298] and an anti-foliation in [194]; in other words, it is a geodesic distribution. The $(GD, F)$ and $(UD, F)$ almost-product structures are actually Riemannian and conformal foliations.

Let us consider some examples of the 36 classes of almost-product structures constructed by using hypersurfaces and products of manifolds given in [185]. There are nine classes of almost-product structures on real (complex) hypersurfaces $M$ in $K^{n+1}$ ($K = \mathbb{C}, \mathbb{H}$). Let $M$ be an orientable real hypersurface in $\mathbb{C}^{n+1}$, $N$ be a unit normal vector field on $M$, and let $J$ be a canonical almost-complex structure on $\mathbb{C}^{n+1}$ (see Fig. 12). The canonical almost-product structure on $M$ is defined by the distributions $V = JN$ and $H = JN^\perp$. Observe that $JH = H$. Every $M$ of this kind belongs to the class $(UF, MD)$.
Fig. 12. The structure $(UF, MD)$ on a hypersurface in $\mathbb{C}^{n+1}$.

**Lemma 2.2 [185].** For $M^{2n+1} \subset \mathbb{C}^{n+1}$, we have the following equivalences (here $x, y \in H$):

1. $M$ is $(UF, GF) \iff h(x, y) = 0$,
2. $M$ is $(GF, GD) \iff h(x, Jx) = 0, h(x, JN) = 0$,
3. $M$ is $(GF, MF) \iff h(x, Jy) = h(Jx, x), h(x, Jx) = 0$,
4. $M$ is $(UF, GD) \iff h(x, Jx) = 0$,
5. $M$ is $(UF, MF) \iff h(x, Jy) = h(Jx, y)$,
6. $M$ is $(GF, MD) \iff h(x, JN) = 0$.

Let $M$ be an orientable real hypersurface in $\mathbb{H}^{n+1}$, $N$ be a unit normal vector field on $M$, and let $\{J_1, J_2, J_3\}$ be a canonical almost-complex structures on $\mathbb{H}^{n+1}$. These elements define the canonical almost-product structure on $M$ with $V$ generated by the vector fields $\{J_1N, J_2N, J_3N\}$ and $H = V^\perp$. Observe that $J_1H = H$.

Every $M$ of this kind belongs to the class $(\Delta, MD)$. Almost-product structures on a complex hypersurface $M$ in $\mathbb{H}^{n+1}$ can be considered analogously.

Conformal transformations of metrics on products of manifolds with a pair of distributions lead to some additional examples of almost-product structures [185]. The rules for metric products of manifolds with Riemannian almost-product structures are as follows:

$$
\begin{array}{ccc}
(M, P) & (M', P') & (M \times M', P + P') \\
(MF, GF) & (GF, MF) & (MF, MF) \\
(MF, GF) & (GF, GD) & (MF, GD) \\
(MF, GF) & (GF, MD) & (MF, MD) \\
(GD, GF) & (GF, GD) & (GD, GD) \\
(GD, GF) & (GF, MD) & (GD, MD) \\
(MD, GF) & (GF, MD) & (MD, MD) \\
\end{array}
$$

(2.37)

Indeed, there are interesting classes of foliations and distributions outside of the above scheme, such as, for example, the following:

1. Foliations with the property that the vector field $P_2(\nabla yz)$ is basic whenever $y$ and $z$ are basic vector fields; these foliations are characterized by the condition

$$
((\nabla z h_2)(y, y) - 2(\nabla y h_2)(z, y), x) + (h_2(y, y), B_1(z, x)) - 2(h_2(z, y), B_1(y, x)) = 0
$$

(2.24)

(see Sec. 3.8).

2. Foliations with constraints on the extrinsic curvature of the leaves, for instance, foliations of negative (nonpositive) extrinsic sectional curvature (see [39]) and some generalizations for partial Ricci curvature (see Sec. 3.7).

3. Weakly harmonic distributions [233], which generalize the concept of minimal foliation.
2.4. Constructions of Totally Geodesic and Totally Umbilic Foliations

2.4.1. Nullity foliations. Totally geodesic and totally umbilic foliations appear naturally as null-distributions (or kernels) in the study of manifolds with degenerate curvature-like tensors and certain differential forms. Examples of such nullity foliations can be divided into two classes.

1. A curvature-like tensor field $R$ on a Riemannian manifold $M$ is a $(1, 3)$-tensor field that satisfies the first and second Bianchi identities. For a curvature-like tensor field $R$, the $k$-nullity space (the nullity space when $k = 0$) at $m \in M$ is defined by the formula

$$N_k(m) = \{ x \in T_m M : R(x, y)z = k((y, z)x - (x, z)y) \text{ for all } y, z \in T_m M \},$$

and its dimension is called the index of $k$-nullity of $R$ at $m$. Let $G$ be an open (nonempty) subset of $M$ on which the index of $k$-nullity of $R$ takes its minimum value. Then the distribution $N_k$ is involutive and geodesic on $G$, and if $M$ is complete, then the maximal integral manifolds are complete. The nullity space of the Riemannian curvature tensor was first studied in 1966 in a number of works and was later generalized to any curvature-like tensor (see [155]).

For a hypersurface $M^n \subset M^{n+1}(k)$ with type number $r(x) \geq 2$ ($r(x)$ is the rank of the second quadratic form at $x$), the index of $k$-nullity $\nu_k(x)$ of the curvature tensor is equal to $n - r(x)$ (see [155]). Since $r(x)$ is equal to 1 or 0 if and only if $M^n$ has constant sectional curvature $k$ at $x$, the Riemannian manifolds of constant $k$-nullity $n - 2$ are of special interest. Every Riemannian manifold $M$ of nullity $n - 2$ is a semisymmetric space, i.e., $R(x, y) \circ R = 0$ $(x, y \in TM)$ holds, where $\circ$ denotes the derivation on the algebra of all tensor fields on $M$. Since the nullity distribution defines a totally geodesic and locally Euclidean foliation, these spaces are said to be foliated semisymmetric (see [31]). (Other semisymmetric spaces are either locally symmetric spaces or 2-dimensional surfaces, or Szabo cones; see [164].) A key problem of classifying 3-dimensional Riemannian manifolds with $k$-nullity 1 with respect to some constant $k \in \mathbb{R}$ is studied in [207] and [126] (for $k = 0$, see [42]). In these papers, a special system of PDE's is solved and explicit formulas for these metrics are given; using these formulas, interesting hypersurfaces in $\mathbb{R}^4(k)$ are constructed.

The decomposition of a Riemannian manifold $M$ whose curvature tensor has a positive index of nullity is studied in [234] and [294].

The nullity space of the Weyl conformal curvature tensor (which does not satisfy the second Bianchi identity) is studied in [97] and [268]: leaves of a certain foliation are totally umbilic and conformally flat. The nullity space of the Bochner curvature tensor (which also satisfies the second Bianchi identity) on a Kählerian manifold is studied in [150]: leaves of the corresponding foliation are Kählerian totally geodesic submanifolds. Contact metric manifolds $(M^{2n+1}, \phi, \xi, \eta, g)$ satisfying the condition that the characteristic vector field $\xi$ belongs to the $k$-nullity distribution, i.e., $R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y)$, are studied in [162] and [214]. Sasaki metrics on the unit tangent bundle $S_1M$ (see Sec. 4.1.2) with a positive index of $k$-nullity (($k > 0$)) are studied in [318].

2. The relative nullity space of the second fundamental form $h$ of the submanifold $M \subset \tilde{M}$ at $m \in M$ is given by

$$\text{ker } h(m) = \{ x \in T_m M : h(x, y) = 0 \text{ for all } y \in T_m M \}.$$
technique is generalized in [229, 135] and [32], where variations of circles in Riemannian manifolds are studied and results on the completeness of spherical foliations induced by the kernel of a certain (spherical) differential form are given.

2.4.2. Double-twisted products. A popular generalization of the product of two (pseudo-)Riemannian manifolds is the warped product of these manifolds introduced in [25]. In fact, it appeared in mathematical and physical literature long before that time; for example, polar coordinates of the Euclidean plane $\mathbb{R}^2$ describe a warped product representation of $\mathbb{R}^2 \setminus \{0\}$, and most of the space-time models are warped products. Warped products allow us to construct new examples of complete Einstein manifolds [24]. In recent studies of spaces with the Ricci curvature bounded from below, a class of warped products $(a, b) \times_\lambda M$ for some fixed $\lambda$ serves as a set of possible smooth rigid models [58, 59]. The result of Cartan stating that a Riemannian manifold $M$ that satisfies the axiom of $s$-planes with fixed $1 < s < \dim M$ is isometric to a space form was later generalized by geometers in two directions, namely, to more general submanifolds (totally umbilic, extrinsic spheres, submanifolds with parallel second fundamental form) and manifolds with additional structures (complex, quaternion, etc.) (see [172]).

The warped product of an $l$-dimensional space form with an arbitrary Riemannian manifold contains a large number of $s$-dimensional totally geodesic submanifolds for any $s > l$; they are inverse images of the projection onto the base of $(s - l)$-dimensional totally geodesic submanifolds in a given space form. In [200] this property is used as a basis of the synthetic axiom of $(l, s)$-planes, which is a generalization of the Cartan axiom: for every point $p \in M$ and any $l$-dimensional direction $V \subset T_p M$, there exists a totally geodesic $s$-dimensional submanifold tangent to $V$. The conjecture (which is partially proved in [200]) states that the axiom of $(l, s)$-planes (for any $1 \leq l < s \leq \dim M - 1$) characterizes the warped products of $l$-dimensional space forms with an arbitrary Riemannian manifold.

For simplicity, we denote by $\{L_i\}$ the canonical foliations on the product manifold $M_1 \times M_2$ with natural projections $p_i$ onto $M_i$, ($i = 1, 2$). Let $P_i$ be the projection of $T(M_1 \times M_2)$ onto $T L_i$, and let $P_i^\perp = \text{Id} - P_i$, ($i = 1, 2$). The following notion is a natural generalization of warped products.

**Definition 2.6** ([218] and [160]). Let $(M_i, g_i)$, $i = 1, 2$, be a Riemannian manifold, and let $\lambda_i$: $M_1 \times M_2 \to \mathbb{R}$ be positive differentiable functions. The double-twisted product $M_1 \times_{(\lambda_1, \lambda_2)} M_2$ is the differentiable manifold $M_1 \times M_2$ with a Riemannian metric defined by

$$(x, y) = \lambda_1^2 g_1(P_1 x, P_1 y) + \lambda_2^2 g_2(P_2 x, P_2 y)$$

for all vectors $x$ and $y$ tangent to $M_1 \times M_2$. In particular, if $\lambda_i$ are independent of the $M_i$-components, then $M_1 \times_{(\lambda_1, \lambda_2)} M_2$ is a double-warped product.

This definition generalizes the Bishop notion of umbilic product $M_1 \times_\lambda M_2$, which in [61] is called the twisted product and which is a special case of the double-twisted product $M_1 \times_{(1, \lambda)} M_2$. If, in this case, $\lambda$ depends only on a point of $M_1$, then $M_1 \times_\lambda M_2$ is a warped product. Note that the conformal change of a Riemannian metric can be interpreted as a twisted product, namely, a product where the first factor $M_1$ consists of only one point. Therefore, formulas and assertions concerning double-twisted products are applicable in many situations. For information on double-twisted products with more than two factors, see [160] and [98].

**Proposition 2.1** [160]. In the double-twisted product $M_1 \times_{(\lambda_1, \lambda_2)} M_2$,

1. the leaves $\{L_i\}$ form a totally umbilic foliation with mean curvature vector field

$$H_i = -P_i^\perp(\text{grad} \log \lambda_i)$$

(in the case of the twisted product $\lambda_1 = 1$, the leaves $\{L_1\}$ are even totally geodesic);

2. the leaves $\{L_i\}$ have parallel (in a normal connection) mean curvature vector if and only if $\lambda_i$ is the product of two positive functions on $M_1$ and $M_2$. 

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Lemma 2.3 [218]. The Levi-Civita connection $\nabla$ and the curvature tensor $R$ of the double-twisted product $M_1 \times_{(\lambda_1,\lambda_2)} M_2$ with $U_i = -\text{grad} (\log \lambda_i)$ and also the related Levi-Civita connection $\tilde{\nabla}$ and the curvature tensor $\tilde{R}$ of the ordinary product of Riemannian manifolds $M_1$ and $M_2$ satisfy the relations

$$\nabla_{xy} = \tilde{\nabla}_{xy} + \sum_i ((P_x, P_y) u_i - (x, u_i) P_y - (y, u_i) P_x),$$

$$R(x, y) = \tilde{R}(x, y) + \sum_i (u_i, u_i) P_x \wedge P_y$$

$$+ \sum_i ((\nabla_x u_i - (x, u_i) u_i) \wedge P_y - (\nabla_y u_i - (y, u_i) u_i) \wedge P_x),$$

where, for all $u, v \in T_p M$, the wedge product $u \wedge v$ denotes the linear mapping $w \rightarrow (v, w)u - (u, w)v$.

Double-twisted and double-warped products can be characterized in terms of the geometry of their canonical foliations.

Proposition 2.2 [218]. Let $g$ be a (pseudo-)Riemannian metric on a smooth manifold $M_1 \times M_2$; assume that the canonical foliations $\{L_1\}$ and $\{L_2\}$ intersect each other at right angles everywhere. Then $g$ is a metric of

1. the double-twisted product $M_1 \times_{(\lambda_1,\lambda_2)} M_2$ if and only if $\{L_1\}$ and $\{L_2\}$ are totally umbilic foliations,
2. the twisted product $M_1 \times_{\lambda} M_2$ if and only if $\{L_1\}$ is a totally geodesic foliation and $\{L_2\}$ is a totally umbilic foliation,
3. the warped product $M_1 \times \lambda M_2$ if and only if $\{L_1\}$ is a totally geodesic foliation and $\{L_2\}$ is a spherical foliation,
4. the ordinary (pseudo-)Riemannian product if and only if $\{L_1\}$ and $\{L_2\}$ are totally geodesic foliations.

Note that for a simply connected manifold $M$, the splitting (i.e., the local metric decomposition; see (4) of Proposition 2.2) is equivalent to the global product structure. On the other hand, a geodesic foliation on the Klein bottle with a flat metric splits only locally. If a complete connected and simply connected Riemannian manifold $M$ has two complementary orthogonal totally geodesic foliations, then the de Rham decomposition theorem (see [155] and [213]) asserts that $M$ is isometric to the product of two leaves through any point $m \in M$; for the case where $M$ is not assumed to be simply connected, see [306].

Manifolds with two complementary orthogonal totally umbilic foliations (totally umbilic orthogonal 2-nets [203]) were considered by many authors (see [25, 98, 195]). In [160] and [218], the “double-twisted decomposition theorems” are proved. Even if a totally umbilic orthogonal 2-net $M$ is a simply connected and complete Riemannian manifold, we cannot expect $M$ to be globally isometric to a double-twisted product. If, however, one of the foliations is totally geodesic, then $M$ splits globally as a twisted or warped product [218]. The case of $M$ with boundary is considered in [67]; for the case of $M$ with additional assumptions concerning the metric, see [99, 100]; for the case of $M$ with an additional complex structure, see [307] and [201]. Finally, for the cases of $M$ with an additional symplectic-type structure, consult [64] and [308].

2.4.3. Totally geodesic foliations on space forms. For $k > 0$, totally geodesic foliations on a 3-sphere are classified in [104]; in the case of $S^n (n \geq 3)$, there exist some interesting subclasses such as skew-Hopf fibrations and fibrations associated with a curvature tensor (see Sec. 2.6).

For $k < 0$, there are no totally geodesic foliations on a complete space form $M(k)$ with a finite volume [323, 324]; the same is true for locally symmetric spaces. For $k < 0$, there are no totally geodesic foliations with compact leaves on the space form $M(k)$ even locally (see [251] for $K_{\text{mix}} < 0$). Some results for totally geodesic foliations on a hyperbolic space $H^n(k)$ are given in [87, 11, 47].

For $k = 0$, a totally geodesic foliation, given locally on $M(0)$ and having compact leaves, splits (see [251] for $K_{\text{mix}} = 0$).

In [13], conformal geodesic one-dimensional foliations on the 3-dimensional space form $M(k)$ are studied. Minimal foliations on the space forms $M(k)$ are studied for $k > 0$ in [144] and [116].
2.5. Riemannian Foliations

The study of Riemannian foliations was initiated by Reinhart [227, 228]. Recent results are systematized in [187] and [285-287].

Leaves of a Riemannian foliation are locally given by level sets of a Riemannian submersion. The second fundamental form $h_2$ of the normal distribution $T_2$ vanishes (i.e., the operator $B_2(x, \cdot)$ is skew-symmetric for all $x \in TL$). Therefore, this distribution on $TM$ is geodesic. In particular, a geodesic that is orthogonal to a leaf at one point of $M$ is orthogonal to $\{L\}$ at every one of its points. This property, which characterizes Riemannian foliations, was one of the initial observations of Reinhart [228] concerning this subject.

Since the curvature of a Riemannian foliation has stronger vanishing properties than that of an arbitrary foliation, its characteristic classes of Riemannian foliation should possess special properties. Many ordinary classes vanish, but there are additional classes that can be defined only for Riemannian foliations (see [227]). The idea advanced in [286] is to characterize the tautness of a Riemannian foliation by cohomological properties. It is based on the following theorem (for simplicity, let $TL$ and $TM$ be orientable).

**Theorem 2.2** ([259] and [279]). A Riemannian metric $g_L$ on leaves (with the volume form $\omega_L$) induces a Riemannian metric $g$ on $M$ for which all leaves are minimal if and only if $\omega_L$ is a restriction of a $\nu$-form $\chi$ to $M$ that satisfies the condition $d\chi(x_1, \ldots, x_{\nu+1}) = 0$, where $\nu (= \dim L)$ of the vector fields among $\{x_i\}$ are sections of $TL \subset TM$.

For example, the foliation $\mathcal{F}(G, H)$ of a Lie group $G$ by a Lie subgroup $H$ is taut [181]. A compact orientable foliation on a compact orientable manifold $M$ is taut iff the holonomy groups of all leaves are finite [227].

The umbilicalizable Riemannian foliation $\{L\}$ on a compact connected manifold is tense, and the following conditions imposed on it are equivalent: (1) $\{L\}$ is taut; (2) $\{L\}$ is geodesible; (3) $\{L\}$ is cohomologically taut (i.e., the cohomology of the basic forms, which are the differential forms that are locally pull-backs of the forms on the local quotient manifold $r^\nu$, satisfies the Poincaré duality) [49].

### 2.5.1. The basic cohomology

Let $\{L\}$ be a foliation on $M$. A differential form $\omega \in \Omega^r(M)$ is basic if

$$i(x)\omega = 0, \quad \Theta(x)\omega = 0, \quad x \in TL,$$

where $i(x)$ denotes the inner product with respect to $x$. In a distinguished chart $(x_1, \ldots, x_\nu; y_1, \ldots, y_{m-\nu})$ of the foliation, this means that

$$\omega = \sum_{\alpha_1 < \cdots < \alpha_r} \omega_{\alpha_1 \cdots \alpha_r} \, dy_{\alpha_1} \wedge \cdots \wedge dy_{\alpha_r},$$

where the functions $\omega_{\alpha_1 \cdots \alpha_r}$ are independent of $(x_1, \ldots, x_\nu)$, i.e., $\frac{\partial}{\partial x_i} \omega_{\alpha_1 \cdots \alpha_r} = 0$. The set $\Omega_B(L) \subset \Omega(M)$ is a subcomplex since the exterior derivative preserves the basic forms $d_B: \Omega_B(L) \to \Omega_B(L)$. Introduce the notation $d_B: \Omega_B(L) \to \Omega_B(L)$. By definition, the basic cohomology is $H^*_B(L) = H^*(\Omega_B(L), \delta_B)$. It plays the role of the de Rham cohomology of the leaf space of a foliated manifold. For the case of a foliation of codimension 1, there are just two groups, $H^0_B(L)$ and $H^1_B(L)$.

The transverse orientation of the foliation $\{L\}$ is defined by the orientation of $TL^\perp$. The transversal volume form $\chi^\perp$ on $TL^\perp$ of a Riemannian foliation is holonomy invariant, i.e.,

$$\Theta(X)\chi^\perp = 0, \quad X \in TL.$$

The holonomy invariance condition shows us that $\chi^\perp \in \Omega_B^{\dim L}(L)$. Consequently, $d\chi^\perp = 0$. It is of interest to examine the cohomology class $[\chi^\perp] \in H^0_B^{\dim L}(L)$, which plays the role of the orientation class for the leaf space of a foliation.

**Theorem 2.3** [287]. Let $\{L\}$ be a transversally oriented foliation on a closed oriented manifold $M$ satisfying one of the following conditions: (1) $\{L\}$ is a minimal foliation with holonomy-invariant transversal-volume form $\chi^\perp$; (2) $\{L\}$ is a taut Riemannian foliation. Then $[\chi^\perp] \neq 0$ in $H^0_B^{\dim L}(L)$. 

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The mean-curvature form (1-form on $TM$) of the foliation is given by
\[ k(y) = g^\perp(H, y), \quad y \in TL^\perp, \]
where $H \subset TL^\perp$ is a mean-curvature vector field. By construction, $i(x)k = 0$ for all $x \in TL$. Furthermore, for every Riemannian foliation on a closed manifold, there is a well-defined cohomology class $[k_B] \in H^1_B(L)$ (there is a bundle-like metric such that $k$ is a basic one-form; see [286]) whose vanishing characterizes the tautness of the foliation. For a transversally orientable Riemannian foliation $\{L\}$ on a closed orientable Riemannian manifold $M$, the following three conditions are equivalent (see [289]):

1. $\{L\}$ is taut;
2. $H^\text{codim}_B(L) \cong \mathbb{R}$;
3. $[k] = 0$.

The sufficient conditions for the tautness of a transversally orientable Riemannian foliation of codimension $q \geq 2$ on a closed orientable manifold can be expressed in terms of the transversal Ricci operator $\tilde{\text{Ric}}: TL^\perp \to TL^\perp$ and the transversal curvature operator $\tilde{\text{R}}: \Lambda^2(TL^\perp)^* \to \Lambda^2(TL^\perp)^*$, by virtue of the following vanishing results (see [286]):

1. If $\tilde{\text{Ric}} > 0$, then $H^\text{codim}_B(L) = 0$,
2. If $\tilde{\text{R}} > 0$, then $H^\text{codim}_B(L) = 0$ for $0 < r < q$.

An infinitesimal automorphism $V$ (i.e., the flow of $V$ maps leaves into leaves) acts on $s \subset TL^\perp$ as follows:

\[ \Theta(V)s = [V, Y^s]. \]  

(2.26)

where $Y^s \subset TM$ is such that $(Y^s)^s = s$. The right-hand side of Eq. (2.26) does not depend on the choice of $Y^s$. Let $V(L)$ be the set of all infinitesimal automorphisms of the foliation $\{L\}$, and let $Y \in V(L)$ be an infinitesimal automorphism of $\{L\}$. The transversal divergence $\text{div}_B Y$ is defined by

\[ \Theta(Y)^\perp = \text{div}_B Y \cdot \chi^\perp. \]

Here we use the fact that $\Theta(Y)^\perp \in \Omega^\text{codim}_B(L) \subset \Lambda^\text{codim}_B(TL^\perp)^*$. Observe that $\text{div}_B Y \in \Omega^0_B(L)$, and in fact, depends only on $Y^\perp$.

By using the Stokes theorem, one can prove the transversal divergence theorems for harmonic foliations with holonomy-invariant transversal-volume form and for Riemannian foliations.

**Theorem 2.4** ([148] and [287]). Let $\{L\}$ be a transversally orientable minimal foliation with holonomy invariant transversal volume form $\chi^\perp$ on a closed Riemannian manifold $(M, g)$, and let $Y$ be an infinitesimal automorphism of $\{L\}$. Then

\[ \int_M \text{div}_B(V^\perp)\, d\text{vol} = 0. \]

The value of this integral in the case of a Riemannian (not necessarily minimal) foliation is given in the following theorem.

**Theorem 2.5** [287]. Let $\{L\}$ be a transversally orientable Riemannian foliation on a closed orientable manifold $(M, g)$, and let $V \subset V(L)$. Then

\[ \int_M \text{div}_B(V^\perp)\, d\text{vol} = \int_M g^\perp(H, V^\perp)\, d\text{vol} \equiv (H, V^\perp)^\perp, \]

(2.27)

where $\perp$ is the global inner product of the sections $H$ and $V^\perp$ in $TL^\perp$.

### 2.5.2. Transversal Killing fields
An infinitesimal automorphism $V \in V(L)$ with the property $\Theta(V)g^\perp = 0$ is said to be transversally metric. If this holds, $V^\perp$ is called a transversal Killing field [287]. For the point foliation (with $\dim L = 0$), this is the usual definition of a Killing vector field. The (transversal) Jacobi operator $\tilde{\mathcal{J}} = \tilde{\Delta} - \tilde{\text{Ric}} : TL^\perp \to TL^\perp$, where $\tilde{\Delta}$ is a transversal Laplacian is also associated with $\tilde{\nabla}$ (see [192]). The condition $\tilde{\mathcal{J}}(Y) = 0$ defines (transversal) Jacobi fields.
The transversal divergence Theorem 2.4 is a key one for the following result.

**Theorem 2.6 [148].** Let \( \{L\} \) be a transversally orientable minimal foliation on a compact Riemannian manifold \((M, g)\) with a bundle-like metric, and let \( Y \) be an infinitesimal automorphism of \( \{L\} \). Then the following properties are equivalent:

1. \( Y \) is a transversal Killing field, i.e., \( \Theta(Y)g^\perp = 0 \),
2. \( Y \) is a transversally divergence-free Jacobi field,
3. \( Y \) is transversally affine, i.e., \( \Theta(Y)\nabla = 0 \).

Moreover, if \( \text{codim } L = 2 \), then the following properties are equivalent:

4. \( Y \) is a transversal conformal field, i.e., \( \Theta(Y)g = \lambda \cdot g^\perp \),
5. \( Y \) is a transversal Jacobi field.

By this theorem, the linear space of transversal Jacobi automorphisms of the (harmonic) Hopf fibration \( p: S^3 \to S^2 \) is isomorphic to the linear space of infinitesimal conformal fields on \( S^2 \); in particular, this space is 6-dimensional.

The operator \( A_Y: TL^\perp \to TL^\perp \) for \( Y \in TL^\perp \) is defined by the law \( A_Y(Z) = -\nabla_Z Y \), where \( Z \in TM \) with \( Z^\perp = \tilde{Z} \) [320]. The following theorem generalizes the results for Killing fields on a Riemannian manifold \( M \) by Kostant and Currás–Bosch.

**Theorem 2.7 [320].** Let \( \{L\} \) be a minimal foliation on a connected orientable complete Riemannian manifold \((M, g)\) with a bundle-like metric. Let \( Y \) be a transversal Killing field with a finite global norm. Then, for each \( m \in M \), \( (A_Y)_m \) belongs to the Lie algebra of the linear holonomy group \( \Psi_T(m) \), where \( \nabla \) is the transversal Riemannian connection of \( \{L\} \).

If, in Theorem 2.7, the transversal Ricci operator \( \tilde{\text{Ric}} \) of \( \{L\} \) is nonpositive everywhere and negative for at least one point of \( M \), then every transversal Killing field with a finite global norm is trivial [321]. Note that any Killing field of a bounded length (for instance, on a compact manifold) preserves a codimension-1 totally geodesic foliation [207, 208].

### 2.5.3. Transversally symmetric Riemannian foliations.

There are Riemannian foliations whose transversal geometry can be locally modeled on a Riemannian symmetric space. A transversal symmetry can be characterized by conditions imposed on the canonical Levi-Civita connection on a normal bundle.

**Theorem 2.8 [287].** Let \( \{L\} \) be a Riemannian foliation on \((M, g)\), and let \( g \) be a bundle-like metric. Then the following conditions are equivalent:

1. \( \{L\} \) is transversally symmetric,
2. local geodesic symmetries (geodesic reflections) on a model space are isometries,
3. \( \nabla_z R(z, y, z, y) = 0 \), \( z, y \in TL^\perp \),
4. \( \nabla_z R(z, y, z, y) + 2R(z, O_2y, z, y) = -6((\nabla_z O)_2y, O_2y), \quad z, y \in TL^\perp \).

All these conditions are purely local and are automatically satisfied for a Riemannian codimension-1 foliation. For a totally geodesic Riemannian foliation, this characterization can be improved in the analytical case with the use of the following results (see [286, 287]): reflections in the leaves are isometric if and only if the geodesic reflections on the model space are (local) isometries.

In a space of constant curvature, the reflections in totally geodesic submanifolds are isometries. Hence, from the results mentioned above, we obtain the following theorem.

**Theorem 2.9 [287].** Let \( \{L\} \) be a Riemannian foliation on a space \((M, g)\) of constant curvature, and let \( g \) be a bundle-like metric. Then \( \{L\} \) is transversally symmetric if and only if

\[
(O_zz, T(O_zz, y)) = 0, \quad y, z \in TL^\perp. \tag{2.28}
\]
Definition 2.9 [75]. The union $F_\alpha$ of two half-horocycles with opposite normals at a common point is called a broken horocycle. The trajectory $D_\alpha$ (containing $x \in l$) that is orthogonal to the family $N$ of all straight lines intersecting the $l$-axis at a given angle $\alpha$ is called the fifth line. (Recall that a straight line, a circle, a horocycle, and an equidistant are the first four lines in $H^2$.)

Theorem 2.11 [75]. There exist exactly three nontrivial metric fibrations (i.e., fibrations with fibers different from one-point fibers and from the whole plane) of $H^2$ with connected fibers: (1) a fibration into horocycles, (2) a fibration into broken horocycles, (3) a fibration into the fifth lines.

2.6. Large-Sphere Foliations

Consider a round sphere $S^3 = \{x \in \mathbb{R}^4: |x| = 1\}$ in a Euclidean space $\mathbb{R}^4$ with an orthonormal basis $\{e_i\}$. A 2-dimensional subspace $\sigma$ of $\mathbb{R}^4$ with an orthonormal basis $\{a, b\}$ intersects $S^3$ along the large circle (geodesic) $\gamma(t) = a \cos t + b \sin t$, which plays the role of a straight line in a sphere. Two large circles in $S^3$ may not intersect, but are always engaged. For every large circle $\gamma \subset S^3$ and every point $x_0 \in S^3 \setminus \gamma$ at a distance $\operatorname{dist}(x_0, \gamma) < \pi/2$, there exist exactly two large circles $\gamma_1, \gamma_2 \subset S^3$ that are equidistant to $\gamma$ (Clifford parallelness), i.e., for all $x \in \gamma_1 \cup \gamma_2$, we have $\operatorname{dist}(x, \gamma) = \operatorname{dist}(x_0, \gamma)$. The set $\gamma^\perp = \{x \in S^3: \operatorname{dist}(x, \gamma) = \pi/2\}$ of points that are at the maximal distance from $\gamma$ is also a large circle; it is orthogonal to $\gamma$ (see Fig. 13). In particular, a surface that is equidistant to $\gamma$, $T(\gamma, s) = \{x \in S^3: \operatorname{dist}(x, \gamma) = s\}$ with $0 < s < \pi/2$, has a locally Euclidean metric and two (orthogonal for $s = \pi/4$) foliations by large circles; $T(\gamma, \pi/4)$, is called the Clifford torus [326]. Each of two continuous partitions of $S^3$ by large circles that are Clifford parallel to the given $\gamma$ are called Hopf fibrations; their base is $S^2$.

The Hopf fibration introduced in [136–137] about sixty years ago had a very considerable effect in topology, since it provided the first example of a homotopically nontrivial map from one sphere to another of lower dimension. It also had a great effect in geometry because the fibers are equidistant geodesics and form a Riemannian foliation. The Hopf fibration can be defined in several equivalent ways.

1. The Hopf fibration $p: S^3 \to S^2 \cong CP^1$ for complex variables $z_1 = x_1 + ix_2, \ z_2 = x_3 + ix_4$ in \( \{C^2 = C^1 \times C^1\} \cap S^3 \) has the form

$$p(z_1, z_2) = \frac{|z_1|^2 - |z_2|^2}{2 \Re(z_1 \bar{z}_2), \ 2 \Im(z_1 \bar{z}_2)};$$

for real variables this $p: (x_1, x_2, x_3, x_4) \to (y_1, y_2, y_3)$ has the form

$$y_1 = x_1^2 + x_2^2 - x_3^2 - x_4^2, \ y_2 = 2(x_1x_3 + x_2x_4), \ y_3 = 2(x_2x_3 - x_1x_4).$$

2. We introduce a complex variable $z$ on $S^2$ via the stereographic projection: $z = \frac{y_1 + iy_2}{1 - y_3}$. Then $p$ can be written in a more compact form $z = \frac{z_1 + iz_2}{z_1 + iz_2}$ as an ordinary map from the unit sphere in $C^2 = \mathbb{R}^4$ into the complex projective line $CP^1 = S^2$ [104].

3. Consider the action $\varphi(t) = (\exp(2\pi i t)z_1, \exp(2\pi i t)z_2)$ of the group $\mathbb{R}^1$ on $S^3$. The orbits of this foliated action form a Hopf fibration.

4. The Hopf fibration can be given as a collection of intersections of $S^3$ with all holomorphic 2-planes \( \{a = x \wedge Jx\} \), where $J$ is any complex structure in $\mathbb{R}^4$, i.e., a linear operator given in some orthonormal basis $\{e_i\}$ by the rule

$$Je_1 = e_2, \ Je_2 = -e_1, \ Je_3 = e_4, \ Je_4 = -e_3.$$ 

Note that the multiplication by $J$ gives us an orientation of fibers.

The projective space $KP^n (K = \mathbb{R}, \mathbb{C}, \mathbb{H})$ can be introduced as a set of orbits for the right action of the group $K^* = K \setminus \{0\}$ on $K^{n+1} \setminus \{0\}$, i.e., $x \sim y \iff$ there exists a $\lambda \in K^*$ such that $x = y\lambda$. This quotient space is not defined for a (nonassociative) Cayley algebra $Ca$. For $K = \mathbb{C}, \mathbb{H}$, the orbits are equidistant (dim $K - 1$)-dimensional large spheres. We set the distance between the points on $KP^n$ equal to the distance between the corresponding orbits on a round sphere $S^n \dim K - 1$ and obtain a canonical metric on the projective spaces.
Condition (2.28) is satisfied both in the case of a totally geodesic foliation (i.e., tensor $T = 0$) and in the case of a Riemannian foliation with an integrable normal bundle (i.e., tensor $\mathcal{O} = 0$); therefore, these foliations are necessarily transversally symmetric. Conversely, the existence of a transversally symmetric foliation imposes strong constraints on the geometry of the foliated space. A typical result is that the transversal symmetry of a foliation defined by a Killing vector field of unit length on a complete simply connected $(M, g)$ implies that $(M, g)$ is a naturally reductive homogeneous space (see [298]). A Riemannian foliation can be compared with transversally symmetric foliations; for results concerning the case of a small basic mean curvature, see [285].

The following classes of Riemannian foliations defined in terms of the eigenvalues and eigenspaces of a transversal Jacobi operator generalize the notions of $\varepsilon$-space and $\Psi$-space introduced by J. Berndt and L. Vanhecke in 1992.

**Definition 2.7** [288]. A Riemannian foliation $\{\mathcal{L}\}$ on $(M, g)$ is a transversal $\varepsilon$-foliation if the eigenvalues of the Jacobi operator $\hat{R}(\gamma', \gamma')$ are constant along each geodesic $\gamma$ that is orthogonal to the leaves of the foliation. $\{\mathcal{L}\}$ is a transversal $\varepsilon$-foliation if the eigenspaces of $\hat{R}(\gamma', \gamma')$ can be spanned by parallel fields of eigenvectors along each geodesic $\gamma$ orthogonal to the leaves of the foliation.

To give examples of transversal $\varepsilon$- and $\Psi$-foliations, it suffices to consider warped products $B \times_f F$ over a $\varepsilon$- or $\Psi$-space $B$. These concepts are used for a new characterization of transversally symmetric foliations.

**Theorem 2.10** [288]. The foliation $\{\mathcal{L}\}$ is transversally symmetric if and only if it is simultaneously a transversally $\varepsilon$- and a transversally $\Psi$-foliation.

The following interesting subclass of transversally symmetric (Riemannian) foliations was introduced and studied in [111–114].

**Definition 2.8.** Locally Killing-transversally symmetric spaces (locally KTS-spaces) are Riemannian manifolds $(M, g)$ endowed with the isometric flow generated by the unit Killing vector field such that the local reflections with respect to the flow lines are isometries.

These foliated spaces were studied in [111–114] and in [110] with the use of the extrinsic and the intrinsic geometry of geodesic spheres and tubular hypersurfaces around flow lines and around geodesics orthogonal to flow lines.

### 2.5.4. Riemannian foliations on space forms.

For $k > 0$, Riemannian submersions from spheres $S^n(k)$ are Hopf fibrations, except for, possibly, the case where $\dim M = 15$ and $\dim L = 7$ (see [82–83, 221–222] and [120, 121]). The class of Riemannian foliations on the spheres $S^n(k)$ is larger than the class of Hopf fibrations (see Sec. 2.6), and for $\dim L \leq 3$, consists only of homogeneous foliations (i.e., foliations generated by an isometric action of a Lie group) [121]. There are no homogeneous 7-dimensional foliations on the spheres $S^n(k)$ [174].

For $k = 0$, a Riemannian submersion of the complete $M(0)$ is the projection of the direct product onto one of its factors (see [304]). Riemannian foliations of a compact space form $M(0)$ split (are locally projections of the direct product onto one of its factors); without compactness, this statement is false even for $\mathbb{R}^2 \setminus \{O\}$: consider, for example, a 1-dimensional foliation generated by glide rotations.

For $k < 0$, there are no Riemannian foliations (and submersions) on the compact space form $M(k)$; the same is true for compact locally symmetric spaces $M$ of negative curvature (see [304]). In contrast, there exist Riemannian submersions from noncompact $\mathbb{H}^n(k)$ with various types of leaves (see [87, 47, 115]).

One-dimensional Riemannian foliations of Einstein spaces are studied in [212].

The concept of a metric fibration as a partition of a metric space to isometric and locally equidistant subsets is studied in [75]. For Riemannian manifolds, this leads one to a special case of Riemannian submersions. The only metric fibrations of $\mathbb{R}^n$ with connected fibers are the unions of parallel $k$-planes ($k = 0, 1, \ldots, n$) [75]. The hyperbolic geometry is much more interesting from this point of view.
$CP^n$ and $HP^n$. In particular, this canonical metric on $CP^1$ transforms it into a round two-sphere of radius $1/2$. Generalized Hopf fibrations (see [106]) and a similar complex Hopf fibration [81]

$$S^1 \subset S^{2n-1} \rightarrow CP^{n-1}, \quad S^3 \subset S^{4n-1} \rightarrow HP^{n-1}, \quad S^7 \subset S^{15} \rightarrow S^8,$$

are Riemannian submersions. Moreover, the fibrations in (2.29), except, perhaps, for one (see Sec. 2.5.4), are unique Riemannian submersions of Euclidean spheres [121]. On a round sphere, the foliation by geodesics is like a large-circle fibration. The simplest of them, a generalized Hopf fibration (2.29a) with fibers $\{S^1\}$, can be given as a collection of intersections of $S^{2n-1}$ with all holomorphic 2-planes $\{\sigma = x \wedge Jx\}$, where $J$ is a complex structure, i.e., the linear operator in $\mathbb{R}^{2n}$ given for some orthonormal basis $\{e_i\}$ by the rule

$$Je_{2i-1} = e_{2i}, \quad Je_{2i} = -e_{2i-1} \quad (1 \leq i \leq n).$$

(2.30)

Let $\mathcal{F}_0(S^{2n-1})$ denote the space of all Hopf fibrations of the sphere $S^{2n-1}$. Each fiber spans the corresponding oriented 2-plane through the origin in $\mathbb{R}^n$, and hence, defines a point in the Grassmannian manifold $G(2, 2n)$.

**Definition 2.10** ([104] for $n = 2$). The skew-Hopf fibration is given by the intersections of $S^{2n-1}$ with all holomorphic 2-planes $\{\sigma = x \wedge Jx\}$, where $J$ is an almost-complex structure, i.e., the linear operator on $\mathbb{R}^{2n}$ given by the rule (2.30) for some affine basis $\{e_i\}$. In other words, the skew-Hopf fibration can be obtained from the Hopf fibration as a result of a nondegenerate linear transformation of $\mathbb{R}^{2n}$ and then by the central projection of the images of fibers back to $S^{2n-1}$.

Let $\mathcal{F}_1(S^{2n-1})$ be the space of all skew-Hopf fibrations of the sphere $S^{2n-1}$. Let $\mathcal{F}(S^{2n-1})$ be the space of all oriented large-circle fibrations of $S^{2n-1}$.

For simplicity, below we will give some facts concerning geodesic foliations on a 3-sphere. The space $\mathcal{F}_0(S^3)$ is 2-dimensional and homeomorphic to a pair of disjoint 2-spheres. The 8-dimensional space $\mathcal{F}_1(S^3)$ is the disjoint union of two copies of $S^2 \times \mathbb{R}^6$. The following two indicated spaces are homogeneous [104]:

$$\mathcal{F}_0(S^3) = O(4)/U(2), \quad \mathcal{F}_1(S^3) = GL(4, \mathbb{R})/GL(2, \mathbb{C}).$$

(2.31)

The space $\mathcal{F}(S^3)$ is infinite-dimensional. Let $V$ be the Hopf unit vector field, and let $D^2$ be a small ball transversal to $V$ at a point $p$. Small $C^1$-perturbations of $V$ on $D^2$ that are identical in a neighborhood of the boundary $\partial D^2$ lead us to different large-circle foliations of $S^3$ [104]. There also exist discontinuous fillings of $S^3$ by large circles: one can fill the closed solid torus $x_1^2 + x_2^2 \geq x_3^2 + x_4^2$ on $S^3$ as for the Hopf fibration and then fill the remaining open solid torus as for the Hopf fibration defined by the second continuous partition of $S^3$ by large circles. Using the isometry $G(2, 4) \equiv S^2 \times S^2$, the Hopf fibration $h \in \mathcal{F}_0(S^3)$ can be characterized by the fact that its orbit space $M_h$ appears inside the Grassmannian as $\{\text{point}\} \times S^2$ or $S^2 \times \{\text{point}\}$. However, not all submanifolds of $G(2, 4)$ can serve as $M_f$ for $f \in \mathcal{F}(S^3)$.

**Theorem 2.12** [104]. A submanifold in $G(2, 4) \equiv S^2 \times S^2$ corresponds to the fibration $f \in \mathcal{F}(S^3)$ if and only if it is the graph of a distance-nonincreasing map $\tilde{f}$ from either of the spheres $S^3$ (a factor in the Grassmannian).
to the other. The fibration $f$ is differentiable if and only if the corresponding map $\tilde{f}$ is differentiable with $|df| \leq 1$.

Hence the space $\mathcal{F}_0(S^3)$ is a deformation retract of the space $\mathcal{F}(S^3)$. The catalogue of large-circle fibrations of the 3-sphere (in Theorem 2.12) is one of the first nontrivial examples in which one has a clear overview of all possible geodesic foliations of a fixed Riemannian manifold. The structure of the space of skew-Hopf fibrations $\mathcal{F}_1(S^3)$ is studied in [94] and [237].

The interest in fibrations of an $n$-sphere by large $\nu$-spheres is due to the Blaschke problem and by extremal theorems in Riemannian geometry (see [82, 83, 94, 104–108, 120–122, 124, 136–137, 144, 174, 237, 244, 256, 265–266, 295, 318]).

A compact Riemannian manifold $M$ is called a C*-manifold if all its geodesics are closed and are of length $\pi$. This class includes Blaschke manifolds for which, by definition, all cut loci $\text{Cut}(p) \subset T_pM$, $p \in M$, are round spheres of constant radius and dimension. Examples are CROSS: spheres or projective spaces. The Blaschke conjecture claims that any Blaschke manifold is isometric to its model CROSS. For a Blaschke manifold, the exponential map $\exp_p: T_pM \to C(p)$, being restricted on the sphere $S_p$ with radius $d(p, C(p))$, defines a large-sphere foliation; for CROSS, this foliation is a Hopf fibration. Since every large-sphere foliation of $S^N$ is homeomorphic to the Hopf fibration [266] (for some special cases, see [105]; some results for differential fibration by large spheres can be found in [124]), a simply connected Blaschke manifold is homeomorphic to its model CROSS.

In cases where the extremal value for the curvature, diameter, or volume of a manifold is considered (under other given conditions), it is often possible to prove that the manifold is isometric to a model space from a finite list. A beautiful example of these extremal theorems is the following.

**Theorem 2.13** (on minimum diameter) [21]. Let $M$ be a complete connected, simply connected Riemannian manifold with sectional curvature $1 \leq K_M \leq 4$ and diameter $\frac{\pi}{2}$. Then $M$ is isometric to CROSS: a sphere of curvature 4 or one of the projective spaces $CP^\infty$, $HP^\infty$, and $CaP^2$, with its canonical metric.

Gluck et al. [107] gave a constructive proof of this theorem; by using Berger's geometric arguments, they showed that the exponential map from the tangent cut locus to the cut locus of a manifold is a fibration of a round sphere by parallel large spheres; hence, it is a Hopf fibration. Then they checked how this fibration encodes the curvature tensor and used it to construct an isometry between two round spheres or the projective space. Later on, the Berger theorem was generalized in several directions (see [21, 309, 122]). One of the key points in these results comes from the study of Riemannian foliations on a round sphere. For 1- and 3-dimensional leaves, they are always Hopf fibrations [120, 121]; a partial classification of Riemannian foliations on $S^15$ with 7-dimensional leaves is given in [309, 221, 222, 174].

Toponogov has studied $V_m(-\infty, 4)$, the set of complete simply connected Riemannian manifolds with sectional curvature $\leq 4$ and injectivity radius $\geq \pi/2$. Since the diameter of these manifolds is bounded from below, the case of the extremal value $\pi/2$ of the diameter is especially interesting.

**Theorem 2.14** [291, 292]. A manifold $M \in V^{2n+1}(-\infty, 4)$ with diameter $\pi/2$ and sectional curvature $\leq 4$ is isometric to a sphere of curvature 4. A manifold $M \in V^{2n}(-\infty, 4)$ with diameter $\pi/2$ is isometric to the sphere of curvature 4 or its geodesics are grouped into the following families (as for projective spaces):

(F1) for every point $p \in M$ and any vector $\lambda \in T_pM$, there exists an a-dimensional ($a = 2, 4,$ or 8, and if $a = 8,$ then $\dim M = 16$) subspace $d(\lambda) \subset T_pM$ such that all geodesics $\gamma \subset M$ ($\gamma(0) = p$, $\gamma'(0) \in d(\lambda)$), form a totally geodesic submanifold $F(p, \lambda)$, which is isometric to the sphere $S^a(4)$;

(F2) for all nonzero vectors $\lambda_1, \lambda_2 \in T_pM$, the submanifolds $F(p, \lambda_1)$ and $F(p, \lambda_2)$ either coincide or their intersection consists of only one point $p$.

Manifolds with properties (F1) and (F2) constitute a special case of Blaschke manifolds. Toponogov [54–56] conjectured that the manifolds in $V^{2n}(-\infty, 4)$ with extremal diameter equal to $\pi/2$ are isometric to CROSS.
Note that tangent \( a \)-planes to the submanifolds \( \{F(p, \cdot)\} \) (in Theorem 2.14) induce an \((a-1)\)-dimensional large-sphere foliation of the round sphere \( S_p \) in the tangent space \( T_p M \). In the case \( K_M > 0 \), for almost every point \( p \in M \), such a fibration \( f_p \) is related to the class \( \mathcal{F}_R \) of analytic geodesic fibrations on \( \text{CROSS} \) by means of the function of sectional curvature in \( T_p M \) [290–292]. For simplicity, it is defined below for the sphere \( S^{2n-1} \).

**Definition 2.11** [290]. The fibration \( f \in \mathcal{F}(S^{2n-1}) \) belongs to the class \( \mathcal{F}_R(S^{2n-1}) \) if there exists a multilinear function ("curvature tensor") \( R: \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R} \rightarrow \mathbb{R} \) with the following properties:

- (R₁) symmetries and the first Bianchi identity:
  \[ R(x, y, z, w) = R(z, w, x, y) = -R(x, y, w, z), \]
  \[ R(x, y, z, w) + R(z, w, x, y) + R(y, z, x, w) = 0, \]

- (R₂) for almost every vector \( x \in S^{2n-1} \), there exists a unique 2-plane \( \sigma \ni x \) with \( R(x, y, x, y) = 1 \) \( \forall y \in \sigma \), \( y \perp x \), \( |y| = 1 \),

- (R₃) if a 2-plane \( \sigma \) contains a fiber of \( f \), then \( R(x, y, x, y) = 1 \) for all \( x, y \in \sigma \) \( y \perp x \), \( |x| = |y| = 1 \), i.e., the sectional curvature of \( R \) is extremal on \( \sigma \).

The surprising fact is that \( \mathcal{F}_R(S^{2n-1}) = \mathcal{F}_1(S^{2n-1}) \), since \( \mathcal{F}_R(S^{2n-1}) = \mathcal{F}_0(S^{2n-1}) \) was expected (see [237, 244]).

The natural strategy to prove that a Riemannian manifold with properties (R₁)–(R₃) is isometric to \( \text{CROSS} \) (and hence, to deduce that the induced large-sphere foliations in the tangent spheres \( \{S_p\} \), \( p \in M \), are Hopf fibrations. This claim was conjectured in [54, Lemma 6] for a foliation with the properties (R₂) and (R₃) and the curvature symmetries (R₁). By considering a curvature tensor at only one point \( p \) of \( M \), one can only show that such a foliation on the tangent sphere \( S_p \), is skew-Hopf fibration (the proof of Lemma 6 in [290] is incorrect). Later, the Toponogov conjecture was proved in a different way [257], namely, by using the global integral geometric methods of Berger and Kazdan (see [23]) and recent topological results for manifolds with closed geodesics.

**Theorem 2.15** [257]. A Riemannian manifold \( M \in V^m(-\infty, 4) \) with extremal diameter \( \pi/2 \) is isometric to \( \text{CROSS} \).

We will use Theorem 2.15 below in Theorem 4.10 of Sec. 4.2.2. From Theorem 2.15, we obtain a generalization of Theorem 2.13 by Berger.

**Corollary 2.1** (on the diameter rigidity). A complete connected, simply connected Riemannian manifold \( M^{2n} \) with sectional curvature \( 0 < K_M \leq 4 \) and diameter \( \text{diam}(M) = \pi/2 \) is isometric to \( \text{CROSS} \).
Chapter 3
FOLIATIONS AND A MIXED CURVATURE

3.1. Foliations and a Curvature

The relations between curvature and topology are fundamental for Riemannian geometry “in the large” (see [96, 48, 156]), where two points of view can roughly be distinguished [96], namely
— the traditional point of view: by assuming some rather strong properties of the curvature (e.g., sign definiteness and pinching), we obtain sharp structure theorems,
— the rough point of view: under weaker assumptions we obtain, e.g., the finiteness results for topological-type manifolds.

Examples of this kind of result can be found in [215] and [300], respectively.

In 1979, Gluck posed the following question: what prevents a foliation on $M$ from being geodesible?

In the last few years, the interest of geometers in similar problems on the existence of adapted metrics and foliations (or distributions) on a manifold with additional curvature restrictions has become stronger.

FC1. Is there a complete Riemannian metric $g$ for a foliated manifold $M$, $\{L\}$ which induces given geometrical and curvature properties of a foliation?

FC2. Is there a foliation $\{L\}$ with given metric and curvature properties for a fixed complete Riemannian manifold $(M, g)$? If these foliations exist, then what is their classification?

There are three types of sectional curvature for a foliation, namely, tangential, mixed (denoted by $K_{\text{mix}}$; a plane that contains a tangent vector to the foliation and a vector orthogonal to the foliation is said to be mixed), and transversal.

We will concentrate our attention in this paper on a mixed curvature such as a sectional, Ricci, or scalar curvature (see [304, 302, 147, 324]) whose geometrical sense follows from the fact that for a totally geodesic foliation, certain components of the curvature tensor along any geodesic $\gamma \subset L$ are contained in the Jacobi equation $Y'' + R(t)Y = 0$ for the Jacobi tensor induced by the foliation and the Riccati equation $B' + B^2 + R(t) = 0$ for the structural tensor $B: TL \times TL^\perp \to TL^\perp$ defined by $B(x, y) = (\nabla_x y)^\perp, \ (x \in TL, y \in TL^\perp)$. The tensors $Y$ and $B$ are related as $Y' = B(\gamma', Y)$; see below. In the case of constant curvature, the solutions to the above Jacobi and Riccati equations (and therefore, the relative behavior of geodesics on nearby leaves) are well known (see Fig. 14).

For a given foliated manifold, a metric of constant tangential, mixed, or transversal curvature can be regarded as canonical. Examples of foliations with $K_{\text{mix}} = \text{const}$ are Hopf fibrations and the following foliations (see Sec. 2.4.1):

1. $k$-nullity foliations on a manifold with degenerate curvature tensor; in [123], these metrics are called partially hyperbolic, parabolic, or elliptic,

2. relative nullity foliations on a submanifold $M$ of the space form $\tilde{M}(k)$.

It is easy to find the following upper bound for the dimension of leaves of a compact totally geodesic foliation with $K_{\text{mix}} > 0$:

$$\dim L < \text{codim} L \quad (\text{i.e.,} \quad \dim L < \frac{1}{2} \dim M)$$

(for the generalization for a partial Ricci curvature, see Sec. 3.7).

The idea is that two compact totally geodesic submanifolds $L_1$ and $L_2$ with $\dim L_1 + \dim L_2 \geq \dim M$ in a Riemannian space $M$ of positive sectional curvature (for example, two large spheres in a round sphere) intersect each other. For dual results, for $K_{\text{mix}} = \text{const} < 0$, on a foliated manifold with finite volume, see [323, 324]. For $K_{\text{mix}} = \text{const} > 0$, Ferus proved the following theorem in 1970; it has a number of corollaries.

**Theorem 3.1** (local in transversal directions) [86] If, for a totally geodesic foliation $\{L\}$, $K_{\text{mix}} = k > 0$ along a complete leaf $L_0$, then $\dim L < \rho(\text{codim} L)$. (For the definition of $\rho(n)$, see Sec. 1.1.)

**Corollary 3.1.** Let $M^n$ be a complete submanifold in the sphere $S^{n+p}$. (1) If the index of relative nullity $\mu(M) > \nu(n)$, then $M^n$ is a totally geodesic submanifold [86]. (2) If the extrinsic sectional curvature is nonpositive and $2p < n - \nu(n)$, then $M^n$ is a totally geodesic submanifold [91].
The sequence \( u(n) \) is defined as \( u(n) = \max \{ t : t < \rho(n - t) \} \). Some values of \( u(n) \) are given by \( u(n) = n - (\text{higher power of 2} \leq n) \); for \( n \leq 24 \), in particular, \( u(n) \leq 8d - 1 \) for \( n < 16^d \), and \( u(2^d) = 0 \). For example, \( u(1) = 0 \), \( u(2) = 0 \), \( u(3) = 1 \), \( u(4) = 0 \), \( u(5) = 1 \), \( u(6) = 2 \), \( u(7) = 3 \), \( u(8) = 0 \), etc.

**Theorem 3.2** [235]. The estimate \( \dim L < \rho (\text{codim} L) \) in Theorem 3.1 is

(a) sharp: for all \( \nu < \rho(n) \), there exists a tube around a \( \nu \)-dimensional large sphere in the round sphere \( S^n \) with a foliation by \( \nu \)-dimensional large spheres (\( \dim L \leq 2 \) for the Kählerian case),

(b) false for nonconstant \( K_{\text{mix}} > 0 \): there exists a tube around a closed geodesic in any one of the projective spaces \( KP^n (n \geq 2; K = \mathbb{C}, \mathbb{H}, \mathbb{Ca}) \) which can be foliated by closed geodesics,

(c) true for ruled submanifolds \( M^{n+1}, \{ L^r \} \) with \( K_{\text{mix}} > 0 \) in the sphere \( S^{n}(k) \).

Thus, the following problems are natural: (1) when, for a given foliated smooth manifold \( M \), does there exist a complete geodesible metric of nonnegative (or positive) mixed curvature, (2) what is the structure of these foliations having large dimension, in particular, when do they split, (3) how can we effectively apply the technique of foliations to submanifolds with generators (ruled, parabolic, etc.) in a Riemannian space with nonnegative curvature. The first problem is interesting even locally for a foliation by closed geodesics, i.e., when \( \dim L = 1 \) and \( \text{codim} L = 2m + 1 \).

**1.44 Problem.** Let \( M = S^1 \times B^{2m+1} \) be the product of the circle and the odd-dimensional ball with the product foliation \( \{ S^1 \times \mathbb{R} \} \). For what maximal \( \delta \in [1/4, 1) \) does there exist a geodesible metric on \( M \) with positive and \( \delta \)-pinched \( \d_{\text{mix}} \)?

For examples with \( \delta \geq (\frac{m}{m+1})^2 \) \( (m \geq 2) \), see Theorem 3.2 (b), (3.25), and Sec. 3.4.2. Toponogov conjectured that for a totally geodesic foliation on a compact Riemannian manifold with \( K_{\text{mix}} > 0 \), the Ferus inequality \( \dim L < \rho (\text{codim} L) \) holds.

### 3.2. Jacobi Vector Fields and the Riccati Equation

The study of the second variation of the length or energy of a curve in a Riemannian manifold leads one to the Jacobi equation. Sometimes, the simple use of the second variation allows us to establish relationships between curvature properties and the structure of a manifold in general. We will give here some basic material from the theory of variations of geodesics; for more details, see [155, 48, 96].

For smooth normal vector fields \( y \) and \( z \) along a geodesic \( \gamma \) (i.e., \( y, z \perp \gamma \)), the *index bilinear form* is defined by the well-known formula

\[
I(y, z) = (y', z)_{\theta(\gamma)} - \int_0^{l(\gamma)} (z, y'' + R_{\gamma'}y) \, ds,
\]

obtained from the second variation of energy \( E(\gamma) = \frac{1}{2} \int_0^{l(\gamma)} |\gamma'(s)|^2 \, ds \) in the direction of \( y \) for the case where \( z = y \). Here \( R_{\gamma'} := R(\cdot, \gamma')\gamma' \) is the Jacobi operator (see, e.g., [22]) and \( y' := \nabla_{\gamma'} y, y'' := \nabla_{\gamma'} \nabla_{\gamma'} y \).

**Definition 3.1.** A vector field \( y \) along a geodesic \( \gamma \) in a Riemannian manifold \( M \) is called the *Jacobi vector field* if it satisfies the *Jacobi differential equation*

\[
y'' + R_{\gamma'} y = 0.
\]

With each rectangle \( \delta: Q \to M \), we associate two vector fields: \( x = d\delta(\frac{\partial}{\partial t}) \) and \( y = d\delta(\frac{\partial}{\partial \sigma}) \). A smooth variation (rectangle) \( \delta: Q \to M \) is called a *geodesic variation* if all \( \tau \)-edges \( \delta(t, \cdot) \) are geodesics (see Fig. 15).
The geometrical sense of a Jacobi vector field is as follows.

**Lemma 3.1** [96]. If $\delta: Q \to M$ is a geodesic variation, then $y = d\delta(\frac{\partial}{\partial x})$ is the Jacobi vector field along every $\tau$-edge. Conversely, every Jacobi vector field can be generated by some geodesic variation.

The tangent and the orthogonal component of the Jacobi vector field $y$ along the geodesic $\gamma$ are also Jacobi vector fields. The Jacobi vector fields orthogonal to the geodesic $\gamma$ are considered in many situations. A conjugate point and a focal point along a geodesic are defined in terms of Jacobi vector fields.

The Jacobi equation (3.3) in the space form $M(k)$ can be decomposed, and its solutions (Jacobi vector fields) can be written in the following simple form:

$$y(t) = \begin{cases} 
\cos \sqrt{k} t \bar{y} + \sin \sqrt{k} t \bar{z} & \text{for } k > 0 \\
\sqrt{k} t \bar{y} + \bar{z} & \text{for } k = 0 \\
\cosh \sqrt{-k} t \bar{y} + \sinh \sqrt{-k} t \bar{z} & \text{for } k < 0,
\end{cases}$$

where $\bar{y}, \bar{z}$ are parallel vector fields along $\gamma$. Consequently, conjugate and focal points are absent when $k \leq 0$.

There exist comparison theorems for the lengths of Jacobi vector fields when the curvature of one manifold is a majorant for the curvature of another manifold. In most cases, the model for comparison is a space of constant curvature (see [48]).

**Definition 3.2.** A tensor field $Y(t): V \to V$ is called the Jacobi tensor if

1. the Jacobi equation

$$Y'' + R_{\gamma} Y = 0 \quad (3.3)$$

holds;

2. $Y(t)$ is nondegenerate in the sense that $\ker Y(t) \cap \ker Y'(t) = \{0\}$ for all $t$ (this condition holds precisely when the action of $Y$ on linearly independent parallel sections of $\gamma^\perp$ gives rise to linearly independent Jacobi vector fields).

A Jacobi tensor $Y$ is called a Lagrange tensor if

3. the Wronskian $W(Y, Y) = (Y')^* Y - Y^* Y'$ is zero (which, in particular, is true when $Y(t) = 0$ for some $t$), or what is equivalent, if the tensor $B := Y'^* Y^{-1}$ is self-adjoint at all points where $Y$ is invertible.

For example, the following is a version of the Meyer theorem in terms of $n$-dimensional Jacobi tensors [81]. If $\operatorname{tr} R_{\gamma} \geq c > 0$ in (3.3) with $Y(0) = 0$, then the first positive value of $t$ at which $Y(t)$ is not invertible satisfies $t \leq \pi \sqrt{n}/c$ and equality occurs if and only if $R_{\gamma} \equiv (c/n) \mathbf{1}d$.

When $Y$ is invertible, we can set $B(t) := Y'(t)Y^{-1}(t)$. If we differentiate $B$ covariantly and substitute the result into Eq. (3.3), then we find that

$$B' + B^2 + R_{\gamma} = 0. \quad (3.4)$$

**Lemma 3.2.** Let the matrix-valued function $Y(t)$ ($\det Y(0) \neq 0$) and the matrix-valued function $B(t)$ satisfy the equation

$$Y'(t) = B(t)Y(t). \quad (3.5)$$

Then $Y(t)$ satisfies the Jacobi equation (3.3) for a certain matrix-valued function $R_{\gamma}$ if and only if $B(t)$ satisfies the matrix Riccati equation (3.4).
The invertible Jacobi tensor $Y$ is a Lagrange tensor if and only if the corresponding tensor $B$ is self-adjoint for some $t$. Moreover, every Lagrange tensor can be obtained by a normal geodesic variation of $\gamma$ along some hypersurface [81]. Important examples are Jacobi tensors, which vanish at some point of a geodesic, and stable Lagrange tensors (when they exist) with the condition $Y = \lim_{s \to \pm \infty} Y_s$, where the Jacobi tensors $Y_s$ satisfy the conditions $Y_s(0) = \text{Id}$ and $Y_s(-s) = 0$. In the latter case, $B(t)$ is the second fundamental form of the (stable) horosphere through $\gamma(t)$ and $\text{tr} B(t)$ is its mean curvature.

According to the decomposition $B = B^+ + B^-$, the matrix Riccati equation (3.3) can be polarized into symmetric and skew-symmetric parts:

$$\begin{cases}
(B^+)' + (B^+)^2 + (B^-)^2 + R(t) = 0, \\
(B^-)' + B^+ B^- + B^- B^+ = 0.
\end{cases}$$ (3.6)

The expansion $\theta$, the sheaf tensor $\sigma$, and the vorticity tensor $\omega$ can be calculated from $B$ as follows: $\theta := \text{tr}(B) = \text{tr} B^+$, $\sigma := B^+ - a \text{Id}$, $\omega = B^-$, where $a = \frac{1}{\dim \gamma^2}$ if $\gamma$ is a Riemannian or time-like geodesic and $a = \frac{1}{2 \dim \gamma^2}$ if $\gamma$ is null. Then, at points where $Y(t) \neq 0$, we have $\theta = \text{tr}(Y' Y^{-1}) = \frac{\text{det} Y'}{\text{det} Y}$. Note that $\sigma = 0$ exactly for any conformal operator $B$; moreover, $\sigma^2 \geq 0$, but $\omega^2 \leq 0$. For Lagrange tensors in Riemannian or Lorentzian manifolds, the parameter values $t$ for which $\text{det} Y(t) = 0$ are isolated [81].

From Eq. (3.6 (a)), it follows that $\theta$ satisfies the following Riccati equation (which is called the Raychaudhuri equation in relativity theory; see [165, 77]):

$$\theta' + a \theta^2 + \{\text{tr} R + \text{tr}(\sigma^2) + \text{tr}(\omega^2)\} = 0. \quad (3.7)$$

In particular, if $Y$ is a Lagrange tensor, then, since $\omega = 0$, $\theta$ satisfies

$$\theta' + a \theta^2 + \{\text{tr} R + \text{tr}(\sigma^2)\} = 0. \quad (3.8)$$

In global Riemannian geometry, it is often better to use the Riccati equation (or inequality) (3.4) than the Jacobi equation (3.3). In [149], a Riccati equation (or inequality) for the shape operator is employed to the study of local differential geometry of level sets of the local distance function. The volume comparison and extremal theorems (for balls, horospheres, and tubes) are also obtained by using the Riccati equation (see [119, 296, 80]); for the case of semi-Riemannian manifolds, see [7]. The Lagrange tensors and the Raychaudhuri equation are employed to the study of hypersurface geometry, geodesic congruences, and the boundary rigidity problem (see [81]).

**Lemma 3.3.** Let the Riccati matrix equation $B' + aB^2 + R(t) = 0$, where $a = \text{const} > 0$ and $R(t)$ is a symmetric continuous $(n \times n)$-matrix, admits a solution $B(t)$ defined for $-\infty < t < \infty$ with a symmetric initial value $B(0)$. Then

$$\liminf_{s \to +\infty} \int_{-s}^{s} \{\text{tr} R(t) + \text{tr}(\sigma^2)\} \, dt \leq 0 \quad (3.9)$$

and the equality is attained if and only if $\text{tr} B(t) = \text{tr} R(t) + \text{tr}(\sigma^2) \equiv 0$. Moreover, if $\text{tr} R(t) \geq 0$, then $B(t) = R(t) \equiv 0$.

The application of Lemma 3.3 to manifolds without conjugate points (when a stable Lagrange tensor exists along every geodesic) is given in [77]. Indeed, inequality (3.9) can be replaced by a more traditional inequality:

$$\liminf_{s \to +\infty} \int_{-s}^{s} \text{tr} R(t) \, dt \leq 0. \quad (3.10)$$

We cannot omit the requirement that $B(0)$ be a symmetric matrix in Lemma 3.3, since, for every $n > 1$, there exists a symmetric constant $(n \times n)$-matrix $R_n > 0$ such that the Riccati equation (3.4) with $R(t) = R_n$
has a continuous solution $B(t)$ for $t \in \mathbb{R}$. For example, if $n = 2m$, then $R_m = E_n$ and

$$B(t) = \begin{pmatrix} B_2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_2 \end{pmatrix}, \text{ where } B_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

(3.11)

this corresponds to the Hopf fibration $p: \mathbb{S}^{2m+1} \rightarrow \mathbb{C}P^m$ with the fiber $\{S^1\}$, and if $n = 2m + 1$, then $R_m = \{1, \frac{1}{2}, \ldots, \frac{1}{2}\}$ is a diagonal matrix, $B_2$ is as in (3.11), and

$$B(t) = \begin{pmatrix} \frac{1}{2}B_2 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \frac{1}{2}B_2 & 0 \\ 0 & \cdots & 0 & B_3 \end{pmatrix}, \text{ where } B_3 = \begin{pmatrix} \cos(t) & 2\sin(t) & 0 \\ 0 & \sin(\frac{1}{2}) & \cos(\frac{1}{2}) \\ \sin(\frac{1}{2}) & -\cos(\frac{1}{2}) & -\sin(\frac{1}{2}) \end{pmatrix}.$$

(3.12)

The Jacobi tensor $Y(t)$ along a geodesic $\gamma$ that corresponds to the matrix $B(t)$ in (3.12) (by virtue of the equations $Y' = B(t) \cdot Y$, $Y(0) = E_n$) can be induced by an $n$-parameter family of geodesics. Since this Jacobi tensor is nondegenerate and geodesics in $\mathbb{C}P^{m+1}$ are simple closed curves, this family forms a foliation near $\gamma$ (see Theorem 3.2 (b)). The solution (3.12) of the Riccati equation (3.4) allows us to infer that the result in [50] (Lemma 3.4 and, hence, Theorem 3.2 stating that Ferus' result is true for nonconstant $K_{\text{mix}} > 0$) is incorrect.

### 3.3. $L$-Parallel Vector Fields

**Definition 3.3** [118]. The horizontal connection $\tilde{\nabla}$ and the horizontal curvature operator $\tilde{R}$ of an almost-product structure are given by

$$\tilde{\nabla}_y z = P_2(\nabla_y z), \quad \tilde{R}(y, z)u = \tilde{\nabla}_{P_2[y, z]} u - \{\tilde{\nabla}_y, \tilde{\nabla}_z\}u, \quad (y, z, u \in T_2).$$

(3.13)

The vertical connection $\hat{\nabla}$ and the vertical curvature operator $\hat{R}$ can be defined in a similar way.

The operators $\tilde{R}$ and $\hat{R}$ satisfy all curvature symmetries and the Bianchi identities.

**Theorem 3.3** [118]. The following Gauss, Codazzi, and Ricci equations for two complementary orthogonal distributions $T_1$ and $T_2$ on $TM$ hold:

(a) $$(R(y, z)u, w) = (\tilde{R}(y, z)u, w) - (O_y u, O_z w) + (O_y w, O_z u) + (O_{yz}, O_{uw}),$$

(b) $$(R(y, z)w, x) = ((\nabla_y O)z w - (\nabla_z O)z w, x) + (O_y z - O_{zy}, T_z w),$$

(c) $$(R(y, x_1)z, x_2) = ((\nabla_{x_1} O)y z + O(y x_1, z), x_2) + ((\nabla y T)z x_2 + T(T_{x_1} y, x_2), z),$$

where $y, z, u, w \in T_2$ and $x_1, x_2 \in T_1$.

The Gauss and Codazzi equations have dual equations. The Ricci equation and the second Bianchi identity yield a formula for $(R(x_1, x_2) y, z)$. All of these equations, taken together, solve the problem of computing the curvature of the distributions $T_1$ and $T_2$ by using the curvature of $M$ as well as the problem of computing the curvature of $M$ through the curvature of the distributions $T_1$ and $T_2$. There are formulas for the covariant derivatives of $R$ corresponding to (3.14) [118], but we omit them.

**Lemma 3.4** [252] The following differential equation holds:

$$((\nabla_{x_1} B_2)(x_2, y, z) + (B_2(x_2, B_2(x_1, y), z) + (\nabla_y B_1)(z, x_1), x_2) + (B_1(z, B_1(y, x_1)), x_2) + (R(y, x_1)x_2, z) = 0.$$

(3.15)
Proof. We have

\[
((\nabla_y T)_{x_1, x_2, z}) = ((\nabla_y B_1)(z, x_2, x_1), T(T_{x_1, y, x_2}, z) = (B_1(z, B_1(y, x_1)), u)
\]

for the tensors \( T \) and \( B_1 \), and similarly, for the tensors \( O \) and \( B_2 \). Therefore, (3.15) follows from (3.14 (c)).

If \( \gamma: I \to M \) is a unit speed \( T_1 \)-geodesic, i.e., \( \gamma' \subset T_1 \) and \( \nabla_{\gamma'} \gamma' \subset T_2 \), then the endomorphism field \( B_2(t) = B_2(\gamma'(t), \cdot) \) of \( T_2 \) along \( \gamma \) satisfies the following Riccati differential equation:

\[
B_2' + B_2^2 + F(t) = 0, \quad (t \in I),
\]

where \((F(t)y, z) = (R(y, \gamma')\gamma', z) + ((\nabla_y B_1)(z, r'), r') + (B_1(z, B_1(y, r')), r')\).

In the case of a totally geodesic foliation, Eq. (3.15) is given in [73],

\[
(\nabla_{x_1} B_2)(x_2, y) + B_2(x_2, B_2(x_1, y)) + R(y, x_1)x_2 = 0,
\]

and the Riccati equation (3.16) takes the form

\[
B_2' + B_2^2 + R(t) = 0, \quad (Y(0) = \text{Id}).
\]

A \( T_1 \)-parallel vector field \( y \) is \( L \)-parallel if the distribution \( T_1 \subset TM \) is tangent to the foliation \( \{L\} \). (By analogy we can define a \( T_2 \)-parallel vector (tensor) field.)

Definition 3.4. A vector field \( y: \gamma \to T_2(\gamma) \) along a unit speed \( T_1 \)-geodesic \( \gamma \subset M \) is said to be \( T_1 \)-parallel if the following first-order ODE holds:

\[
P_2(\nabla_{\gamma'} y) = B_2(\gamma', y).
\]

Let \( \{e_i(0)\} \) be an orthonormal basis of \( T_2 \) at \( m \in \gamma(0) \). We continue it along \( \gamma \) as a solution to the differential equation \( P_2(\nabla_{\gamma'} e_i) = 0 \) and obtain an orthonormal frame field \( \{e_i\} \) in \( T_2 \) along \( \gamma \). Let \( \{y_i\} \) with \( y_i(0) = e_i(0) \) be \( T_1 \)-parallel vector fields along \( \gamma \). This yields the \( T_1 \)-parallel tensor (endomorphism) field \( Y: T_2(\gamma) \to T_2(\gamma) \) along \( \gamma \) given by \( y_i = Ye_i \). Clearly, \( P_2(\nabla_{\gamma'} Y) = P_2(\nabla_{\gamma'} Y)e_i \) and

\[
P_2(\nabla_{\gamma'} Y) = B_2(\gamma', Y) \quad (Y(0) = \text{Id}).
\]

Since the ODE (3.19) is homogeneous, it follows that a \( T_1 \)-parallel vector field has no zeros or is identically zero. For a foliation \( \{L\} \) (i.e., for \( TL = T_1 \)), \( L \)-parallel fields coincide with the horizontal basic vector fields \( \{y\} \) and are locally defined on the leaves (globally along the leaves with trivial holonomy or for a submersion \( \pi: M \to B \) with \( \ker \pi = T_1 \)) and satisfy the equation

\[
P_2(\nabla_{\gamma'} y) = B_2(\gamma', y), \quad x \in TL.
\]

For the totally geodesic foliation \( \{L\} \) (i.e., \( TL = T_1 \)), the concept of an \( L \)-parallel Jacobi field (i.e., induced by a foliation) is introduced in [73]. Let \( \{L\} \) be a totally geodesic foliation, and let \( \gamma \) be a geodesic of \( M \) tangent to \( L \). Then the foliation generates a \((\dim M - \dim L)\)-dimensional linear subspace \( J_{\gamma}(L) \) in the \((2 \dim M)\)-dimensional space of all Jacobi vector fields of \( M \) along \( \gamma \). This subspace can be parametrized by elements of \( T_{\gamma(0)} L^\perp \), and the initial values of these Jacobi vector fields are

\[
y(0) \in T_{\gamma(0)} L^\perp, \quad \nabla_{\gamma(0)} y = B_2(\gamma'(0), y(0)).
\]

Therefore, \( J_{\gamma}(L) \) coincides with the space of \( L \)-parallel vector fields along \( \gamma \).

The Jacobi vector fields \( y \) from \( J_{\gamma}(L) \) are generated by one-parameter families of geodesics of \( M \) each of which is a geodesic in a leaf \( L \). Note that a nontrivial Jacobi vector field \( y \) from \( J_{\gamma}(L) \) has no zeros, since the ODE is homogeneous (see (3.21)):

\[
\nabla_{\gamma'} y(t) = B_2(\gamma', y(t)).
\]

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This fact implies numerous global results about totally geodesic foliations (for instance, Corollary 3.1!). In particular, many authors prove the completeness of leaves of certain foliations on open submanifolds of complete manifolds in the case where the foliation is given by integral manifolds of the null spaces of certain tensor fields or differential forms (see Sec. 2.4.1).

For an unbilic or a spherical foliation, the variational interpretation of $L$-parallel vector fields is given in [223] and [135]. The case of an arbitrary foliation $\{L\}$ is considered in [303], where certain ergodic properties of a geodesic flow of foliation are studied. $L$-Jacobi fields occur as variation fields of a varying leaf geodesic $c$ among leaf geodesics. The second-order differential operator

$$J_c y = -y'' - R(y, c')c' + (\nabla_y h_1)(c', c') + 2h_1(P_1 y', c')$$

is similar to the operator $-P^2_2 \nabla_c (P_2 \nabla_c y) - \Delta_c y$ (see (3.24) below) acting on the space of vector fields along $c$. Of course, the $L$-Jacobi operator $J_c$ depends on the curvature of $M$ as well as on the second fundamental form of $\{L\}$. The operator $J_c$ plays the role of the second variation formula for the arc length $L$ and the energy $E$ of the leaf curves, but leaf geodesics appear as critical geodesics for $L$ and $E$ only for certain so-called admissible variations [309].

For the structural tensor $B_0(t)$ and the $T_1$-parallel vector field $y(t)$ (the tensor $Y(t)$) along a $T_1$-geodesic, we have Eqs. (3.19) and (3.16) (see Lemma 3.4). The “missing” Jacobi-type equation for these tensors can be obtained from the following lemma.

**Lemma 3.5** [251]. Let $M$ be a Riemannian manifold with two complementary orthogonal distributions $T_1$ and $T_2$ and let $y: \gamma \to T_2(\gamma)$ ($y(0) \neq 0$) be a $T_1$-parallel vector field along a $T_1$-geodesic $\gamma \subset M$. Then

$$P_2 \nabla_{\gamma'}(P_2 \nabla_{\gamma'} y) + \Delta_{\gamma'} y = 0,$$

(3.23)

where the tensor $\Delta_{\gamma'}: T_2(\gamma) \to T_2(\gamma)$ is given by the formula

$$(\Delta_{\gamma'} y, z) = (R(y, \gamma')\gamma', z) + ((\nabla_y B_1)(z, \gamma'), \gamma') + (B_1(z, B_1(y, \gamma'), \gamma').$$

(3.24)

For a geodesic distribution $T_1$ (in particular, for the case of a Riemannian foliation $\{L\}$ with $TL = T_2$, where the geodesic $\gamma$ is orthogonal to the leaves of the foliation), we have

$$(\Delta_{\gamma'} y, z) = (R(y, \gamma')\gamma', z) + (A_1(z, A_1(y, \gamma'), \gamma')$$

(3.25)

and the tensor $\Delta_{\gamma'}$ is symmetric. For a totally geodesic foliation $\{L\}$ with $TL = T_1$, $\Delta_{\gamma'}$ coincides with the Jacobi operator restricted to $T_2$.

For the geodesic distribution $T_1$ and any geodesic $\gamma \subset M$, ($\gamma' \subset T_1$), all $T_1$-parallel vector fields are Jacobi vector fields. Moreover, for the Riemannian foliation $\{L\}$ (with $TL = T_1$), all $T_2$-parallel vector fields can be induced by variations of $T_2$-geodesics [287] and [303].

**Theorem 3.4** (see [287]). Let $\gamma(t) \subset M$ be a geodesic tangent to a geodesic distribution $T_1$, and let the orthogonal distribution $T_2$ be integrable. Then the following conclusions hold:

(a) an ordinary Jacobi vector field $y$ along $\gamma$ is orthogonal to $T_1$ if and only if it satisfies the following initial conditions at $m = \gamma(0)$:

$$y(0) = y_0 \in T_2(m), \quad \nabla_{\gamma'(0)} y = B_2(\gamma'(0), y_0) + B_1(y_0, \gamma'(0));$$

(3.26)

(b) a $T_2$-parallel vector field $y$ along $\gamma$ is an ordinary Jacobi vector field orthogonal to $T_1$ if and only if it satisfies the following initial conditions at $m = \gamma(0)$:

$$y(0) = y_0 \in T_2(m), \quad P_2 \nabla_{\gamma'(0)} y = B_2(\gamma'(0), y_0).$$

(3.27)
3.4. Foliations by Geodesics with $K_{\text{mix}} > 0$

From the properties of the Riccati equation (see Lemma 3.3), it follows that a totally geodesic foliation with complete leaves satisfying $K_{\text{mix}} > 0$ has the nowhere integrable orthogonal distribution. Moreover, a totally geodesic foliation with complete leaves and integrable orthogonal distribution $TL^\perp$ satisfying $\text{Ric}_t \geq 0$ (i.e., the mixed Ricci curvature is nonnegative) splits locally ([1, 46] for $K_{\text{mix}} \geq 0$).

The known rigidity and splitting results for foliations of nonnegative curvature in mixed directions are not as systematic as the facts in the theory of Riemannian manifolds of nonnegative curvature; results for foliations usually contain strong additional assumptions, for example, the normal distribution is integrable or totally geodesic, the (co)dimension is equal to one, the curvature is constant, or the results are for a specific class of foliations such as $(U F, U F)$ or $(G F, F)$.

The proof of Theorem 3.1 with the best estimate $\dim L < \rho (\text{codim } L)$ for the dimension of a totally geodesic foliation with $K_{\text{mix}} > 0$ is based on the following scheme, which was used later by other geometers such as K. Abe, M. Magid, S. Tanno, P. Dombrowski, A. Borisenko, and the author.

3.4.1. Ferus scheme. 1. O'Neill and Stiel [206] have studied submanifolds with a positive index of relative nullity in space forms $\tilde{M}(k)$ and have used the following property of the matrix Riccati equation with $K_{\text{mix}} = k = const. $

\[ B' + B^2 + kE = 0 \]  

(3.28)

This is true for the structural tensor $B(t) = B(\gamma'(t), \cdot)$ of a relative nullity foliation $\{L\}$ along a geodesic $\gamma \subset L$ (see Sec. 2.4.1): the eigenvectors of the solution $B(t)$ do not depend on $t$; every eigenfunction $\lambda(t)$ satisfies the scalar Riccati equation $\lambda'(t) + \lambda^2(t) + k = 0$, and hence, for $k > 0$, a solution cannot be extended to the whole real line $-\infty < t < \infty$.

2. The key fact for the existence of an eigenvector for the structural tensor $B: TL \times TL^\perp \to TL^\perp$ of a totally geodesic foliation $\{L\}$ (see Lemma 3.6 below) is the following: in the case $\dim L \geq \text{codim } L$, for any point $m \in L$, there exist nonzero vectors $x \in T_mL$ and $y \in T_mL^\perp$ and also a real number $\lambda$ with the property $B(x, y) = \lambda y$.

The first property (see Eq. 3.28)) is not true if $kE$ is replaced by a nonconstant symmetric matrix $R(t) > 0$. Moreover, the inequality $\dim L < \text{codim } L$ without additional conditions is not true for geodesic foliations with $K_{\text{mix}} > 0$ (see Sec. 3.4.2).

As a rule, a linear operator defined on an even-dimensional vector space $\mathbb{R}^n$ has no eigenvectors. The situation changes drastically when we consider a $\nu$-parameter family of linear operators, and it becomes even more complicated when linear operators commute with an additional complex structure.

Lemma 3.6 [86]. Let $V_1$ and $V_2$ be real vector spaces of dimension $\dim V_1 = \nu$ and $\dim V_2 = n$, respectively, and let $B: V_1 \times V_2 \to V_2$ be a bilinear operator. If, for any nonzero vector $x \in V_1$, the endomorphism $B(x, \cdot): V_2 \to V_2$ has no real eigenvalues, then $\nu < \rho(n)$.

Proof. We can assume that $V_1$ and $V_2$ are Euclidean spaces. Let $\{e_i\} \subset V_1$ be an orthonormal basis. Then the $\nu$ continuous vector fields $w_i(y) = B(e_i, y) - (B(e_i, y), y)y$, $1 \leq i \leq \nu$, are tangent to the unit sphere $S^{n-1} \subset V_2$. If these vector fields are linearly dependent at some point $y \in S^{n-1}$, i.e., $\sum_i \lambda_i w_i(y) = 0$, then $B(\sum_i \lambda_i e_i, y) = [\sum_i \lambda_i (B(e_i, y), y)]y$; this is impossible by the hypothesis of the lemma. Therefore, $\nu < \rho(n)$.

Lemma 3.7 [1]. Let $J$ be a complex structure on real even-dimensional vector spaces $V_1$ and $V_2$. Let $B: V_1 \times V_2 \to V_2$ be a bilinear operator with the property

\[ B(Jx, y) = B(x, Jy) = JB(x, y). \]  

(3.29)

Then, for some nonzero vector $x \in V_1$, the endomorphism $B(x, \cdot): V_2 \to V_2$ has an eigenvector.

Lemmas 3.6 and 3.7 are used as estimates of the dimension of a foliation (or distribution) with $K_{\text{mix}} = \text{const} > 0$ ($B$ is a part of the structural tensor (2.5)), and also as estimates of the index of relative nullity of submanifolds in $S^N$ and $CP^N$. We complete these results with the following lemma.
Lemma 3.8 [242]. Let \( J \) be a complex structure on real vector spaces \( V_1 \) and \( V_2 \) of even dimension \( \dim V_1 = \nu \) and \( \dim V_2 = n \), respectively. Let \( B : V_1 \times V_2 \rightarrow V_2 \) be a bilinear operator with the property
\[
B(Jx, y) = JB(x, y).
\] (3.30)

If, for any nonzero vector \( x \in V_1 \), the endomorphism \( B(x, \cdot) : V_2 \rightarrow V_2 \) has no real eigenvalues, then \( \nu = 2 \) and \( n \) is divisible by 4.

Lemma 3.8 allows us to prove, for example, that (1) a complete Kählerian manifold \( M^{2n} \) with \( k \)-null index \((k > 0)\)
\[
\mu_k(M) > \begin{cases} 
2, & n \text{ is odd,} \\
0, & n \text{ is even,} 
\end{cases}
\]
is isometric to \( CP^n \), (2) there are no (Riemannian) submersions from \( CP^7 \) and \( HP^3 \); this is assumed in [83].

3.4.2. Foliations on closed geodesics with \( K_{\text{mix}} \equiv 1 \). The Hopf fibrations serve as an example of foliations of odd-dimensional Riemannian manifolds by closed geodesics \( \{L = S^1\} \) with \( K_{\text{mix}} = \text{const} > 0 \). In Sec. 3.1, the problem on the existence of an even-dimensional Riemannian manifold foliated by closed geodesics with \( (\delta \cong 1) \)-pinched positive \( K_{\text{mix}} \) is formulated.

Theorem 3.5 [253]. For any \( \delta \in (0, 1) \), there exists a Riemannian manifold \( M^{2n+2} \), where \( n \geq \frac{\sqrt{5}}{1-\sqrt{\delta}} \), with a fibration by closed geodesics \( \{L = S^1\} \) and with a positive \( \delta \)-pinched mixed sectional curvature.

We assume that Theorem 3.5 holds as \( \delta \to 1 \) when the dimension \( n \) is fixed. In proving Theorem 3.5, on \( M^{2n+2} \) we construct a metric with mixed curvature \((\frac{n}{n+1})^2 \leq K_{\text{mix}} \leq 1 \) and with the length of geodesics \( l(L) = 2\pi(n+1) \) tending to infinity as \( \delta \to 1 \). Can one obtain, for any \( \delta \in (0, 1) \), an example with the length \( l(L) \leq c \) (for instance, with \( c = 2\pi \)) under the standard assumption that \( \delta \leq K_{\text{mix}} \leq 1 \)?

Theorem 3.5 directly follows from Lemmas 3.9 and 3.10.

Lemma 3.9. Assume that a symmetric matrix \( R(t) \) and a nondegenerate matrix \( Y(t) \) are of order \( n \times n \) and are \( T \)-periodic and satisfy the Jacobi equation
\[
\dot{Y}(t) + R(t) \cdot Y(t) = 0, \quad (0 \leq t \leq T).
\]

Then there exists a Riemannian metric on the product \( M^{n+1} = S^1 \times B^n(r) \) of the circle \( S^1 \) by the \( n \)-dimensional ball \( B^n(r) \) of radius \( r \) with the following properties:

(a) the closed curves \( \{\gamma_z(t) = (t, Y(t)z)\}_{z \in B^n(r)} \) are geodesics;
(b) the components of the mixed curvature \( R(\cdot, \gamma_o) \gamma_o \) along \( \gamma_o(t) \) are expressed by the formula \( R(\gamma_z, \gamma_o) \gamma_o = R(t)z, \ z \perp \gamma_o; \)
(c) the Jacobi tensor of the foliation \( \{\gamma_z\} \) has the form \( Y(t) \) for some parallel orthonormal basis along \( \gamma_o \).

Lemma 3.10. For any \( n \in \mathbb{N} \), there exists a square matrix \( Z_{n,n+1}(t, s) \) (see Table 2) of order \( 2n + 1 \) with entries
\[
z_{jk} = \begin{cases} 
a_{jk} \cos(\frac{t}{n}) + b_{jk} \sin(\frac{t}{n}), & j \leq n, \\
a_{jk} \cos(\frac{s}{n+1}) + b_{jk} \sin(\frac{s}{n+1}), & n + 1 \leq j \leq 2n + 1, 
\end{cases}
\]
satisfying the condition \( \det Z_{n,n+1}(t, t) = \cos(t - s) \). In particular, the determinant of the matrix \( Y_{2n+1}(t) := Z_{n,n+1}(t, t) \) is identically equal to 1.

In fact, the matrix \( Z_{1,2}(t, s) \) with \( \delta = 0.25 \) is given in (3.12). Consider some more examples, where, for simplicity, we set \( C_n := \cos(\frac{1}{n}) \) and \( S_n := \sin(\frac{1}{n}) \). The matrix \( Y_n(t) := Z_{2,3}(t, t) \) satisfies the condition
\[ \delta = \frac{4}{9} \approx 0.44 \text{ and} \]

\[ Y_4(t) = \begin{pmatrix}
0 & S_3 & -C_3 & 0 & 0 \\
S_3 & C_3 & 0 & 3C_3 & 0 \\
C_3 & 3S_3 & 0 & S_3 & 0 \\
0 & 0 & -2S_2 & C_2 & S_2 \\
0 & 0 & 0 & 0 & S_2 & C_2
\end{pmatrix}. \]

The matrix \( Y_1(t) := Z_{3,4}(t, t) \) satisfies the condition \( \delta = \frac{9}{16} \approx 0.56 \) and

\[ Y_1(t) = \begin{pmatrix}
0 & S_3 & -C_3 & 0 & 0 & 0 \\
S_3 & C_3 & 0 & 3C_3 & 0 & 0 \\
C_3 & 3S_3 & 0 & S_3 & 0 & 0 \\
0 & 0 & 0 & C_4 & S_4 & -2S_4 & 0 \\
0 & 0 & 0 & C_4 & S_4 & -2S_4 & 0 \\
0 & 0 & 0 & C_4 & S_4 & -2S_4 & 0
\end{pmatrix}. \]

We do not give here the matrices \( Y_6(t) := Z_{4,5}(t, t) \) with \( \delta = \frac{16}{25} \approx 0.64 \), \( Y_{11}(t) := Z_{5,6}(t, t) \) with \( \delta = \frac{25}{36} \approx 0.7 \), etc. to save room [253]. Neither do we give here the matrices \( B_{2n+1}(t) = Y_{2n+1}(t)' \cdot (Y_{2n+1}(t))^{-1} \), where \( n = 2, 3, 4 \ldots \).

### 3.5. Splitting Foliations with Nonnegative Mixed Curvature

#### 3.5.1. The area of an \( L \)-parallel vector field.

The solution \( Y(t) \subset \mathbb{R}^n \) of the Jacobi matrix equation \( y'' + R(t)y = 0 \) with constant curvature matrix \( R(t) = kE > 0 \) can be written in the form \( y(t) = y(0) \cos(\sqrt{k}t) + \frac{y'(0)}{\sqrt{k}} \sin(\sqrt{k}t) \). If the initial values \( y(0) \) and \( y'(0) \) are linearly independent, then the curve \( y(t) \) is an ellipse in the 2-plane \( y(0) \wedge y'(0) \) and the area of the parallelogram \( y(t) \wedge y'(t) \) is constant. We introduce and study a similar function (called the *area of an \( L \)-parallel vector field*) along the geodesics of a Riemannian manifold with a foliation (or distribution).

The following comparison lemma is used in Theorem 3.7 below for estimation of the length of \( L \)-parallel vector fields in terms of mixed-sectional-curvature pinching.

**Definition 3.5.** For a Riemannian manifold with complementary orthogonal distributions \( T_1 \) and \( T_2 \), the *turbulence along a \( T_1 \)-geodesic \( \gamma \) (a rotation component of the tensor \( B_2 \)) is defined by

\[ a(T_1, \gamma) = \sup \{ \langle B_2(x, y), z \rangle : x \in T_1(\gamma), y, z \perp T_1(\gamma), y \perp z, |x| = |y| = |z| = 1 \}. \]
Fig. 16. The behavior of the $L$-parallel field $y(t)$ along the “extremal” geodesic.

For a foliation $\{ L \}$, the turbulence along a leaf $L_0$ is defined by

$$a(L_0) = \sup \{ (B_2(x, y), z) : x \in TL_0, y, z \in TL_0^\perp, y \perp z, |x| = |y| = |z| = 1 \}$$

(see [205] and [147] for Riemannian submersions or foliations).

It follows from the relation $a(L) = 0$ for all leaves that the orthogonal distribution $TL^\perp$ is tangent to a totally umbilic foliation.

**Theorem 3.6** (local). Let $\{ L^\nu \}$ be a totally geodesic foliation on a Riemannian manifold $M^{n+\nu}$, and let there exist a point $m \in M$ such that along any leaf geodesic $\gamma : [0, \pi/\sqrt{k}] \to M$, $\gamma(0) = m$, we have

$$k_2 \geq K(\gamma', y) \geq k_1 > 0, \quad y \in TL^\perp(\gamma),$$

and one of the following inequalities holds:

(1 - $k_1/k_2$) $\cdot \max\{a(TL, \gamma)^2/k, 1\} \leq 0.337,$

$$\frac{2(|\nabla B_0^+| + |B_2^+|^2(a(TL, \gamma) + |B_2^+|))}{1} + (k_2 - k_1)|B_0^+|$$

$$\times \max\{a(TL, \gamma)^2/k, 1\} \leq 0.0082 k_2 \quad (\text{with } k_1 \geq 0.583 k_2),$$

where $k = (k_1 + k_2)/2$ and $B_2^+y$ is a component of $B_2^+y$ that is orthogonal to the vector $y$. Then $\nu < \rho(n)$, and if $TL$ is invariant with respect to the Kählerian structure on $M$, then $\nu = 2$ and $n$ is divisible by 4.

Theorem 3.6 is not true without conditions (3.34), but their coefficients 0.337 and 0.0082, respectively, can be obtained by the method for proving the theorem and can presumably be increased. Condition (3.34a) holds if $k_1 = k_2$, and condition (3.34b) holds if the foliation is conformal. The assumption in Theorem 3.6 that the distribution $TL$ is geodesic and integrable is made to simplify the formulation.

**The idea of the proof of Theorem 3.6. Step 1.** Let us assume the contrary, namely, that there exist unit vectors $x_0 \in T_1(m)$, $y_0 \in T_2(m)$ and a real $\lambda \leq 0$ with the property $B(x_0, y_0) = \lambda y_0$ (see Lemmas 3.6 and 3.8). The “extremal” leaf geodesic $\gamma : [0, \pi/\sqrt{k}] \to M$ ($\gamma'(0) = x_0$) and the $L$-parallel Jacobi field $y(t)$ along $\gamma$ with the initial value $y(0) = y_0$ play a crucial role. We decompose this vector field into a standard and a “small” term

$$y(t) = \left( \cos(\sqrt{k}t) + \frac{1}{\sqrt{k}} \sin(\sqrt{k}t) \right) y_0 + u(t), \quad \text{where } u(0) = u'(0) = 0.$$ (For $k_2 = k_1$, we have $u(t) \equiv 0$, and hence, the $L$-parallel Jacobi vector field $y(t)$ vanishes at the point $t_0 = \cot^{-1}(\sqrt{k})/\sqrt{k}$.

This contradiction completes the proof of Theorem 3.1 by Ferus. For conformal foliation, i.e., a foliation with $B_2^+ = 0$ and $K_{\text{mix}} > 0$, see [235].) In the general case $k_2 > k_1$ with assumption (3.34) (actually, for $\delta \geq 0.5821$), we prove (using Lemma 3.11) that the function $|y(t)|$, “the length of the $L$-parallel vector field $y(t)$,” has a local minimum at $t_1 \in (0, \pi/\sqrt{k})$ (see Fig. 16).

**Step 2.** We note that the function $V(t)$, “the area of a parallelogram on the vectors $y(t), y'(t)$,” changes “slowly” along the geodesic $\gamma$, that is

(a) $|V(t)| \leq \frac{1}{2}(k_2 - k_1)|y(t)|^2$,

(b) $|V(t)^2| \leq \frac{2(|B_2^+|^2(a(TL, y) + |B_2^+|) + |B_0^+|(k_2 - k_1))|y(t)|^4$
(this function is constant for $k_2 = k_1$ or for a conformal foliation). We obtain a contradiction here, since the function $V(t)$ cannot increase from the zero value $V(0)$ to the "large" value $V(t_1)$ on the given interval of length $\leq \pi/\sqrt{k}$.

In [253], Theorem 3.6 is proved by a different method.

**Lemma 3.12.** Let $M$ be a Riemannian manifold with a compact totally geodesic foliation $\{L\}$ and $K_{mix} \leq 0$. Then $M$ splits along $L$ and $L^\perp$.

The condition of Lemma 3.12 imposed on $K_{mix}$ of $M$ can be replaced by a weaker condition of nonnegativity of the mixed Ricci curvature.

Theorem 3.6 and Lemma 3.12 imply the following theorem.

**Theorem 3.7 (decomposition).** Assume that $M^{n+\nu}$ is a Riemannian manifold with a compact totally geodesic foliation $\{L\}$ of curvature $k_2 \geq K_{mix} \geq k_1 \geq 0$ and one of the following inequalities hold:

\[
(k_2 - k_1) \cdot \max\{a(L_0)^2, k\} \leq 0.337 k_2 k,
\]

\[
2(|B^L_2| + |B^L_3|^2 (a(L_0) + |B^L_2|)) + (k_2 - k_1)|B^L_2|^2 \times \max\{a(L_0)^4, k^2\} \leq 0.0082 k^2 k^3, \quad (\text{with } k_1 \geq 0.583 k_2),
\]

where $k = (k_1 + k_2)/2$, $L_0$ is a leaf, and $B^L$ is a component of $B^L$ orthogonal to the vector $y$. If $\nu \geq \rho(n)$ ($\nu > 2$ for $n$ divisible by 4 in the case of Kählerian $M$ and $\{L\}$), then $k_1 = k_2 = 0$ and $M$ splits along $L$ and $L^\perp$.

The assumption in Theorem 3.7 that the leaves are totally geodesic is given for simplicity. In Chap. 4, Theorem 3.7 is used to obtain results concerning the metric decomposition of ruled and parabolic submanifolds.

**3.5.2. The volume of an $L$-parallel vector field.** Here we develop the method used in Sec. 3.5.1.

Assume that $x \in T_mL$ is a unit vector, $\nu \in [2, \dim L]$ is an integer, $\gamma \subset L$, $\gamma(0) = x$, is a unit speed geodesic, and $Y: L \to TL^\perp$ is an $L$-parallel (Jacobi) vector field in a neighborhood of $\gamma$. We define the following quantities:

\[
a_{\gamma}(Y) = \sup\{|\nabla_u Y|/|Y| : u \in T_{\gamma}L, \ |u| = 1\},
\]

\[
R_{\gamma}(Y) = \sup\{|R(u, Y)v)/|Y| : u \perp v, \ |u| = |v| = 1\},
\]

where $\nabla_u$ is a component of the given vector that is orthogonal to $Y$. Obviously, $a_{\gamma}(Y) \leq a(L)$.

**Definition 3.6.** For any $\nabla$-parallel $\nu$-dimensional subbundle $Z$ of $T\gamma L$ that contains $\gamma'$, the nonnegative function $V_{\nu+1}(Y)$ is defined as a $(\nu + 1)$-dimensional volume of the parallelepiped $Y \wedge \nabla_{e_1} Y \wedge \ldots \wedge \nabla_{e_\nu} Y$ in $T_{\gamma}L^\perp$, where $(e_i)_{1 \leq \nu}$ is an orthonormal basis in $Z$. (The values $V_{\nu+1}(Y)$ do not depend on the choice of the basis in $Z$. For brevity, we will call $V_{\nu+1}(Y)$ a $(\nu + 1)$-dimensional volume of $Y$.)

**Lemma 3.13 [251].** If the mixed curvature $k_2 \geq K(u, Y) \geq k_1$, $u \in Z$, then the derivative of the $(\nu + 1)$-dimensional volume satisfies the inequality

\[
|\nabla_z V_{\nu+1}(Y)| \leq ((\nu - 1)R_{\gamma}(Y) + \frac{1}{2}(k_2 - k_1)) a_{\gamma}(Y)^{\nu-1}|Y|^\nu+1.
\]

Note that $V_{\nu+1}(Y) = \text{const}$ if $M$ is a real space form. In the proof of Theorem 3.6, we considered a special case of (3.40) for the function $V_2(Y)$, which is the area of the parallelogram $Y \wedge Y'$ in $T_{\gamma}L^\perp$ (i.e., for $\nu = 1$), and obtained inequality (3.35). Lemma 3.13 is used in Theorem 3.8 (and Theorems 4.24-4.25) in the case $\nu = 2$. 

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For a foliated Riemannian manifold $M$, $\{L\}$ we denote
\[
\delta_R = \sup\{(R(x,y)u,z) : x,u \in TL, y,z \in TL^\perp, x \perp u, y \perp z, |x| = |y| = |u| = |z| = 1\}.
\]

(3.41)

**Theorem 3.8.** Let $M^{n+\nu}$ be a Riemannian manifold with a compact totally geodesic foliation $\{L^\nu\}$ satisfying $k_2 \geq K_{\text{mix}} \geq k_1 \geq 0$, and let one of the following conditions hold:
\[
(k_2 - k_1 + 2\delta_R) d_L \exp(3|B^+| d_L) a(L) \leq 2 k_1,
\]

(3.42)

\[
(k_2 - k_1 + 2\delta_R) a(L) \max\{a(L)^2, k\}\sqrt[3]{2} \leq 0.004 k^2 k_2, \text{ with } k_1 \geq \frac{1}{2} k_2,
\]

(3.43)

where $d_L$ is the maximal diameter of the leaves. If $\nu \geq \rho(n)$, then $k_1 = k_2 = 0$ and $M$ splits along $L$ and $L^\perp$.

Conditions (3.42) and (3.43) hold automatically in the case $\delta_R = 0$ (say, for a conformally flat metric) and $k_1 = k_2$. In contrast to (3.36) and (3.37), condition (3.42) does not contain an a priori lower bound for the mixed-curvature pinching.

**Proof.** Step 1. By Lemma 1.2, there exists a leaf $L_0$ with a trivial holonomy. All $L$-parallel fields are globally defined on $L_0$.

Let (3.42) hold. By Lemma 3.6, for any point $m \in L_0$, there exist unit vectors $X_0 \in T_m L_0$ and $Y_0 \in T_m L_0^\perp$ and a real number $\lambda \leq 0$ with the property $B(x_0, y_0) = \lambda y_0$. Let $Y : L_0 \to TL^\perp$, $(Y(m) = y_0)$ be an $L$-parallel (globally defined) vector field, $m' \in L_0$ be a minimum of the length function $|Y| : L_0 \to \mathbb{R}^+$ ($m'$ exists on a compact $L_0$), and let $\gamma : [0, l] \to L_0 (\gamma(0) = m, \gamma(l) = m', l \leq \text{diam} L)$ be the minimal geodesic between the points $m$ and $m'$. Let $x_1(t) = \gamma'(t)$ be the velocity vector field and $x_2(t) \in T_{\gamma(t)} L$ be a unit vector field that is parallel along $\gamma$ and has the properties
\[
x_2(0) \perp x_1(0) \text{ and the vectors } \{x_2(0), x_1(0), x_0\} \text{ are linearly dependent}.
\]

(3.44)

Let $y(t) = Y(\gamma(t))$ be the restriction of $Y$ to $\gamma$, and let $y(t) = \nabla x_2(t) y$.

Step 2. Let $V_3(t), 0 \leq t \leq l$, be the volume of the parallelepiped $y(t) \land y_1(t) \land y_2(t)$ in $T_L L^\perp$. By Lemma 3.13 (for $\nu = 2$) and the inequality $R(Y) \leq \delta_R$, we obtain
\[
|V_3(t)| \leq ((k_2 - k_1)/2 + \delta_R) a(L)|y(t)|^3.
\]

(3.45)

Let $\{\partial_1, \partial_2\}$ be an orthonormal basis of the plane $x_1(l) \land x_2(l)$ with the property $\nabla_{\partial_1} Y \perp \nabla_{\partial_2} Y$. By virtue of the well-known formula for the second derivative of the length of a Jacobi vector field, we have $|\nabla_{\partial_2} Y| \geq \sqrt{k_1} |y(l)|$, and, therefore,
\[
V_3(l) = |y(l)| \cdot |\nabla_{\partial_1} Y| \cdot |\nabla_{\partial_2} Y| \geq k_1 |y(l)|^3.
\]

(3.46)

(Since $\gamma(l)$ is a point of minimum of the function $|Y|$, the vector $y_l = \nabla x_2 Y$ is orthogonal to the vector $y(l)$.) Therefore, using the condition $V_3(0) = 0$, we obtain
\[
k_1 |y(l)|^3 \leq V_3(l) \leq \int_0^l |V_3(t')| dt \leq a(L)((k_2 - k_1)/2 + \delta_R) \int_0^l |y(t)|^3 dt.
\]

(3.47)

Step 3. The relation $(y(t)^2)' = 2(y(t), y_1(t)) = 2(B^+(x_1(t), y(t)), y(t))$ implies $|(y(t)^2)'| \leq 2|B^+| |y(t)|^2$, and, therefore,
\[
|y(t)| \leq |y(l)| \exp(|B^+| l).
\]

(3.48)

From (3.48) and (3.47), we obtain an inequality that is opposite to (3.42).

Step 4. The extremal situation above implies $k_1 = k_2 = 0$ or $a(L) = 0$. In the latter case, the structural tensor $B$ is conformal, in particular, the distribution $TL^\perp$ is integrable. From the inequality $K_{\text{mix}} \geq 0$, by using Lemma 3.3, we obtain the relation $B = 0$, and again, $k_1 = k_2 = 0$.
Step 5. Since \( \{L\} \) is a compact totally geodesic foliation with the condition \( K_{\text{mix}} = 0 \), Lemma 3.12 completes the proof.

### 3.5.3. Foliations with \( K_{\text{mix}} > 0 \) and with conditions on \( L \)-Jacobi fields.

**Theorem 3.9.** Let \( \{L\} \) be a foliation into closed geodesics on a Riemannian manifold \( M^{n+1} \) with \( k_2 \geq K_{\text{mix}} \geq k_1 > 0 \), and let the following conditions hold:

1. \( k_1/k_2 \geq \frac{A}{A+1.51} \), where \( A = \max\{a(L)/\sqrt{k}, 1\} \) and \( k = (k_1 + k_2)/2 \);
2. the sets of local minima of the functions \( \{y^2\} \) (where \( \{y\} \) are \( L \)-parallel Jacobi fields along the leaves) are connected;
3. the 1-dimensional subspaces in \( TL \) are invariant under the complete parallel displacement along every leaf \( L \);
4. the length of every leaf \( L \) does not exceed \( \pi/\sqrt{k} \).

Then the dimension \( n \) is even.

Theorem 3.9 can be generalized to compact foliations of dimension \( \nu > 1 \). Conditions (1)–(4) of Theorem 3.9 hold for geodesics and Jacobi fields in the projective space \( \mathbb{R}P(k) \). In the case \( k_1 = k_2 \), the curvature tensor has a standard form and every function \( y^2(t) \) (where \( y(t) \) is a Jacobi vector field) can be written as

\[
a_1 \cos(2\sqrt{k}t) + a_2 \sin(2\sqrt{k}t) + a_0 \quad (0 \leq t \leq \pi/\sqrt{k})
\]

and hence, (2) holds. If \( k_1/k_2 \) is close to 1, then the “implicit” property (2) can be expressed in terms of the curvature tensor, but, in this paper, we do not calculate this expression. In view of \( A \geq 1 \), condition (1) implies \( k_1/k_2 \geq 0.399 \); the constant 0.399 is obtained by the method used for proving Theorem 3.9 and can be improved.

**Idea of the proof.** Let \( m \in L \), and let \( S^{n-1} \subset T_mL^1 \) be the unit sphere centered at the point \( m \in L \). Assume that the length of every nonzero \( L \)-Jacobi field \( y(t) \) along \( L \) has a unique point of global minimum. Since the mixed curvature is positive, the derivative \( y'(t) \) at a point of minimum \( t \) is a nonzero vector that is orthogonal to \( y(t) \). We transport all these vectors \( y'(t) \) along \( L \) to the initial point \( m \) and obtain a continuous line field on the sphere \( S^{n-1} \). Hence \( n \) is even. (If the length of some nonzero \( L \)-Jacobi field \( y(t) \) along \( L \) has more than one point of global minimum, then, by using conditions (1) and (4), we can change this line field in a neighborhood of its singularity locus in order to obtain a continuous line field on the whole sphere \( S^{n-1} \).)

### 3.6. Integral Formulas for Mixed Curvature

**3.6.1. Integral formulas for mixed curvature.** The first integral formula for the curvature of a foliated manifold was obtained by G. Reeb.

**Theorem 3.10.** Let \( H \) be the mean-curvature function of a transversally orientable foliation of codimension 1 on a closed oriented Riemannian manifold \( M \). Then \( \int_M H \, d\text{vol} = 0 \), where \( d\text{vol} \) is the volume form of \( M \).

If a foliation has a basic projectable mean-curvature function, then, according to Theorem 3.10, the mean-curvature is zero along one of its leaves, and this leaf is minimal [299]. The following problem is studied in [209, 210] and [301]: what functions on \( M \) can serve as the mean curvature functions of a given codimension-1 foliation?

For a given Riemannian manifold \( M \) with a codimension-1 foliation, let \( K(m) \) be the determinant of the second fundamental form of the leaf through the point \( m \in M \). A foliation minimizing the total curvature \( \int_M |K| \) is said to be tight. The integral formulas of these foliations are studied with the intensive use of integral geometry in [166] and [168]. For example, the following problem is open: prove (directly!) that \( S^n \) does not admit a tight foliation; for \( n = 3 \) [210], the corresponding result was obtained by using the fact of existence of the Reeb component.

Let \( \{x_i\} \subset T_1 \) and \( \{y_j\} \subset T_2 \) be local orthonormal bases. The **mixed scalar curvature** is defined by

\[
s_{\text{mix}} = \sum_{i,j} (R(y_j, x_i)x_i, y_j).
\]
For the distributions on a space form $M(k)$, we have $s_{\text{mix}} = k(\dim T_1)(\dim T_2)$. Note that

\begin{align*}
\sum_{i,j} (B_2(x_i, B_2(x_i, y_j), y_j))_{ij} &= ||B_2^+||^2 - ||B_2^-||^2 = ||h_2||^2 - ||A_2||^2, \\
\sum_{i,j} (B_1(y_j, B_1(y_j, x_i), x_i))_{ij} &= ||B_1^+||^2 - ||B_1^-||^2 = ||h_1||^2 - ||A_1||^2.
\end{align*}

(3.50)

The contraction of Eq. (3.15)

\begin{align*}
&((\nabla x, B_2)(x_i, y_j), y_j) + (B_2(x_i, B_2(x_i, y_j), y_j) \\
&+ ((\nabla y, B_1)(y_j, x_i), x_i) + (B_1(y_j, B_1(y_j, x_i), x_i) + (R(y_j, x_i)x_i, y_j) = 0
\end{align*}

(3.51)

with respect to $i$ and $j$, by virtue of the formulas

\begin{align*}
\text{div}_1 H_2 &= \text{div} H_2 + |H_2|^2, \\
\text{div}_2 H_1 &= \text{div} H_1 + |H_1|^2,
\end{align*}

(3.52)

yields the equation

\begin{align*}
\text{div} (H_1 + H_2) &= s_{\text{mix}} + ||h_1||^2 + ||h_2||^2 - |A_1|^2 - |A_2|^2 - |H_1|^2 - |H_2|^2.
\end{align*}

(3.53)

For similar formulas with more than two orthogonal distributions, see [14].

In certain special cases of a Riemannian almost-product manifold, $s_{\text{mix}}$ in formula (3.53) has a simpler form, for example,

\begin{align*}
\begin{cases}
(UD, UD) : & -\text{div}(H_1 + H_2) + ||A_1||^2 + ||A_2||^2 + \frac{v_1}{v_2} |A_1|^2 + \frac{v_2}{v_1} |A_2|^2, \\
(GD, GD) : & -\text{div}(H_1 + H_2) + ||A_1||^2 + ||A_2||^2, \\
(GD, MD) : & \text{div}(H_1 + H_2) + ||A_1||^2 + ||A_2||^2 - |h_2|^2, \\
(F, F) : & -\text{div}(H_1 + H_2) - ||h_1||^2 - ||h_2||^2 + |H_1|^2 + |H_2|^2, \\
(MF, MF) : & -\text{div}(H_1 + H_2) - ||h_1||^2 - ||h_2||^2.
\end{cases}
\end{align*}

(3.54)

The integration of Eq. (3.53) along a compact manifold $M$ leads one (by the Green's theorem) to integral formulas that have many interesting global consequences.

**Theorem 3.11.** Let $T_1$ and $T_2$ be complementary orthogonal distributions on a closed orientable Riemannian manifold $M$. Then

\begin{align*}
\int_M \left\{ s_{\text{mix}} + ||B_1^+||^2 - ||B_2^+||^2 - ||B_1^-||^2 - ||B_2^-||^2 - |H_1|^2 - |H_2|^2 \right\} = 0.
\end{align*}

(3.55)

Integral formulas for foliations that have a form similar to formula (3.55) are introduced in [220]. In [232], Eq. (3.55) is deduced for a Riemannian almost-product manifold in terms of similarly defined differentials $\nabla P$, $\delta P$, and $dP$ of the tensor $P$ (see Definition 2.5):

\begin{align*}
\int_M \left\{ 4 s_{\text{mix}} + |\nabla P|^2 - |dP|^2 - |\delta P|^2 \right\} = 0.
\end{align*}

(3.56)

In [302], Eq. (3.55) is derived by a direct computation. In [275], Eq. (3.55) has the form

\begin{align*}
\int_M \left\{ 64 s_{\text{mix}} + ||L||^2 - ||N||^2 - |H|^2 \right\} = 0,
\end{align*}

(3.57)

where $N, L, \text{ and } H$ are given by formulas (2.20), (2.22), and (2.23), respectively, and

\begin{align*}
\frac{1}{64} ||L||^2 &= ||h_1||^2 + ||h_2||^2, \\
\frac{1}{64} ||N||^2 &= ||A_1||^2 + ||A_2||^2, \\
\frac{1}{64} |H|^2 &= |H_1|^2 + |H_2|^2.
\end{align*}

(3.58)

(3.59)
This method of integral formulas is applied in \([274, 276]\) to degenerate maps \(f: M \to N\) of nonconstant rank \(r(f) < \dim M\) (see Example 2.1). Note that Eq. (3.53) can be compared with the following formula for a Riemannian foliation \([189]\):

\[
|h_1|^2 = |A_2|^2 + (\Delta P_1, P_1) - s_{\text{mix}},
\]

where \(\Delta P_1\) is the Laplacian of the orthogonal projection \(P_1\). See also \([95, 195, 319]\).

Equation (3.55) gives us decomposition criteria for compact manifolds with Riemannian almost-product structure under the constraints on the sign of \(s_{\text{mix}}\).

**Corollary 3.2** \([274, 276]\). (1) Let \(T_1\) and \(T_2\) be complementary orthogonal umbilic distributions on a closed oriented Riemannian manifold \(M\) with \(s_{\text{mix}} \leq 0\). Then \(M\) splits along \(T_1\) and \(T_2\).

(2) Let \(T_1\) and \(T_2\) be complementary orthogonal geodesic and umbilic distributions on a closed orientable Riemannian manifold \(M\) with \(s_{\text{mix}} \geq 0\). If both \(T_1\) and \(T_2\) are integrable, then \(M\) splits along \(T_1\) and \(T_2\).

Some global results are connected with the integration of Eq. (3.53) over a compact leaf \(L_0\):

\[
\int_{L_0} \left( s_{\text{mix}} + \text{div} H_1 + \|B_1^+\|^2 + \|B_2^+\|^2 - \|B_2^-\|^2 \right) = 0.
\]

**Corollary 3.3.** A compact minimal foliation \(\{L\}\) on a Riemannian manifold \(M\) with an integrable orthogonal distribution and \(s_{\text{mix}} \geq 0\) splits along \(L\) and \(L^\perp\); if the manifold \(M\) is simply connected, then it is the product \(M = L \times L^\perp\). (In particular, a minimal foliation \(\{L\}\) on a Riemannian manifold \(M\) with the integrable orthogonal distribution and \(s_{\text{mix}} > 0\) has no compact leaves \([302]\).)

**3.6.2. The integral of a mixed scalar curvature along a leaf.** Using the method of matrix Riccati ODE’s, we can deduce an integral inequality for \(s_{\text{mix}}\) along a complete leaf; the consequences are rigidity and splitting results of Riemannian manifolds with a foliation (distribution) whose mixed Ricci or scalar curvature is nonnegative and whose norms of integrable tensors are bounded from above.

Let \(M\) be a Riemannian manifold with complementary orthogonal distributions \(T_1\) and \(T_2\). Let \(n = \dim T_2\) and \(\nu = \dim T_1\); denote by \(\{y_j\}\) a local orthonormal basis of \(T_2\) and by \(\{x_i\}\) a local orthonormal basis of \(T_1\) with \(x_1 = e\). Let

\[
|A_i|(e) = \sup\{|A_i(e, v)| : |v| = 1\},
\]

\[
|A_i|(p) = \sup\{|A_i|(e) : e \in T_i(p), |e| = 1\}.
\]

Note that for \(e \in T_1(p)\) we have

\[
\begin{align*}
\sum_j |B_1^-(y_j, e)|^2 & = \sum_j |B_1^-(y_j, e, x_i)|^2 = \sum_i |A_1(e, x_i)|^2 \leq (\nu - 1)|A_1|^2(e), \\
\sum_j |B_2^+(e, y_j)|^2 & = \sum_i |A_2(y_i, y_j)|^2 \leq n(n - 1)|A_2|^2(p).
\end{align*}
\]

The mixed Ricci curvatures are defined as follows:

\[
\text{Ric}_1(x) = \sum_j K(x, y_j), \quad \text{Ric}_2(y) = \sum_i K(x_i, y).
\]

**Theorem 3.12** (rigidity). Let \(M\) be a complete Riemannian manifold equipped with two complementary orthogonal distributions \(T_1\) and \(T_2\), \((\dim T_1 = \nu\) and \(\dim T_2 = n)\), and let \(T_1\) be geodesic. Then

(a) for every \(T_1\)-geodesic \(\gamma(t) \subset M\), we have

\[
\liminf_{t \to +\infty} \int_{-\infty}^t \left( \text{Ric}_1(\gamma') - (\nu - 1)|A_1|^2(\gamma) - n(n - 1)|A_2|^2(\gamma) \right) dt \leq 0,
\]

and equality holds for every \(\gamma\) if and only if \(T_2\) is a geodesic distribution and the skew-symmetric operators \(B_2\) and \(B_1\) have "constant torsion" in the following sense:

\[
B_2(x, B_2(x, y)) = -|A_2|^2(p)x^2y, \quad B_1(y, B_1(y, x)) = -|A_1|^2(p)y^2x;
\]

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(b) if the equality holds in (3.63) for every \( \gamma \), and in addition, \( \nu \geq \rho(n) \) (if \( T_1 \) is invariant with respect to the Kählerian structure on \( M \), then, for \( \nu > 2 \), we assume that \( n \) is divisible by \( 4 \)), then \( T_2 \) is also integrable, and, hence, is tangent to a Riemannian totally geodesic foliation; moreover, \( K_{\text{mix}} \geq 0 \).

In [273] the hyperdistribution \( T_1 \) such that property (3.64) is satisfied by \( B_1 \) is referred to as having constant torsion.

Proceeding from [125], where manifolds without conjugate points are studied, we obtain an integral formula for \( s_{\text{mix}} \) along a complete leaf.

**Theorem 3.13 (rigidity and splitting).** Let \( \{L^r\} \) be a foliation on a Riemannian manifold \( M^{n+v} \) with complete minimal leaves. Assume that the quadratic form
\[
\operatorname{Ric}_1(x) + \operatorname{trace}(B_2^{-1}(x, B_2^2(x, \cdot)) - h_1(x, B_1^+(\cdot, x)) - (\nabla h_1)(x, x))
\]
on a unit tangent bundle \( S_{L_0} \) of every leaf \( L_0 \) has an integrable positive or negative part. Then
\[
\int_{L_0} \left( s_{\text{mix}} - n(n-1)|A_2|^2 \right) d\text{vol} \leq 0, \tag{3.65}
\]
and equality holds for every leaf if and only if \( \{L\} \) is a Riemannian totally geodesic foliation. Moreover, if equality holds in (3.65) for all leaves and \( \nu \geq \rho(n) \) (if \( T_L \) is invariant with respect to the Kählerian structure on \( M \), then for \( \nu > 2 \), we assume that \( n \) is divisible by \( 4 \)), then \( M \) splits along \( L \) and \( L^\perp \).

In the following corollary that generalizes Theorem 3.19 in [46], the distribution \( T_L^\perp \) is integrable and the foliation splits without any additional assumption on the dimension. Moreover, the integrability of \( T_L^\perp \) can be assumed only along some submanifold that is transversal to \( T_L \). For a compact leaf \( L_0 \), see Corollary 3.3.

**Corollary 3.4 (splitting).** Let \( M \) be a foliated Riemannian manifold with complete minimal leaves \( \{L\} \) and integrable orthogonal distribution \( T_L^\perp \). Assume that the function \( \operatorname{Ric}_1(x) - \operatorname{trace}(h_1(x, B_1^+(\cdot, x)) + \operatorname{trace}(\nabla h_1)(x, x)) \) on \( S_{L_0} \) has an integrable positive or negative part along every leaf \( L_0 \). Then
\[
\int_{L_0} s_{\text{mix}} d\text{vol} \leq 0 \tag{3.66}
\]
and equality holds (for every leaf) if and only if \( M \) splits along \( L \) and \( L^\perp \).

From Theorem 3.12 and independently of Theorem 3.13, we obtain the following corollaries.

**Corollary 3.5 (splitting) [243].** Let \( M \) be a foliated Riemannian manifold with complete totally geodesic leaves \( \{L\} \), and let
\[
\operatorname{Ric}_1(x) \geq n(n-1)|A_2|^2(p), \quad x \in T_p M.
\]
If \( \nu \geq \rho(n) \) (if \( T_L \) is invariant with respect to the Kählerian structure on \( M \), then, for \( \nu > 2 \), we assume that \( n \) is divisible by \( 4 \)), then \( M \) splits along \( L \) and \( L^\perp \).

**Corollary 3.6 (rigidity) [147].** Let \( \{L\} \) be a Riemannian foliation on a complete Riemannian manifold \( M \). If, for every geodesic \( \gamma \) orthogonal to the leaves, we have
\[
\operatorname{Ric}_2(\gamma') \geq (n-1)|A_2|^2(\gamma'),
\]
then \( \{L\} \) is a totally geodesic Riemannian foliation.

The inequality in Corollary 3.6 is actually sharp: all assumptions are met; for instance, see [129] for a foliation \( \{L\} \) with \( (T_L = T_2) \) of codimension-1 with \( \operatorname{Ric}_1 \geq 0 \).

**3.6.3. Foliations with an almost-geodesic orthogonal distribution.** Using the method of matrix Riccati ODE’s, we obtain rigidity result for foliations of “large” dimension whose mixed sectional curvature is nonnegative and whose norms of \( h_2 \) are bounded from above.
Let \( \{L\} \) be a Riemannian foliation with totally geodesic leaves. Then \( B_1 = B_2^+ = 0 \), and it follows from (3.15) that
\[
\begin{align*}
\nabla_x B_2^- &= 0 \\
(B_2^+ (x, B_2^+ (x, y)) &= (R(x, y)y, x).
\end{align*}
\]
Hence \( K_{\text{mix}} \geq 0 \). If we assume that \( K_{\text{mix}} > 0 \), then we find from (3.67b) that the skew-symmetric operator \( B_2^- \) is nondegenerate, and therefore, has no eigenvectors, i.e., in this case, the Furs inequality \( \nu < \rho(n) \) holds. This local property of a Riemannian foliation (and its modification for a conformal foliation with complete totally geodesic leaves) plays a crucial role in this section.

Let \( |h_i| = \sup \{|h_i(v, w)| : |v| = |w| = 1, v, w \in T_1 \} \).

**Theorem 3.14** (rigidity). Let \( M \) be a complete Riemannian manifold with a geodesic distribution \( T_1 \) of dimension \( \dim T_1 = \nu \). Assume that its orthogonal distribution \( T_2 \) is of dimension \( \dim T_2 = n \), and let the following inequality hold:
\[
K_{\text{mix}} \geq |h_2|^2 + |A_1|^2.
\]
If \( \nu \geq \rho(n) \) (if \( T_1 \) is invariant with respect to the Kählerian structure on \( M \), then, for \( \nu > 2 \), we assume that \( n \) is divisible by 4), then the distribution \( T_2 \) is geodesic.

**Corollary 3.7.** Let \( M \) be a complete Riemannian manifold with a geodesic distribution \( T_1 \) of dimension \( \dim T_1 = \nu \). Assume that its orthogonal distribution \( T_2 \) is of dimension \( \dim T_2 = n \), and let there exist a point \( p \in M \) such that for every \( T_1 \)-geodesic \( \gamma: \mathbb{R} \to M \) with \( \gamma(0) = p \), the following inequality holds:
\[
\liminf_{s \to +\infty} \int_{-s}^{s} \left\{ K(\gamma', y_t) - |B_2^+ (\gamma', y_t)|^2 + |B_2^- (\gamma', y_t)|^2 \right\} > 0,
\]
where \( y_t \in T_2(\gamma') \) is parallel along \( \gamma \). Then \( \nu < \rho(n) \), and if \( T_1 \) is invariant with respect to the Kählerian structure on \( M \), then \( \nu = 2 \) and \( n \) is divisible by 4.

A special case of Theorem 3.14 (and Corollary 3.7) arises when \( T_1 \) is tangent to a totally geodesic foliation. Corollary 3.7 (and Theorem 3.14) can be formulated in a more general form for the distribution \( T_2 \) that is "close" to umbilic. In this case, we replace the expression \( B_2^+ (\gamma')y \) in (3.69) by \( B_2^+ (\gamma')y - \beta (\gamma')y \), where \( \beta: T_1 \to \mathbb{R} \) is a regular linear functional.

### 3.7. Foliations with Positive Partial Mixed Curvature

Some authors (in [315, 270, 271], and later in [195, 196]) studied curvature functions on a Riemannian manifold that fill a gap between the sectional curvature and the Ricci curvature. For \( q + 1 \) orthonormal vectors \( V = \{x_0, x_1, \ldots, x_q\} \), the (partial) \( q \)-dimensional Ricci curvature of \( M \) is, by definition,
\[
\text{Ric}^q (V) = \sum_{i=1}^{q} K(x_0, x_i).
\]
The 1-dimensional Ricci curvature is the same as the sectional curvature, and the \((M-1)\)-dimensional partial curvature of \( M \) is the ordinary Ricci curvature. For a foliated Riemannian manifold \( M, \{L\} \), there exist two mixed \( q \)-dimensional Ricci curvatures \( \text{Ric}_1^q (y_0, x_1, \ldots, x_q) \) when \( q \leq \dim L \) and \( \text{Ric}_2^q (x_0, y_1, \ldots, y_q) \) when \( q \leq \text{codim} L \), where \( x_i \in TL, y_i \in TL^\perp \). If \( \text{Ric}^q \) is nonnegative (positive) and \( q' > q \), then \( \text{Ric}^{q'} \) is also nonnegative (positive) by virtue of the identity
\[
\sum_{i=1}^{q'} K(x, z_i) = \frac{q'}{k C_q^k} \sum_{1 \leq i_1 < \cdots < i_k \leq q'} \sum_{j=1}^{k} K(x, z_{i_j}),
\]
or by induction, using \( \sum_{i=1}^{q+1} K(x, z_i) = \frac{1}{q} \sum_{i=1}^{q+1} \sum_{j \neq i} K(x, z_j) \). By the Meyer theorem, a complete Riemannian manifold with \( \text{Ric}_M^q \geq c > 0 \) is compact. Moreover, if \( \text{Ric}_M^q \geq qc \) (resp., \( \text{Ric}_M^q \leq qc \)) for \( M^n \), then
The class of Riemannian manifolds with positive curvature $\text{Ric}^g$ is wider than the class of manifolds with positive sectional curvature. For example, since $M^{2n} = S^n(1) \times S^n(1), n > 2$, has Ricci curvature $\text{Ric}_M = \text{Ric}_{M_1}^{2n-1} = n - 1$ and sectional curvature $K_M = \text{Ric}_M^1 \in [0, 1]$, its $(n + 1)$-dimensional Ricci curvature is positive: $\text{Ric}_M^{n+1} \geq 1$.

If the radius of the circle of $S^2$ is “small,” then there exists a large circle of $S^2$ that is “far” from it. Morvan [193] started from this elementary fact and generalized the result of [93] by giving an upper bound for the distance between two submanifolds of a Riemannian space with positive sectional curvature. The generalization of the result in [193] for the case $\text{Ric}^g > 0$ and the version for foliations are given below (see [255]).

Let $\|h_L\|$ be the supremum of the norms of the second fundamental forms of the leaves, and let $\text{diam}^+ L$ be the maximal distance between the leaves of the foliation $\{L\}$.

**Theorem 3.15.** Let $\{L^\nu\}$ be a compact foliation on a Riemannian manifold $M^{\nu+n}$ with $\text{Ric}_M^\nu > c > 0$ for some $q < \nu$. Then

\[
(\text{diam}^+ L)^2 \leq \begin{cases} 
\frac{2c}{\epsilon} \|h_L\| + \frac{\epsilon^2}{4} & \text{if } \nu \leq n - 1, \\
\frac{2c}{\epsilon} \|h_L\| + \frac{(\nu-\nu+n-1)\epsilon^2}{c} & \text{if } n - 1 < \nu < n - 1 + q, \\
\frac{2c}{\epsilon} \|h_L\| & \text{if } \nu \geq n - 1 + q.
\end{cases}
\]

For the Riemannian foliation in Theorem 3.15, the condition on curvature can be weakened up to $\text{Ric}_M^\nu \geq c > 0$, since the minimal geodesics between any two of the leaves is orthogonal to all leaves.

The following corollary for $q = 1$ is proved in [281] by using the ideas of [93].

**Corollary 3.8.** Let $M^{n+n}$ be a Riemannian manifold with compact totally geodesic foliation $\{L^\nu\}$, and let $\text{Ric}_M^\nu(L)$ be positive along some leaf. Then $\nu < n - 1 + q$.

Let us consider classes of submanifolds with additional conditions imposed on the second fundamental form.

**Definition 3.7** [269]. A submanifold $M$ in a Riemannian manifold $\tilde{M}$ is $k$-saddle if, for every normal vector $\xi \in TM^k$, the second quadratic form $A_\xi$, which is reduced to a diagonal form, has $\leq (k - 1)$ coefficients of the same sign. A foliation $\{L\}$ on a Riemannian manifold $M$ is $k$-saddle if every leaf is a $k$-saddle submanifold.

Totally geodesic submanifolds are 1-saddle. A classical saddle surface $M^2$ in $\mathbb{R}^3$ (for instance, a surface of negative Gaussian curvature) is a 2-saddle submanifold. In [34], another definition (the value of $k$ differs by 1) of a $k$-saddle submanifold is given.

Since the maximal linear subspace of the cone

\[\sum_{i=1}^k a_i x_i^2 - \sum_{j=k+1}^r a_j x_j^2 = 0, \quad a_s > 0,\]

is of dimension $\min\{k, r - k\}, [34]$, we have from Theorem 3.15 that the following assertion holds.
Corollary 3.9. Let $M^{n+r}$ be a Riemannian manifold with a compact $k$-saddle foliation $\{L^r\}$, and let $\text{Ric}_\mathbb{L}(L) > 0$ for some $q \leq \nu - k + 1$. Then $\nu < q + n + 2k - 3$. Moreover, if $\text{Ric}(M) \geq c > 0$, then

$$\text{diam}^\perp L \leq \left\{ \frac{\pi}{2} \sqrt{\frac{q}{c}} \frac{\nu + n + 2k - 3}{c} \right\}$$

if $\nu \leq n + 2k - 3$,

$$\frac{\pi}{2} \sqrt{\frac{q - \nu + n + 2k - 3}{c}}$$

if $n + 2k - 3 < \nu < q + n + 2k - 3$.

Definition 3.8 ([38] for $s = 1$). A foliation $\{L\}$ on a Riemannian manifold $M$ is of nonpositive (resp., negative) extrinsic $s$-dimensional Ricci curvature if the $s$-dimensional Ricci curvature of its leaves is not greater (resp., less) than the $s$-dimensional Ricci curvature of the ambient space along the leaves.

Lemma 3.14 ([102] for $s = 1$). A submanifold $M^n \subset \mathbb{M}^{n+p}$ with nonpositive extrinsic $s$-dimensional Ricci curvature is $k$-saddle with $k = p + s$.

By virtue of Lemma 3.14 and Corollary 3.9, we have

Corollary 3.10. If $\{L^r\}$ is a compact foliation on $M^{n+r}$ of nonpositive extrinsic $s$-dimensional Ricci curvature and $\text{Ric}_\mathbb{L}(L) > 0$ for some $q \leq \nu - n - s + 1$, then $\nu < 3n + q + 2s - 3$.

Similar results hold for Kählerian manifolds of positive partial bisectional curvature (see [151, 152] for totally geodesic submanifolds).

3.8. Transversal Totally Geodesic Foliations and Submersions

Let us consider some generalizations of classes of Riemannian foliations and submersions. Let $\nabla$ and $\hat{R}$ be a horizontal connection and the curvature of a foliation, respectively.

Definition 3.9. A foliation $\{L\}$ is transversal totally geodesic if, for any basic vector fields $y$ and $z$, the derivative $\nabla_y z$ is also a basic field. A foliation $\{L\}$ is transversal curvature invariant if, for any basic vector fields $y$, $z$ and $w$, the vector field $R(y, z)w$ is also basic.

The condition that the right-hand sides of (3.71) and (3.72) vanish characterizes transversal totally geodesic and curvature-invariant foliations, respectively. The class of transversal curvature invariant foliations includes the class of transversal totally geodesic foliations, and the latter includes the class of Riemannian foliations. The characteristics of these classes of foliations (and submersions) in terms of their second fundamental form $h_2$ and the tensor $T$ can be obtained from the following lemma.

Lemma 3.15. For a foliation $\{L\}$, we have

$$(\nabla_x (\nabla_y z) - O_{\nabla_y z}(x, w) = ((\nabla_w h_2)(z, y) - (\nabla_y h_2)(z, w))$$

$$- (\nabla_z h_2)(y, w), x) + (h_2(y, z), T_x w) - (h_2(z, w), T_x y) - (h_2(z, w), T_{xy}).$$

(3.71)

$$(\nabla_x (\hat{R}(y, z) w) - O_{\hat{R}(y, z) w}(x, u) = ((\nabla_y \nabla_u h_2)(z, w) - (\nabla_y h_2)(w, u))$$

$$- (\nabla_z \nabla_u h_2)(y, w) + (\nabla_z \nabla_u h_2)(w, u) + (\nabla_z \nabla_w h_2)(y, u), x)$$

$$+ (\nabla_y h_2)(z, w) - (\nabla_y z)(y, w), T_u u) - (\nabla_y h_2)(z, u)$$

$$+ (\nabla_z h_2)(y, u), T_x w) - (\nabla_z h_2)(w, u), T_x z) + (\nabla_z h_2)(w, u), T_{zy})$$

$$- (h_2(z, u), (\nabla_T x z u) + (h_2(z, w), \nabla_T x u) - (h_2(w, u), (\nabla_T x z Z$$

$$- (\nabla_z T) z y) - (h_2(y, w), (\nabla_T x z u) + (h_2(y, u), (\nabla_T x z u)).$$

(3.72)

Note that the condition that the right-hand sides of relations (3.71) and (3.72) vanish characterizes transversal and totally geodesic and curvature-invariant foliations, respectively. A foliation $\{L\}$ with a parallel
the second fundamental form of the distribution $TL^1$, i.e., $(\nabla_w h_2)(y, z) = 0$, is transversal totally geodesic. Symmetric submanifolds (i.e., submanifolds having a parallel second fundamental form) are classified in some cases (see [176]). A foliation with the condition $(\nabla_y \nabla_z h_2)(v, w) = 0$ is transversal curvature invariant. Hypersurfaces in $\mathbb{R}^n$ with the condition $\nabla \nabla h = 0$ include symmetric submanifolds and also cylinders over a plane curve — the clothoid $k(s) = as + b$ (see [176]).

Example 3.1 (existence). (1) For a twisted product $B \times_f P$, the condition that the right-hand side of (3.71) vanishes if all submanifolds $\{\pi^{-1}(b)\}$ have a parallel second fundamental form (i.e., their mean-curvature vector $H$ is parallel); or equivalently, $f(b, p) = \lambda(b)\mu(p)$. In this case, $\pi_2$ is a transversal totally geodesic submersion, which is not Riemannian for $\lambda(b) \neq \text{const}$.

(2) Let $\{e_i\}_{i=1}^1$ be a local basis of the vertical distribution for the twisted product $B \times_f P$. The mean-curvature vector $H$ of the leaves $\pi^{-1}(b)$ is given by the formula $H = \sum_{i=1}^1 (e_i \log f) e_i$. For any horizontal vectors $y$ and $z$, we have $(\nabla_z ((\nabla_y H)^T)) = -\sum_{i=1}^1 z(e_i \log f) e_i$. Therefore, the property $(\nabla_y \nabla_z h_2)(v, w) = 0$ follows from the system $z(y(e_i \log f)) = 0$, which has the solution $f(b, p) = \mu_0(p)\lambda(b)\exp(\sum_{i=1}^1 b_i\mu_i(p))$. Here $b_i$ are local coordinates on $B$, and we assume that the support of nonconstant functions $\mu_i(p)$ lies in the chart.

We consider a submersion $\pi: M \to \tilde{M}$ of smooth manifolds $M$ and $\tilde{M}$ with compact connected leaves $\{L_b = \pi^{-1}(b)\}_{b \in \tilde{M}}$. Let $M$ be equipped with a Riemannian metric, and let $\dim \tilde{M} > 1$. Along the leaf $L_b$, the horizontal vector fields with the Lebesgue integrable function of the square of length form the vector space $\tilde{M}_b$ with inner product

$$\langle s_1, s_2 \rangle = \int_{L_b} \langle s_1, s_2 \rangle \, d\text{vol}.$$  

The basic vector fields along $L_b$ form a $(\dim \tilde{M})$-dimensional linear subspace $T_b \tilde{M} \subset \tilde{M}_b$, and hence, $\tilde{M}$ with metric $(\cdot, \cdot)$ is a Riemannian manifold (see [23] for the case where $M$ is a unit tangent bundle of $C_l$-manifold and $\tilde{M}$ is a manifold of geodesics with metric $(\cdot, \cdot)$).

Let $P^{L_2}: \tilde{M}_b \to T\tilde{M}$ and $P^{L_2\perp}: \tilde{M}_b \to T\tilde{M}^\perp$ be orthogonal projections. The second fundamental form $h^L: T\tilde{M} \times T\tilde{M} \to TM$ of a submersion $\pi$ (i.e., of a subbundle $T\tilde{M} \subset \tilde{M}$) is defined by the formula

$$h^L(\tilde{Y}, \tilde{Z}) = P^{L_2\perp}(\tilde{\nabla}_Y \tilde{Z}).$$

The following two indices of the nullity of a submersion can be considered.

Definition 3.10. (1) The nullity-space of $h^L$ at a point $b \in \tilde{M}$ is defined by $N_b = \{\tilde{Y} \in T_b \tilde{M}: h^L(\tilde{Y}, \tilde{Z}) = 0 \forall \tilde{Z} \in T_b \tilde{M}\}$. The integer $\mu(h^L) = \min\{\dim N_b : b \in \tilde{M}\}$ is called the index of nullity of $h^L$.

(2) The nullity-space of $h_2$ at a point $b \in M$ is defined by $\tilde{N}_b = \{\tilde{Y} \in T_b \tilde{M}: h_2(\tilde{Y}, \tilde{Z}) = 0 \forall \tilde{Z} \in T_b \tilde{M}\}$. The integer $\mu(h_2) = \min\{\dim \tilde{N}_b : b \in \tilde{M}\}$ is called the index of relative nullity of $h_2$ for the submersion $\pi$.

The relation $\mu(h^L) = \dim \tilde{M}$ (resp., $\mu(h_2) = \dim \tilde{M}$) indicates the transversal totally geodesic (resp., a Riemannian) submersion $\pi$.

Theorem 3.16. Let $\pi: M \to \tilde{M}^p$ be a transversal curvature-invariant submersion of a compact Riemannian manifold $M$ with compact connected leaves $\{L\}$ that satisfies the following conditions: $\tilde{K} > 0$ and $\mu(h^L) > \frac{1}{2} p$. Then $h^L = 0$ and the submersion $\pi$ is transversal totally geodesic.

Theorem 3.17. Let $\pi: M \to \tilde{M}^p$ be a transversal totally geodesic submersion of a compact Riemannian manifold $M$ with compact connected leaves $\{L\}$. We assume that the transversal partial curvature $\tilde{\text{Ric}}_q > 0$ and the condition $\mu(h_2) \geq \frac{1}{2}(p + q - 1)$ holds. Then $\mu(h_2) = p$ and $\pi$ is a Riemannian submersion.
Chapter 4

FOLIATIONS AND SUBMANIFOLDS

4.1. Submanifolds with Generators in Riemannian Spaces

4.1.1. Ruled and (strongly) parabolic submanifolds. Ruled submanifolds in real space forms are studied in [15, 16, 12, 281-283, 175, 3, 71, 157, 182-183, 239, 241, 245, 260]; for \( M = CP^n \) or \( M = HP^n \), see [8, 180, 178, 153, 216, 278, 219, 325, 90]; for the Minkowski space, see [65] and [72].

Definition 4.1. A ruled submanifold is a \( C^2 \)-smooth submanifold \( M \subset \mathbb{R}^n \) (\( n > 0 \)) of a Riemannian space \( \mathbb{R}^n \) endowed with a \( C^1 \)-foliation \( \{ L \} \) having \( \nu \)-dimensional complete leaves (generators or rulings) that are totally geodesic in \( \mathbb{R}^n \). If \( M \) is a Kählerian manifold, \( M \) and \( \{ L \} \) are invariant, then \( M \) is a complex ruled submanifold. A ruled submanifold with stationary normal space \( TM^\perp \) along the rulings is said to be developable. If a developable ruled submanifold is locally isometric to the product \( L \times L^1 \), then it is called cylindrical.

It follows from (2.1) that vectors tangent to the rulings of a ruled submanifold are asymptotic, that is,

\[
h(x, x) = 0, \quad x \in TL,
\]

and the sectional curvature along rulings \( K_L = K_{|L} \). For a developable ruled submanifold, we have

\[
h(x, y) = 0, \quad x \in TL, \quad y \in TM.
\]

Hence the curvature \( K_{\text{mix}} \) coincides with the curvature of the ambient space and the rulings are tangent to relative nullity subspaces (see Sec. 2.4.1). Moreover, a Riemannian space of positive or negative curvature does not contain cylindrical ruled submanifolds (in the sense of Definition 4.1).

Submanifolds of constant nullity were introduced by Chern and Kuiper in 1952 and are also known as strongly parabolic (A. Borisenko), with constant rank (N. Janenko), tangentially degenerate (V. Ryzhkov, M. Akivis), and \( k \)-developable (A. Pogorelov, V. Toponogov, and S. Shefel). In 1972, Borisenko introduced a larger class of parabolic submanifolds.

Definition 4.2. The rank at the point \( m \in M \) of a submanifold \( M \subset \mathbb{R}^n \) is the integer \( r(m) = \max \{ r(\xi): \xi \in T_mM^\perp \} \), where \( r(\xi) \) is the rank of the second quadratic form \( A_\xi \) for the normal vector \( \xi \). We call \( r(M) = \max \{ r(m): m \in M \} \) the rank of \( M \).

A submanifold \( M \subset \mathbb{R}^n \) is

(a) strongly parabolic if \( \mu(M) > 0 \) [63];
(b) parabolic if \( r(M) < \dim M \) [34].

Since \( \ker h = \bigcap_\xi \ker A_\xi \) for any point \( m \in M \) (we can choose \( \text{codim} M \) normal vectors that form a basis in \( TM^\perp \)), it follows that \( \mu(M) \leq \dim M - r(M) \). For a totally geodesic submanifold, the relations \( \mu(M) = \dim M - r(M) = 0 \) hold. For a hypersurface, we have \( \mu(M) = \dim M - r(M) \). Obviously, a \( k \)-saddle submanifold \( M \subset \mathbb{R}^n \) (see Definition 3.7) is parabolic and \( r(M) \leq 2(k - 1) \). For a submanifold \( M \subset \mathbb{R}^n \), we have [63]

\[
\mu(M) \leq \nu(M) \leq \mu(M) + \text{codim} M,
\]

where \( \nu(M) \) (see Sec. 2.4.1) is the index of nullity of the curvature tensor.

Strongly parabolic submanifolds have a ruled structure under certain conditions imposed on the curvature tensor of the ambient space (see Sec. 2.4.1).

Theorem 4.1. [179]. Let \( M \subset \mathbb{R}^n \) be a (complete) submanifold satisfying condition (2.4) and \( \mu(M) > 0 \). Then the regularity domain \( \tilde{G} = \{ m \in M: \mu(m) = \mu(M) \} \) is foliated by (complete) totally geodesic submanifolds (rulings) in \( \mathbb{R}^n \) and the normal space to \( M \) is stationary along the leaves.

From Theorem 4.1 and Corollary 3.8, we obtain the following corollary.
Corollary 4.1 ([34] for $q = 1$). Let $M^n$ be a complete curvature-invariant submanifold in a Riemannian space $\tilde{M}^N$ satisfying the condition $\mu(M) < n$. Then

\[
\mu(M) < \begin{cases} 
\frac{1}{2}(n + q - 1) & \text{if } \text{Ric}^q(M) > 0 \text{ and } M \text{ is compact,} \\
\frac{1}{2}(N + q - 1) & \text{if } \text{Ric}^q(\tilde{M})|_M > 0 \text{ and } \tilde{M} \text{ is compact.} 
\end{cases}
\]

4.1.2. The Sasaki metric on a tangent and a normal bundle. A metric on a tangent bundle of a Riemannian manifold induced by a translation of tangent vectors was constructed in [263, 264].

Let $(x^i)$ be local coordinates on a Riemannian manifold $M$ with the metric tensor $ds^2 = g_{ij}dx^idx^j$. Then, for the induced local coordinates $(x^i, \xi^i)$ on the tangent bundle $TM$, the Sasaki metric is $ds^2 = g_{ij}dx^idx^j + g_{ij}D\xi^iD\xi^j$, where $D\xi^i = \partial \xi^i + \Gamma^i_{jk}\xi^j dx^k$ is a covariant derivative of the tangent vector field. In other words, if $d\theta$ is the angle between the tangent vector $\xi$ at the initial point $x$ and the vector $\xi + d\xi$ at the point $x + dx$ after the translation to the initial point along a minimal geodesic, then $ds^2 = ds^2 + |\xi|^2d\theta^2$ (see [44]). This construction can be applied to an arbitrary vector bundles over a manifold with connection. For more general constructions of the metrics on fiber bundles, see [84, 85].

Theorem 4.2 [44]. The index of nullity $\nu(TM)$ of the curvature tensor of a Sasaki metric is even. The manifold $M^n$ is the metric product $M_{\frac{n-\nu(TM)}{2}} \times \mathbb{R}^{\nu(TM)/2}$, and $TM^n$ is the metric product $TM_{\frac{n-\nu(TM)}{2}} \times \mathbb{R}^{\nu(TM)}$.

The proof is based on the existence of $\nu(TM)/2$ parallel linearly independent vector fields on $M$. The converse of this theorem is not true, i.e., the strong parabolicity of $M$ does not necessarily imply the strong parabolicity of a Sasaki metric on $TM$. The following hypothesis is studied in [317]: for a Sasaki metric on a tangent bundle $TM$, the index of (spherical) $k$-nullity (where $k > 0$) $\nu_k(M)$ is zero if dim $M \geq 3$. For a submanifold $M \subset \tilde{M}$, the following two (usually different) metrics on $TM$ can be considered: a Sasaki metric and a metric induced by a Sasaki metric on $\tilde{T}M$. For example, for a cylinder $M^2 \subset \mathbb{R}^3$, a Sasaki metric on $T(M^2)$ is flat, but the induced metric on $T(M^2)$ from $T(\mathbb{R}^3) = \mathbb{R}^6$ has a nonzero curvature.

Lemma 4.1 [44]. A Sasaki metric on $\tilde{T}M$ induces a Sasaki metric on $TM$ if and only if one of the following two equivalent properties holds:

1. $TM$ is a totally geodesic submanifold in $\tilde{T}M$;
2. $M$ is a totally geodesic submanifold in $\tilde{M}$.

The equivalence of properties (1) and (2) follows from the fact that the projection $p: TM \rightarrow M$ is a Riemannian submersion.

For $M \subset \mathbb{R}^n$, the tangent bundle $TM$ can be regarded as a submanifold in the Euclidean space $T\mathbb{R}^n = \mathbb{R}^{2n}$, which has an index of relative nullity $\mu(TM)$.

Theorem 4.3 [44]. For $M \subset \mathbb{R}^n$, we have $2\mu(M) \geq \mu(TM) \geq \mu(M)$. Moreover, $M$ is a cylindrical submanifold with $(\mu(TM) - \mu(M))$-dimensional ruling, and $TM$ is cylindrical with $2(\mu(TM) - \mu(M))$-dimensional ruling.

The proof of Theorem 4.3 is based on the analysis of the second quadratic forms of $TM$ in $T\tilde{M}$.

A vector field $\xi$ on a manifold $M^n$ can be identified with an $n$-dimensional submanifold $\xi(M)$ in $TM$. This submanifold $\xi(M)$ is totally geodesic in $TM$ if and only if $\xi$ is parallel; moreover, for a compact $M$, the submanifold $\xi(M)$ is minimal in $TM$ if and only if $\xi$ is parallel (see [44]). The volume of a unit vector field $\xi$ is an $n$-volume of the submanifold $\xi(M)$ calculated for a Sasaki metric on the unit tangent bundle $S_1M$, i.e.,

\[\text{Vol}(\xi) = \int_M \sqrt{\det(I + (\nabla \xi)^T(\nabla \xi))} \, d\text{vol},\]

where a covariant derivative $\nabla \xi$ is interpreted as a linear operator on $TM$. For example, a unit vector field of minimal volume on a round 3-sphere is tangent to leaves of the Hopf fibrations [108]. For more general
results concerning \( \text{Vol}(\xi) \), see [229]. Note that \( S_1 M \) has a constant mean curvature in \( TM \) and its volume \( \text{Vol}(S_1 M^n) = \text{Vol}(M^n) \cdot \text{Vol}(S^{n-1}) \).

The problem of classifying the totally geodesic (umbilic) submanifolds of \( TM \) and \( S_1 M \) when \( M \) is a space form or CROSS is very important. Another important problem is to estimate the codimension of an isometric embedding of \( TM \) with a Sasaki metric into a Euclidean space [44].

A Sasaki metric on the normal bundle \( TM^\perp \) of the submanifold \( M \subset \widetilde{M} \) was introduced by Borisenko (see [43] and [223]); it was used to study the extrinsic geometry of submanifolds in a Riemannian space. For induced local coordinates \( (x^i, \xi^i) \) on a normal bundle \( TM^1 \) of a submanifold \( M \subset \widetilde{M} \), a Sasaki metric is defined by \( ds^2 = g_{ij} dx^i dx^j + g_{ij} D^1 \xi^i D^1 \xi^j \), where \( D^1 \xi^i \) is a normal component of the covariant derivative in \( \widetilde{M} \) of a normal vector field \( (\xi^i) \) in a tangent direction [44].

Let \( \nu_H(TM^\perp) \) and \( \nu_V(TM^\perp) \) be the horizontal and the vertical (intrinsic) index of nullity of the curvature tensor of a Sasaki metric.

**Theorem 4.4** [43]. (a) There exist \( \nu_v(TM^\perp) \) linearly independent normal vector fields on the submanifold \( M \), which are parallel in a normal connection. (b) The submanifold \( M \) in \( \mathbb{R}^n \) is foliated by \( \nu_H(TM^\perp) \)-dimensional intrinsically flat submanifolds, which are totally geodesic in \( M \) with flat normal connection.

A normal vector field \( \xi \) on a submanifold \( M^n \subset \widetilde{M} \) can be regarded as the \( n \)-dimensional submanifold \( \xi(M) \) in \( TM^1 \) with Sasaki metric. If \( \xi \subset TM^1 \) is parallel in a normal connection, then the submanifold \( \xi(M) \) is totally geodesic in \( TM^1 \) if and only if \( M \) is a totally geodesic submanifold [316].

From the geometrical point of view, it is interesting to study not only the **spherical tangent bundle** \( S_p M \) but also the **spherical normal bundle** \( S_p M^\perp \), which contains vectors of fixed length \( p \) and is a hypersurface in a normal vector bundle. The formulas for the Riemannian, Ricci, or scalar curvature of Sasaki metrics on \( TM \) and \( TM^\perp \) (resp., \( S_p M \) and \( S_p M^\perp \)) are similar [44]. If \( K_M > 0 \), then the sectional curvature of \( S_p M^\perp \) is positive for sufficiently small \( p \).

The (strong) parabolicity of a given submanifold \( M \subset \widetilde{M} \) ensures the ruled structure of a Sasaki metric on \( TM^\perp \). Let \( \xi_0 \) be a normal vector at a point \( q_0 \) with maximal rank \( r(\xi_0) = r(M) \). Then the rank is constant for normal vectors \( \xi \) that are close to \( \xi_0 \). The kernel of \( \xi \) satisfies \( \sum_{i=1}^n \xi^i A^i_{ij} x^j = 0 \), where \( A^i_{ij} \) are the coefficients of the second quadratic forms with respect to the orthogonal basis of normal space. Since the rank of the above system is constant, the solution space \( L(q, \xi) \) depends regularly on the point and the normal vectors. Therefore, a horizontal lifting \( \tilde{L}(\tilde{q}) \) of the plane \( L(q, \xi) \) to the point \( \tilde{q} = (q, \xi) \) of the normal bundle forms a differentiable family on the neighborhood of the point \( \tilde{q}_0 = (q_0, \xi_0) \).

**Theorem 4.5** [43]. The horizontal distribution \( \tilde{L}(\tilde{q}) \) on the normal bundle of the parabolic submanifold \( M \) in the Riemannian space \( \widetilde{M} \) is integrable if, at points of \( M \), the curvature operator \( \tilde{R} \) of \( \widetilde{M} \) satisfies the condition

\[
\tilde{R}(x, y) \xi \left\{ \begin{array}{lc}
0, & (x, y \in TM, \xi \in TM^\perp), \\
\parallel J \xi, & (x, y \in TM, \xi \in TM^\perp) \end{array} \right.
\] (4.4)

The fibers are totally geodesic submanifolds of the normal bundle with a Sasaki metric, and their projections on \( M \) are totally geodesic submanifolds of \( \widetilde{M} \).

The integrability of the lift of a distribution from a Riemannian submanifold to its tube with a not necessarily constant radius is studied in [159].

**Corollary 4.2** [43]. Let \( M^n \subset \widetilde{M} \) be a (complete) submanifold satisfying conditions (4.4), and let \( 0 < r(M) < n \). Then, for any normal vector \( \xi \in T_p M \) with maximal rank \( r(\xi) = r(M) \), there exists a (complete) ruling \( L(\xi) \subset M \) (\( T_p L(\xi) = \ker A_\xi \)) along which the normal vector \( \xi \) is stationary, i.e., \( \xi \) remains normal under \( \tilde{\nabla} \)-parallel displacement along any geodesic in \( L(\xi) \).
A parabolic submanifold of small codimension is strongly parabolic [33], that is,
\[ \mu(M) \geq \dim M - \frac{1}{2} r(M)(\text{codim} M + 1). \]
The following result completes Corollary 4.1.

**Corollary 4.3** ([37] for \( q = 1 \)). Let \( M^n \) be a complete not totally geodesic submanifold in a Riemannian space \( \bar{M}^N \) satisfying conditions (4.4) and \( r(M) > 0 \).

Then \( r(M) > \begin{cases} \frac{1}{2}(n - q + 1) & \text{if } \text{Ric}^q(M) > 0 \text{ and } M \text{ is compact}, \\ \frac{1}{2}(N - q + 1) & \text{if } \text{Ric}^q(M)|_M > 0 \text{ and } \bar{M} \text{ is compact}. \end{cases} \)

### 4.2. Submanifolds with Nonpositive Extrinsic \( q \)-Dimensional Ricci Curvature

#### 4.2.1. Radius of a submanifold and nonexistence of immersions

For a symmetric bilinear map \( h: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p \) and an orthonormal system of \( q + 1 \) vectors \( \{x_i\}_{0 \leq i \leq q} \subset \mathbb{R}^n \), we define the extrinsic \( q \)-dimensional Ricci curvature by
\[
\text{Ric}^q_h(x_0; x_1, \ldots, x_q) = \sum_{i=1}^q \left( h(x_0, x_0), h(x_i, x_i) \right) - h^2(x_0, x_i),
\]
where \( (, ,) \) is the inner product in \( \mathbb{R}^p \). \( \text{Ric}^q_h \) is called an extrinsic sectional curvature (see [33, 34]). In view of the relation
\[
\text{Ric}^q_h(x_0; x_1, \ldots, x_q) = \sum_{i=1}^q \text{Ric}^q(x_0, x_i),
\]
the inductive formula similar to (3.70),
\[
\text{Ric}^{q+1}_h(x_0; x_1, \ldots, x_{q+1}) = \frac{1}{q} \sum_{i=1}^q \text{Ric}^q_h(x_0; x_1, \ldots, \hat{x}_i, \ldots, x_{q+1}),
\]
holds, where the symbol \( \hat{\ } \) means the absence of the corresponding vector. Therefore, for \( q \in \{1, n-1\} \), it follows from \( \text{Ric}^q_h \leq 0 \) (\( \text{Ric}^q_h \equiv 0 \)) that \( \text{Ric}^{q+1}_h \leq 0 \) (resp., \( \text{Ric}^{q+1}_h \equiv 0 \)).

**Lemma 4.2** ([211] for \( q = 1 \)). Let \( h: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p \) be a symmetric bilinear map, and let the following inequalities hold for some real \( c > 0 \):
\[
\text{Ric}^q_h \leq qc^2, \quad |h(x, x)| > cx^2 \quad (x \neq 0).
\]
Then \( p > n - q \).

From Lemma 4.2, it follows that the symmetric bilinear map \( h: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p \) satisfying the conditions \( p \leq n - q \) and \( \text{Ric}^q_h \leq 0 \) has an asymptotic vector. Lemma 4.2 can be extended to the Hilbert case \( n = \infty \); see [199] for \( q = 1 \).

For a submanifold \( M \subset \bar{M} \) with second fundamental form \( h \) and orthonormal system of \( q + 1 \) vectors \( \{x_i\}_{0 \leq i \leq q} \subset T_mM \), we have
\[
\text{Ric}^q(x_0; x_1, \ldots, x_q) = \text{Ric}^q_h(x_0; x_1, \ldots, x_q). \quad (4.5)
\]
In particular, for a submanifold \( M \) in a Euclidean space, we have \( \text{Ric}^q = \text{Ric}^q_h \).

It is proved in [190] that if \( \bar{M} \) is a complete simply connected Riemannian space with sectional curvature \( a \leq K \leq b \leq 0 \) and \( M \) is a compact Riemannian manifold with \( K \leq a - b \), then \( M \) does not admit an isometric immersion in \( \bar{M} \) unless \( \dim M \geq 2 \dim M \). On the other hand, an isometric immersion of an \( n \)-dimensional compact Riemannian manifold with sectional curvature less than \( \frac{1}{d^2} \) into \( \mathbb{R}^{2n-1} \) can never be contained in a ball of radius \( d \) [142]. In [141], the Otsuki lemma (i.e., Lemma 4.2 for \( q = 1 \)) is used and a nonembedding
A theorem that generalizes these results is proved. Theorem 4.6 below generalizes the result of [141] to the case of partial Ricci curvature.

The positive continuous function $C(b, d)$, $b \leq 0$, $d > 0$, is defined as in [141]:

$$C(b, d) = \begin{cases} \frac{1}{2} \sqrt{-b} \ coth (d \sqrt{-b}) & \text{if } b < 0, \\ \frac{1}{2} & \text{if } b = 0. \end{cases}$$

This function is monotonically decreasing with respect to both $b$ and $d$; obviously, $C^2(b, d) > -b$. Let $M$ be a compact submanifold in $\tilde{M}$ with distance function $\tilde{d}$. We denote by $B(x, r) = \{y \in \tilde{M} : \tilde{d}(x, y) \leq r\}$ a closed ball and assume that

$$r(M) = \inf \{r : M \subseteq B(x, r)\} = \inf \{\max\{\tilde{d}(x, y) : y \in M\}, x \in \tilde{M}\}.$$ There exists a point $x_0 \in \tilde{M}$ such that $M \subseteq B(x_0, r(M))$. Moreover, there exists a point $y_0 \in M$ such that $\tilde{d}(x_0, y_0) = r(M)$. $r(M)$ is called the radius of $M$ in $\tilde{M}$, and $B(x_0, r(M))$ is the minimum ball containing $M$ [141]. Generally speaking, there can be several minimal balls containing $M$. For example, there are two balls (hemispheres) when $M$ is a large circle in a 2-sphere $\tilde{M}$. However, there is only one minimal ball for a compact manifold that is immersed in a Euclidean space [141].

**Theorem 4.6** ([141] for $q = 1$). Let $\tilde{M}^{n+p-q}$ be a complete simply connected Riemannian space with sectional curvature $\tilde{K} \leq b < 0$, and let $d > 0$ be a constant. Let $M^n$ be a compact Riemannian manifold such that at every point of $M^n$ there exists a $p$-dimensional subspace in the tangent space for which the inequality

$$\frac{1}{q} \text{Ric}_M^a \leq a + C^2(b, d)$$

(4.6)

holds. Then $M^n$ does not admit an isometric immersion into $\tilde{M}^{n+p-q}$ in a ball of radius $<d$.

**Corollary 4.4** ([63] for $q = 1$). (a) Let $M^n$ be a compact Riemannian manifold with $\text{Ric}_M^a \leq 0$. Then $M^n$ cannot be isometrically immersed into $\mathbb{R}^{2n-q}$. (b) Let $M^n$ be a Riemannian manifold with $\text{Ric}_M^a < 0$. Then $M^n$ cannot be isometrically immersed into $\mathbb{R}^{2n-q-1}$.

The product $M^{2n} = M^n(-1) \times M^n(-1)$ of hyperbolic space forms has $\text{Ric}_M^{a+1} \leq -1$. By Corollary 4.4, $M^{2n}$ cannot be isometrically locally immersed into $\mathbb{R}^{2n-2}$. Corollary 4.4 can be improved in the case of embeddings into a cylinder of a Euclidean space.

**Definition 4.3.** A hypersurface $C(s, r)$ in $\mathbb{R}^{N+1}$ that is congruent to $\sum_{i=1}^{s+1}(x_i)^2 - r^2 = 0$, where $s \leq N$, is a circular cylinder of radius $r$ with an $s$-dimensional parallel and an $(N-s)$-dimensional ruling (or axis). (For $s = N$, we obtain a hypersphere of radius $r > 0$.) A hypersurface $C(r)$ in $\mathbb{R}^{N+1}$ that is congruent to $\sum_{i=1}^{N}(x_i)^2 - r^2(x_{N+1})^2 = 0$, $x_{N+1} \geq 0$, is an $n$-dimensional circular cone.

The system $\sum_{i=1}^{N}(x_i)^2 = r^2d^2$, $x_{N+1} = d$, defines a parallel of the cone $C(r)$ at the level $d$. The projection of the point $p \in \mathbb{R}^{N+1}$ onto the axis of the cone $C(r)$ (the $(N+1)$th coordinate of $p$ in the canonical coordinate system) is called a level of the point $p$ with respect to the cone $C(r)$. We say that the submanifold $M^n$ in $C(r)$ has the level $d$ if $d$ has minimum value along all the levels (with respect to $C(r)$) of points $m \in M^n$. Note that the submanifold $M^n$ in $C(r)$ does not contain straight lines of a Euclidean space; hence, the rank of its second quadratic forms $r(M) = n$. Hypersurfaces in a cone of a Euclidean space are studied in [189].

**Theorem 4.7.** Let $M^n$ be a compact manifold satisfying the condition $\text{Ric}_M^a \leq qe^2$ for some constant $c > 0$. Then an isometric immersion of $M^n$ into $\mathbb{R}^{n+p}$ ($p \leq n - q$) cannot be inside a cylinder of radius $r = \frac{1}{c}$ with a $(2p + q - 1)$-dimensional parallel.

The following theorem is dual to Corollary 4.4.
Theorem 4.8. (a) Let a complete Riemannian manifold \( M^n \) satisfying the condition \( \text{Ric}_M \leq qc^2 \), where \( q \in [1, n-1] \) and \( c = \text{const} > 0 \), be isometrically embedded in \( \mathbb{R}^{2n-q} \) inside the cone \( C(r) \). Then the level of \( M^n \) in \( C(r) \) is not less than \( (\text{tr} R - 1/r - 1)/c \).

(b) Let \( M^n \subset \mathbb{R}^{2n-q-1} \) be a complete submanifold, where \( q \in [1, n-2] \), in a cone \( C(r) \), and let for some constant \( c > 0 \) and any vectors in \( TM^n \), which are orthogonal to the axis of the cone, the condition \( \text{Ric}_M \leq qc^2 \) hold. Then the level of \( M^n \) in the cone is greater than \( \frac{1}{rc} \).

(c) A complete Riemannian manifold \( M^n \) satisfying \( \text{Ric}_M \leq 0 \) for some \( q \in [1, n-1] \) cannot be isometrically immersed into the cone \( C(r) \) of the Euclidean space \( \mathbb{R}^{2n-q} \).

Another generalization \( \bar{\gamma}_s \) of the notion of extrinsic sectional curvature (as the extrinsic analog of the \( s \)-dimensional sectional curvature \( \gamma_s \), in particular, the Lipschitz–Killing curvature of Riemannian manifold) is given in [33].

Definition 4.4 [33]. For an \( s \)-dimensional subspace \( V \subset T_p M \) with an orthonormal basis \( \{u_1, \ldots, u_s\} \), the number \( \bar{\gamma}_s(V) \) is defined by the formula

\[
\bar{\gamma}_s(V) = \frac{1}{2s!k_s} \left( \sum_{i,j \in S_s} \varepsilon_i \varepsilon_j \left[ (h(u_{i1}, u_{j1}), h(u_{i2}, u_{j2})) - (h(u_{i1}, u_{j2}), h(u_{i2}, u_{j1})) \right] \right),
\]

where \( S_s \) is the set of permutations of order \( s \) and \( \varepsilon_i \) is the sign of the permutation \( i = (i_1, \ldots, i_s) \). The number \( \bar{\gamma}_s(M) = \sup \{ \bar{\gamma}_s(V) : V \subset TM, \dim V = s \} \) is called the extrinsic \( s \)-dimensional curvature of \( M \).

Obviously, the curvature \( \bar{\gamma}_2 \) is equal to \( \text{Ric}_h \). If the ambient space is Euclidean, then \( \bar{\gamma}_2 \) coincides with the intrinsic curvature \( \gamma_s \).

Below, we give a generalization of the Otsuki lemma in a way different from that of Lemma 4.2.

Lemma 4.3 [33]. Let \( h: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p \) be a symmetric bilinear map of Euclidean spaces satisfying the property \( \bar{\gamma}_s(h) \leq 0 \). If \( p < \frac{n-1}{s-1} \), then \( h \) has an asymptotic vector.
4.2.2. The index of relative nullity and extremal theorems.

Lemma 4.4 ([91] for \( q = 1 \)). Let \( h: \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p \) be a symmetric bilinear map of Euclidean spaces with \( \text{Ric}^h \leq 0 \). Then \( \mu(h) \geq n - 2p - q + \delta_{1q} \) and there exists an asymptotic subspace \( T \subset \mathbb{R}^n \) of \( h \) such that \( \dim T \geq n - p - q + \delta_{1q} \).

Lemma 4.4 is a bridge from submanifolds with nonpositive extrinsic curvature to ruled and parabolic submanifolds, since it gives us an estimate of the index of relative nullity \( \mu(M) \) of the submanifold \( M^n \subset \tilde{M}^{n+p} \) satisfying \( \text{Ric}^h \leq 0 \) and with "small" codimension (for \( q = 1 \), see [91] and [33]): \( \mu(M) \geq n - 2p - q + \delta_{1q} \).

It is shown in [69] that any isometric immersion of a Kähler manifold into a complex space form \( \tilde{M}^N(c), \ c \neq 0 \), with a positive index of relative nullity should be holomorphic. From the proof of this theorem and Lemma 4.4, we obtain the following:

Corollary 4.5 ([91] for \( q = 1 \)). Let \( M^{2n} \) be a Kähler manifold, and let a point \( m_0 \in M \) be such that \( \text{Ric}^h(m_0) \leq qc \), where \( c > 0 \) is a constant. Then there is no isometric immersion of \( M^{2n} \) into the real space form \( \tilde{M}^{2m+p}(c) \) for \( p < n - q + \delta_{1q} \).

From [1] and [69], it follows that any isometric immersion of a complete Kähler manifold into \( CP^N \) with a positive index of relative nullity should be totally geodesic. From the above results and Lemma 4.4, we obtain

Corollary 4.6 ([86] and [91] for \( q = 1 \)). Let \( f: M^n \to S^{n+p} \) (or \( f: M^{2n} \to CP^{n+p} \)) be an isometric immersion of a complete Riemannian (resp., Kähler) manifold with \( \text{Ric}^h \leq 0 \). If \( 2p < n - \nu(n) - q + \delta_{1q} \) (resp., \( 2p < n - q + 1 \)), then \( f \) is a totally geodesic embedding.

The next theorem follows from Lemma 4.4, Theorem 4.1, and Corollary 4.1 (see also Theorem 3.1).

Theorem 4.9. Let \( M^n \subset \tilde{M}^{n+p} \) be a complete curvature-invariant submanifold with \( \text{Ric}^h \leq 0 \). Then \( M \) is a totally geodesic submanifold if any one of the following conditions holds:

1. \( M \) is compact with \( \text{Ric}^h(M) > 0 \) and \( 4p \leq n - s - 2q + 2\delta_{1q} \),
2. \( \tilde{M} \) is compact with \( \text{Ric}^h(\tilde{M})_{|M} > 0 \) and \( 5p \leq n - s - 2q + 2\delta_{1q} \),
3. \( 2p < n - \nu(n) - q + \delta_{1q} \) and for some \( k = \text{const} > 0 \)

\[ \tilde{R}(x,y)x = -ky(x,x), \quad x, y \in TM. \tag{4.7} \]

See Corollary 3.1 (b) for a submanifold \( M \) in the sphere \( S^{n+p}(k) \) for which condition (3) holds with \( q = 1 \); for a submanifold satisfying the stronger (than (2.4)) condition (4.4) with \( q = s = 1 \), see [37]. A similar result in the case where (3) is assumed for submanifolds in a Hilbert hypersphere is true without any additional assumptions on their finite codimension; for \( q = 1 \), see [199].

Theorem 4.10 directly follows from Theorem 4.9.

Theorem 4.10. Let \( M^n \subset \tilde{M}^{n+p} \) be a compact, simply connected, curvature-invariant submanifold. Then \( M^n \) is a totally geodesic submanifold isometric to a unit sphere if one of the following properties holds:

1. \( \text{Ric}^h(M) \leq s \leq \text{Ric}^h(\tilde{M})_{|M} \) for some \( s < n - 1 \) and \( 2p < n - \nu(n) - s + \delta_{1s} - 1 \),
2. \( K(\tilde{M})_{|M} \equiv 1, \ K_M \leq 1, \ \text{inj}(M) \geq \pi, \text{ and } 2p < n - 1. \)
The property $\text{inj}(M) \geq \pi$ in the case (2) follows from the inequality $K_M > 0$ if $n$ is even and from the inequality $K_M \geq \frac{1}{4}$ when $n$ is odd (see Sec. 2.6).

If the curvature of the manifold $\tilde{M}$ is subjected to stronger restrictions, then we obtain the following extremal theorem.

**Theorem 4.11.** Let $M^n$ be a compact curvature-invariant submanifold in a simply connected Riemannian space $\tilde{M}^{n+p}$ satisfying the conditions

\[
\frac{9}{4} \geq K_{\tilde{M}} \geq 1, \quad \text{Ric}^*(M) \leq s \text{ for some } s < n - 1 \text{ and } 2p \leq n - s - 2 + \delta_1.
\]

Then $\tilde{M}^{n+p}$ is isometric to a unit sphere.

The next theorem follows directly from Theorems 4.10 and 4.11.

**Theorem 4.12.** Let $M^n$ be a compact curvature-invariant submanifold in a simply connected Riemannian space $\tilde{M}^{n+p}$ with $\frac{9}{4} \geq K_{\tilde{M}} \geq 1$, and let one of the following conditions hold:

1. $\text{Ric}^*(M) \leq s$ for some $s < n - 1$, and $2p < n - \nu(n) - s + \delta_1$,
2. $K_M \leq 1$, $\text{inj}(M) \geq \pi$, and $2p < n - 1$.

Then $\tilde{M}^{n+p}$ is isometric to a unit sphere and $M^n$ is totally geodesic.

Theorems 4.9-4.12 generalize Theorems 3–6 in [37], which were first obtained in [34] for submanifolds in CROSS.

### 4.3. Submanifolds with Generators in Euclidean and Lobachevsky Spaces

#### 4.3.1. Submanifolds with generators in a Euclidean space

It is important to study the relationships between classes of submanifolds with degenerate second fundamental form (strongly parabolic, cylindrical) and metrics with degenerate curvature tensor (for example, with a positive index of nullity, products of manifolds). There is a natural commutative diagram, where the arrows indicate (exact) inclusion maps:

\[
\begin{align*}
\text{strongly parabolic submanifolds} & \quad \text{(1)} \quad \text{strongly parabolic metrics} \\
\text{cylindrical submanifolds} & \quad \text{(3)} \quad \text{cylindrical metrics} \\
\text{strongly parabolic submanifolds} & \quad \text{(2)} \quad \text{strongly parabolic metrics} \\
\text{cylindrical submanifolds} & \quad \text{(4)} \quad \text{cylindrical metrics}
\end{align*}
\]

(1). For a submanifold $M^n \subset \mathbb{R}^N$, the inequality $\nu(M) \geq \mu(M)$ holds with equality for hypersurfaces. The problem on conditions under which a Riemannian manifold $M^n$ admits an isometric immersion into $\mathbb{R}^N$ with the equality $\nu(M) = \mu(M)$ attained is studied in [42]; the following results have been obtained for the basic case of 3-dimensional manifolds with nullity 1.

(a) There exist 3-dimensional analytic metrics with $\nu(M) = 1$ that do not admit an isometric immersion with $\mu(M) = 1$ into $\mathbb{R}^N$, for instance,

\[
g_{11} = \exp(2x_2) + x_2^2 + x_3^2, \quad g_{12} = x_3, \quad g_{13} = -x_2, \quad g_{22} = g_{33} = 1, \quad g_{23} = 0.
\]

For the above metric, the sectional curvature on planes that are orthogonal to a 1-dimensional nullity foliation is $-1$.

(b) The class of 3-dimensional analytic metrics with $\nu(x) \equiv 1$ that admit isometric immersions with $\mu(M) = 1$ into $\mathbb{R}^N$ for $N = 4$ (and for $N > 4$) depends on two functions of two variables. (The class of 3-dimensional analytic metrics with $\nu(x) \equiv 1$ depends on three functions of two variables. Also recall the results of A.N. Kolmogorov, V.I. Arnold, and others on the possibility of representing functions as compositions of functions of a fewer number of variables.)

(2). The distinction of the congruence classes of isometric immersions $f$ of a connected Riemannian manifold $M^n$ into the Euclidean space $\mathbb{R}^{n+1}$ for $n \geq 3$ is the classical rigidity problem for hypersurfaces. In this case, the type number of $M$ at $x$, denoted by $t(x)$, coincides with the rank of the Gaussian map of $f$ and the rank of the second quadratic form $r(x)$. It is well known from the Beez–Killing theorem that $f$ is rigid if $r(M) \geq 3$ and highly deformable in the flat case $r(M) \leq 1$. The situation for the constant $r(M) = 2$ (in this 1754
case $M$ has a developable ruled structure with $(n - 2)$-dimensional ruling) is much more complicated. In [267] and [57], a detailed local analysis is given; the deformations constitute either a discrete set or a 1-parameter family or an infinite-dimensional family in the ruled case $n \geq 4$. The compact case was studied in [262]. In [68], Sacksteder's theorem is extended as follows.

**Theorem 4.13** [68]. Let $f: M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be an isometric immersion of a complete Riemannian manifold that does not contain an open set $M^3 \times \mathbb{R}^{n-3}$ with an unbounded $M^1$. Then $f$ admits (nondiscrete) isometric deformations only along ruled strips (i.e., ruled domains in $M^n$ with $(n - 1)$-dimensional rulings). Furthermore, if $f$ is nowhere completely ruled and the set of totally geodesic points does not disconnect $M$, then $f$ is rigid.

If the ambient space is the round sphere $S^{n+1}$, then the local rigidity problem is reduced to a special aspect of the Euclidean case by considering cones (see [267]). Complete submanifolds $M^n$ are always rigid for $n \geq 4$ but not for $n = 3$ (see [68]). For similar problems in terms of the (degenerate) Grassmannian image of a submanifold in a Euclidean space, see [40].

**Example 4.1** [261]. A hypersurface $M^3 \subset \mathbb{R}^4$ given by the equation $x_4 = x_3 \cos(x_1) + x_2 \sin(x_1)$ has an index of relative nullity $\mu(x) = 1$ but is not a cylinder.

**Theorem 4.14** [3]. Let $M^{n+1}$ be a hypersurface in the Euclidean space $\mathbb{R}^{n+2}$ satisfying the following condition:

\begin{equation}
(a) \text{Then } M \text{ is foliated by } (n - 1)\text{-dimensional rulings.}
\end{equation}

(b) Let $m$ be a point in $M$ through which at least two $n$-dimensional rulings pass. Then there exists an open neighborhood $U$ of $m$ in $M$ that is the Riemannian product of the ruled surface $N^2$ by $\mathbb{R}^{n-2}$. Moreover, if two $n$-dimensional rulings pass through every point of $M$, then $M$ is the product of $\mathbb{R}^{n-2}$ by a double-ruled surface $N^2$ in $\mathbb{R}^3$.

It follows from condition (4.9) that $\mu(M) \geq n - 2$. Note that a minimal hypersurface $M^n$ in $\mathbb{R}^{n+1}$ that admits a foliation by Euclidean $(n - 1)$-planes is either totally geodesic or is the product $M^3 \times \mathbb{R}^{n-2}$, where the surface $M^2$ is the standard helicoid in $\mathbb{R}^3$ [27].

**Example 4.2** [3]. The hypersurface $M^3$ of $\mathbb{R}^4$ given by

\begin{equation}
\bar{f}(u, v, w) = (u \cos w, u \sin w + \cos w, w + v \sin w, w)
\end{equation}

is foliated by planes ($\bar{f}$ is linear with respect to $u, v$), but is not a product of a surface by $\mathbb{R}^1$ (the leaves tangent to ker $h$); the 2-dimensional distribution (ker $h$) is not integrable.

(3) The cylinder theorems for strongly parabolic embeddings into a Euclidean space with additional conditions on the sign of the curvature were obtained first in [201] for $K_M = 0$ and then in [127] for $K_M \geq 0$, and later in [6] for $\text{Ric}_M \geq 0$. For more general results, see Sec. 4.3.1.

**Theorem 4.15** [36]. Let $M^n \subset \mathbb{R}^N$ be a complete submanifold of constant index of relative nullity, and let $0 < r(M) < n$. Then $M$ is cylindrical with an $(n - r(M))$-dimensional ruling.

Cylinder theorems for parabolic submanifolds of "small" rank $r(M) \leq 3$ and with additional conditions for an asymptotic subspace, but without any assumption concerning the sign of the curvature, were also obtained in [36].

**Definition 4.5** [36]. The subspace $A_m \subset T_m M$ of maximal dimension that consists of asymptotic vectors is called asymptotic. The dimension of an asymptotic subspace is called the order of planarity at the point $m$.

**Theorem 4.16** [36]. Let $M^n \subset \mathbb{R}^N$ be a complete submanifold of constant index of relative nullity, and let $\text{rank } r(M)$ be equal to (a) $2$, (b) $3$; assume that the order of planarity at each point is equal to (a) $n - 2$, (b) $n - 3$, respectively. Then $M^n$ is cylindrical with an (a) $(n - 2)$-, (b) $(n - 3)$-dimensional ruling, respectively.
The cylindricity of embeddings of metric products \( M_1 \times M_2 \) into \( \mathbb{R}^N \) of small codimension are considered in [189, 10, 7].

Let \( M^n = M^n_1 \times M^n_2, \ n_1, n_2 \geq 2, \) be a product of two connected complete Riemannian manifolds. If none of the \( M_i \) is everywhere flat or contains a Euclidean strip (i.e., an open submanifold that is isometric to the product \( I \times \mathbb{R}^{n_i-1} \), where \( I \subset \mathbb{R} \) is an open interval), then any isometric immersion \( f: M^n \to \mathbb{R}^{n+2} \) is a product of hypersurface immersions [10, 189]. This means that there exist an orthogonal factorization \( \mathbb{R}^{n+2} = \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_2 \) and isometric immersions \( f_i: M^n_i \to \mathbb{R}^{n+1} \) (\( i = 1, 2 \)) such that \( f(x_1, x_2) = (f_1(x_1), f_2(x_2)) \). This remarkable global theorem has been proved for any number of factors whenever the codimension equals that number. In [17], the problem on conditions under which \( f: M^n = M^n_1 \times M^n_2 \to \mathbb{R}^{n+2} \) is not a product of hypersurface immersions is studied.

**Theorem 4.17** [17]. Let \( f: M^n = M^n_1 \times M^n_2 \to \mathbb{R}^{n+2} (n_i \geq 2) \) be an isometric immersion of a complete connected Riemannian manifold, where none of the factors is everywhere flat. Then there exists a dense open subset each of whose points lies in the neighborhood of the product \( U = U_1 \times U_2 \) with open \( U_i \subset M_i \) such that \( f_U: U_1 \times U_2 \to \mathbb{R}^{n+2} \) is one of the following types:

(i) \( f_U \) is a product of immersions;

(ii) each \( U_i \) is isometric to \( I_i \times \mathbb{R}^{n_i-1} \) (\( i = 1, 2 \)) and \( f_U = g \times \text{Id} \), where \( g: I_1 \times I_2 \to \mathbb{R}^4 \) is an isometric immersion and \( \text{Id}: \mathbb{R}^{n-2} \to \mathbb{R}^{n-2} \) is the identity map;

(iii) only one of the \( U_j \) is isometric to \( I_j \times \mathbb{R}^{n_j-1} \) and \( f_U = \tilde{f} \times \text{Id}: (U_i \times I_j) \to \mathbb{R}^{n+2} \) (\( i \neq j \)), where \( \text{Id}: \mathbb{R}^{n_i-1} \to \mathbb{R}^{n_i-1} \) is the identity map and \( \tilde{f}: U_i \times I_j \to \mathbb{R}^{n+3} \) is the composition \( \tilde{f} = h \circ g \) of isometric immersions \( g: U_i \times I_j \to V \) and \( h: V \to \mathbb{R}^{n+3} \), where the set \( V \subset \mathbb{R}^{n+2} \) is open.

For type (i), either we have a product of hypersurfaces or one of the factors is totally geodesic. Types (ii) and (iii) are not necessarily different, since the immersion \( g \) in (ii) may, in fact, be a composition. A complete local classification of flat surfaces in \( \mathbb{R}^4 \) that are nowhere compositions of isometric immersions is given in [56]. The example in [10] relates immersions of types (i) and (ii). The case where one of the factors of \( M \) is everywhere flat is also studied.

**Definition 4.6** [17]. An isometric immersion \( f: N^{n+m} \to \mathbb{R}^N \) is an \( m \)-cylinder if there exists a Riemannian manifold \( M^n \) such that \( N^{n+m}, \mathbb{R}^N \) and \( f \) have the orthogonal factorizations \( N^{n+m} = M^n \times \mathbb{R}^m, \mathbb{R}^N = \mathbb{R}^{n-m} \times \mathbb{R}^m, \) and \( f = \tilde{f} \times \text{Id}, \) where \( \tilde{f}: M^n \to \mathbb{R}^{n-m} \) is an isometric immersion and \( \text{Id}: \mathbb{R}^m \to \mathbb{R}^m \) is the identity map.

Recall the characterization of complete cylinders due to Hartman [127]:

**Lemma 4.5.** Let \( f: M^n \to \mathbb{R}^N \) be an isometric immersion of a connected complete Riemannian manifold satisfying the condition \( \text{Ric}_M \geq 0 \) such that \( f(M) \) contains \( m \) linearly independent lines passing through one point. Then \( f \) is an \( m \)-cylinder.

**Theorem 4.18** [17]. Let \( M^n \) be a complete connected Riemannian manifold satisfying the condition \( \text{Ric}_M \geq 0 \), without flat points, and let \( f: M^n \times \mathbb{R}^m \to \mathbb{R}^{n+m+2} \) be a \( C^1 \)-regular isometric immersion. Then \( f \) is either an \( m \)-cylinder or an \((m-1)\)-cylinder \( f = \tilde{f} \times \text{Id}: (M^n \times \mathbb{R}) \times \mathbb{R}^{n-1} \to \mathbb{R}^{n+m+2}, \) and there exist a flat Riemannian manifold \( N^{n+2} \) and isometries \( g: M^n \times \mathbb{R} \to N^{n+2} \) and \( h: N^{n+2} \to \mathbb{R}^{n+3} \) such that \( \tilde{f} = h \circ g \) is a composition. Furthermore, if \( M^n \) is simply connected, in the latter case we can consider \( N^{n+2} \) as an open subset of \( \mathbb{R}^{n+2}, \) and then \( g = \tilde{g} \times \text{Id}, \) where \( \text{Id}: \mathbb{R} \to \mathbb{R} \) is the identity map and \( \tilde{g}: M^n \to \mathbb{R}^{n+1} \) is an embedding whose image is a convex hypersurface.

Theorem 4.18 is not true without the assumption on the 1-regularity (see [17] and [133]). For \( m = 1, \) a weaker result, but without the 1-regularity assumption, is given in [198].
A smooth ruled submanifold \( M^{r+n} \subset \mathbb{R}^N \) can be given (locally) in parametric form by

\[
\mathbf{r}(u_1, \ldots, u_n; v_1, \ldots, v_\nu) = \mathbf{p}(u_1, \ldots, u_n) + \sum_{i=1}^{\nu} v_i \mathbf{a}_i(u_1, \ldots, u_n),
\]

(4.11)

where \( \{\mathbf{a}_i(u_1, \ldots, u_n)\} \) are bases of rulings and \( \mathbf{p}(u_1, \ldots, u_n) \) is the base submanifold. In particular, for a cylindrical submanifold, the vectors \( \{\mathbf{a}_i\} \) are constant. A one-parameter family of rulings corresponds to a submanifold \( M^{r+1} \subset M \) of the form

\[
\mathbf{r}(u_1, \ldots, u_n; t) = \mathbf{r}(u_1(t), \ldots, u_n(t); v_1, \ldots, v_\nu),
\]

(4.12)

where \( \{u_i(t)\} \) is a smooth curve in \( \mathbb{R}^n \) [41]. The Gauss equation (2.3a) for a ruled submanifold \( M^{r+n} \subset \mathbb{R}^N \) yields \( K_{\text{mix}} \leq 0 \). Therefore, if \( M^{r+1} \) is a ruled submanifold in the Euclidean space \( \mathbb{R}^N \) with zero scalar curvature, then \( M \) is developable.

**Theorem 4.19** [41]. *Let \( M^{r+n} \) be a ruled submanifold in \( \mathbb{R}^N \). Then \( M \) is cylindrical if and only if all 1-parameter families of rulings have zero scalar curvature.*

Note that the developable ruled submanifold \( M^{r+n} \subset \mathbb{R}^N \) with the integrable distribution \( TL^\perp \) is cylindrical.

### 4.3.2. Submanifolds with generators in Lobachevsky space

Foliations (fibrations) on the Lobachevsky space \( \mathbb{H}^n \) were studied in [47, 154, 75, 115, 323, 19]. Ruled submanifolds in \( \mathbb{H}^n \) were studied in [6, 196, 87, 283, 39, 173, 16, 15, 11, 52, 47, 18].

There are many strongly parabolic submanifolds in \( \mathbb{H}^n \); recall that they have the ruled structure. Any strongly parabolic submanifold in the Euclidean space \( \mathbb{R}^n \) with singular points outside a unit ball corresponds (by using the Cayley–Klein interpretation) to a complete strongly parabolic submanifold in \( \mathbb{H}^n \). In [283], the nondeveloped ruled submanifolds \( M^{r+n} \subset \mathbb{H}^n \) with \( \nu \)-dimensional ruling are studied. In [6], the problem on classification of all isometric immersions of a hyperbolic space form \( \mathbb{H}^{n-1} \) into another hyperbolic space form \( \mathbb{H}^n \) is treated. Since the rank of the shape operator \( A \) is at most one, the subset \( U = \{ x \in \mathbb{H}^n : \text{rank}(A) = 1 \} \) consists of at most a countable number of open connected components in \( \mathbb{H}^n \) each of which is \( C^\infty \)-foliated by complete totally geodesic hypersurfaces (a relative nullity foliation) \( \{L_\alpha\}_{\alpha \in A} \).

**Definition 4.8** [39]. Let \( A \subset \mathbb{H}^n \) be either a horosphere or an equidistant submanifold or a totally geodesic submanifold in \( \mathbb{H}^{n+p-1} \), and let \( F^{l-v} \) be a complete submanifold in \( \mathbb{H}^{n+p-\nu} \). The union of the complete geodesics orthogonal to \( F^{l-v} \) in \( \mathbb{H}^{n+p-\nu} \) forms a submanifold \( F^{l-v} \). The union of the subspaces \( \mathbb{H}^{n-1} \) orthogonal to \( \mathbb{H}^{n+p-1} \) and passing through all points of \( F^{l-v} \) is a complete ruled submanifold \( M^{l} \subset \mathbb{H}^{n+p} \) with \( \nu \)-dimensional ruling, called a cylindrical submanifold.

Cylindrical isometric immersions of \( \mathbb{H}^2 \) into a hyperbolic space \( \mathbb{H}^3 \) are studied in [196]; see also [11].

A cylindrical submanifold \( M^{l} \) in \( \mathbb{H}^{n+p} \) is strongly parabolic with index of relative nullity \( \mu(M) \geq \nu \); the orthogonal distribution (to rulings) is integrable, and every base \( F^{l-v} \) is totally umbilic. If the base \( F^{l-v} \)
belongs to an equidistant or a totally geodesic submanifold (the first case), then $M^t$ is a cylinder with a $(\nu - 1)$-dimensional ruling over a conical submanifold $F^{l-\nu+1}$ with the vertex outside the absolute (a cylinder if the vertex is $\infty$). If the base belongs to a horosphere (the second case), then $M^t$ is a cylinder with a $(\nu - 1)$-dimensional ruling over the conic submanifold $F^{l-\nu+1}$ with the vertex on the absolute.

The normal curvature $k_n$ of any geodesic on the base is constant and $0 \leq k_n < 1$ in the first case or $k_n = 1$ in the second case, if $F^{l-\nu}$ lies on a horosphere. The metric of a cylindrical submanifold $M^t$ in $\mathbb{H}^{l+p}$ is semi-reducible and can be written in the form

$$ds^2 = ds_1^2 + \varphi^2(x_1, \ldots, x_{\nu})ds_2^2,$$

where $ds_1^2$ is a metric in $\mathbb{H}^\nu$ that depends on the variables $x_1, \ldots, x_\nu$; $ds_2^2$ is a metric on some base $F^{l-\nu}$ dependent on the variables $x_{\nu+1}, \ldots, x_l$; the function $\varphi$ has the form $\exp(-x_1)$ or $\cosh(x_1)$ in a special coordinate system [39].

Next we consider the relationship between cylindrical submanifolds and metrics.

**Definition 4.9** [39], [173]. A complete metric on a Riemannian manifold $M$ with a constant index of (intrinsic) $k$-nullity $\nu_k(M) > 0$ (see Sec. 2.4.1) is $k$-cylindrical if there exists a $(\dim M - \nu_k(M))$-dimensional submanifold orthogonal to the distribution of $k$-nullity that is totally umbilic for $\nu_k(M) < \dim M - 1$, and for $\nu_k(M) = \dim M - 1$ is a curve of constant curvature.

**Theorem 4.21** [39]. A $k$-cylindrical metric with $k < 0$ is foliated by $\nu_k(M)$-dimensional complete totally geodesic submanifolds ($k$-nullity foliation) and can be written in the form (4.13) with the function $\varphi$ equal to $\exp(-x_1)$ or $\cosh(x_1)$.

**Theorem 4.22** [39]. Let $M^t \subset \mathbb{H}^p(-1)$ be a strongly parabolic submanifold with $l > \mu(M) > 0$, and let there exist a submanifold orthogonal to the relative nullity distribution that is totally umbilic for $\mu(M) < l - 1$, and for $\mu(M) = l - 1$ is a curve of constant curvature. Then $M$ is a cylindrical submanifold that is represented (in the Cayley–Klein interpretation) by a cylinder with a $\mu(M)$-dimensional ruling over a conical submanifold with vertex on the absolute or outside it.

In [173], a subclass of strongly parabolic metrics that is in a one-to-one correspondence with some subclass of strongly parabolic submanifolds in a Lobachevsky space is studied.

**Theorem 4.23** [173]. Let $M^t$ have a $k$-cylindrical metric with $k < 0$, and let one of the following conditions hold:

1. $\varphi = \exp(-x_1)$ and the base $ds_2^2$ admits an isometric immersion into $\mathbb{H}^{l-\nu+p}$ with zero index of relative nullity,
2. $\varphi = \cosh(x_1)$ and the base $ds_2^2$ admits an isometric immersion into $\mathbb{H}^{l-\nu+p}$ with zero index of relative nullity.

Then $M$ admits an isometric immersion into $\mathbb{H}^{l+p}$ as a cylindrical submanifold (with a $\nu$-dimensional ruling and the same codimension as for an immersion of a base).

### 4.3.3. Cylindricity of submanifolds in a Riemannian space of nonnegative curvature.

We can generalize Theorem 4.15 in different ways.

For a submanifold $M \subset \bar{M}$, the expression $\overline{\Ric}^T(x)$ with $x \in TM$ is defined by the same formula as $\Ric(x)$ for $M$ but by using the sectional curvature of $\bar{M}$:

$$\overline{\Ric}^T(x) = \sum_i K(x, y_i),$$

where $\{y_i\}$ is an orthonormal basis in $TM$ at a given point.

**Lemma 4.6.** Let $M^n \subset \bar{M}$ be a complete submanifold, and let

$$\Ric(x) \geq \overline{\Ric}^T(x), \quad x \in T_mM,$$

(4.14)
for some point \( m \in M \) with \( r(m) = r(M) \). Then \( \mu(M) = n - r(M) \), and the second quadratic form \( A_H \) of the mean-curvature vector \( H \) at \( m \) is positive definite on the subspace \((\ker h)^\perp\).

**Definition 4.10.** Let \( M \subset \tilde{M} \) be a complete curvature-invariant submanifold, and let \( k \geq 0 \) be a constant. For any point \( m \in M \) with \( \mu(m) = \mu(M) > 0 \), we consider the set \( \Gamma(m, h) \), which is the sheaf of geodesics \( \gamma: [0, \pi/\sqrt{k}] \to M, \gamma(0) = m, \gamma'(0) \in \ker h \) (where \( h \) is the second fundamental form of \( M \)) and define a nonnegative number

\[
\alpha(m, h) = \sup\{(\nabla_x z, y') : y, z \parallel \ker h, y \perp z, |y| = |z| = 1, y \in \Gamma(m, h)\}.
\]

This \( \alpha(m, h) \) is an analog of the turbulence \( \alpha(L) \) for a relative nullity foliation on the regularity domain \( \tilde{G} \subset M \) and is well defined by virtue of Theorem 4.1. Similar to \( \delta_R \) (see Sec. 3.5), let \( \delta_R \) be a nonnegative number:

\[
\delta_R = \sup\{(\tilde{R}(x, y)u, z) : x, u \in \ker h, y, z \in TM \cap (\ker h)^\perp, x \perp u, y \perp z, |x| = |y| = |u| = |z| = 1\}.
\]

For example, \( \delta_R = 0 \) holds for a conformally flat metric on \( M \) along \( M \).

Theorem 4.24 (and Theorem 4.25 below) is based on the method developed in Sec. 3.5.2 by using the volume of \( L \)-parallel vector fields.

**Theorem 4.24.** Let \( M^n \subset \tilde{M} \) be a complete analytic curvature-invariant submanifold with \( 0 < r(M) < n \) satisfying the condition

\[
k_2 \geq \tilde{K}(x, y) \geq k_1 \geq 0, \quad x, y \in TM,
\]

and let there exist a point \( m \in M \) of maximal rank satisfying (4.14) and one of the following inequalities:

\[
(k_2 - k_1) \max \{\alpha(m, h)^2, k\} \leq 0.3k_2k, \quad (4.16a)
\]

\[
(k_2 - k_1 + 2\alpha(m, h)\max \{\alpha(m, h)^2, k\})^{1/2} \leq 0.004k^2k_2 \text{ (with } k_1 \geq k_2), \quad (4.16b)
\]

where \( k = (k_1 + k_2)/2 \). Then \( k_1 = k_2 = 0 \) and \( M \) is a cylindrical submanifold with a flat \( \mu(M) \)-dimensional generator.

The sequence \( \nu(n) \) is defined in Sec. 3.1.

**Theorem 4.25.** Let \( M^n \subset \tilde{M} \) be a complete analytic curvature-invariant submanifold, and let either \( M^n \) or \( \tilde{M} \) be compact, and \( n > \mu(M) = \nu(n) \) (\( n > \mu(M) > 0 \) for the Kählerian case),

\[
k_2 \geq \tilde{K}(x, y) \geq k_1 \geq 0, \quad x, y \in T\tilde{M},
\]

and one of the inequalities (4.16a) or (4.16b) hold for a certain point \( m \) with \( \mu(m) = \mu(M) \). Then \( k_1 = k_2 = 0 \) and \( M \) is a cylindrical submanifold with a flat \( \mu(M) \)-dimensional generator.

**Corollary 4.7.** Let \( M^n \subset \tilde{M} \) be a complete curvature-invariant submanifold satisfying condition (4.15) for \( k_1 > 0 \) and condition (4.16a) or (4.16b) for a certain point \( m \in M \) with \( \mu(m) = \mu(M) \). Also assume that one of the following properties holds:

(a) \( \mu(M) > \nu(n) \), \( \mu(M) > 0 \) for the Kählerian case,

(b) (6.2) for the point \( m \in M \),

(c) \( \text{Ric}_h^M \leq 0 \) for some \( q \in [1, n - 2] \) and

\[
2p < \begin{cases} n - \nu(n) - q + \delta_1q, \\ n - q + \delta_1q \end{cases} \text{ for the Kählerian case.}
\]

Then \( M^n \) is a totally geodesic submanifold.
Theorems 4.24 and 4.25 and Corollary 4.7 generalize the results of [36] and [1, 2], (see also Sec. 4.3.1); moreover, for the Kählerian case, the holomorphic bisectional curvature of $\bar{M}$ in (4.15) or (4.17) can be considered. Case (c) of Corollary 4.7 follows from (a), since $\text{Ric}_h \leq 0$ implies the inequality for $\mu(M)$ given in (a); see Lemma 4.4. The analytic condition in Theorems 4.24 and 4.25 can be replaced by the requirement that the function $\mu(m)$ be constant. Note that the sectional curvature in Theorems 4.24 and 4.25 for (4.16b) is 0.5-pinched, in contrast to (4.16a) with weaker requirements for the curvature tensor but with greater (equal to 0.7) pinching of curvature.

The following theorem is based on the procedure of the Riccati ODE from Sec. 3.2.

**Theorem 4.26.** Let $M^n \subset \bar{M}$ be a complete analytic curvature-invariant submanifold satisfying the following conditions: $0 < \tau(M) < n$, $\bar{K}_{1M} \geq 0$, and

$$\text{Ric}(x) \geq \text{Ric}^T(x), \quad x \in TM.$$  

(4.18)

Then $M$ is a cylindrical submanifold with an $(n - \tau(M))$-dimensional generator.

**Corollary 4.8.** Let $M^n \subset \bar{M}$ be a complete curvature-invariant submanifold satisfying the conditions (4.18) and $\bar{K}_{1M} > 0$, and let $\tau(M) < n$. Then $M$ is a totally geodesic submanifold.

Theorem 4.26 and its Corollary 4.8 complete the results of [2] (where the cases $\bar{M} = \mathbb{R}^n$ and $\bar{M} = S^n$ are studied) and also the results from [36]. From Theorems 4.24, 4.25, and 4.26 and the results from Sec. 4.2.2, we can deduce cylinder theorems for submanifolds with nonpositive extrinsic curvature with “small” codimension and some additional conditions as well as for $k$-saddle submanifolds.

4.4. Decomposition of Ruled and Parabolic Submanifolds

4.4.1. Ruled and parabolic submanifolds in CROSS and the Segre embedding. The ruled submanifolds in $S^m$ and $\mathbb{R}^m$ are related by a geodesic map of the hemisphere $S^m_+ = \{x_0 > 0\}$,

$$q = (x_0, \ldots, x_m) \in S^m_+ \rightarrow \tilde{q} = (x_1, \ldots, x_m)/x_0 \in \mathbb{R}^m.$$

**Lemma 4.7** (see [34]). Let $M$ be a submanifold in the hemisphere $S^m_+ = \{x_0 > 0\}$, and let $\tilde{M}$ be its image in $\mathbb{R}^m$ under a geodesic mapping. Assume that $\xi$ is a normal vector to $M$ at a point $q$ and $\tilde{\xi}$ is a normal vector to $\tilde{M}$ at a point $\tilde{q}$. Then (1) the orthogonal projection of $\xi$ onto the hyperplane $x_0 = 1$ is a normal vector $\tilde{\xi}$, i.e.,

$$\xi = (0, \xi_1, \ldots, \xi_m) \rightarrow \tilde{\xi} = (\xi_1, \ldots, \xi_m)/\sqrt{1 - \xi_0^2},$$

(2) $A_{ij}(\xi, q) = cA_{ij}(\xi, \tilde{q})$, where $A_{ij}(\xi, q)$ and $A_{ij}(\xi, \tilde{q})$ are coefficients of the second quadratic forms of $M$ and $\tilde{M}$ for the normal vectors $\xi$ and $\tilde{\xi}$ respectively.

Therefore, locally (strongly) parabolic submanifolds in $S^m$ and $\mathbb{R}^m$ are in one-to-one correspondence; see also [9]. In contrast to the sphere $S^m$, where there is only one type of totally geodesic submanifold (large spheres with all possible dimensions), other CROSS (except for $\text{RP}^m$) have different types of totally geodesic submanifolds:

$$\text{CP}^m: \{\text{RP}^n, \text{CP}^n\}, \quad n \leq m;$$
$$\text{HP}^m: \{\text{RP}^n, \text{CP}^n, \text{HP}^n, S^3\}, \quad n \leq m;$$
$$\text{CaP}^2: \{\text{RP}^n, \text{CP}^n, \text{HP}^n, S^4\}, \quad n \leq 2, \quad i = 3, 5, 6, 7, 8.$$  

These submanifolds in $\text{KP}^m$ ($\mathbb{K} = \mathbb{C}, \mathbb{H}$) are related by the mapping exp to some subspaces $\mathbb{K}^{\bar{n}+1}$ contained in $\mathbb{K}^{\bar{n}+1}$ [23]: (1) the type of the numbers $\mathbb{K} \subset \mathbb{K}$ can be restricted, (2) the dimension can be decreased, $\bar{n} \leq n$.  

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In [34], orthogonal projections of the Riemannian symmetric spaces $S^n$, $CP^m$, and $HP^m$ onto the totally geodesic submanifolds $S^n$, $CP^3$, and $HP^3$, respectively, are studied; submanifolds $M^n$ with regular projection are said to be uniquely projectable.

**Definition 4.11** [34]. A submanifold $M^n$ in CROSS is $t$-uniquely projectable if there exists a totally geodesic submanifold

1. $S_0^{n+t} \subset S^n$,  
2. $CP_0^{(n+t)/2} \subset CP^n$, or  
3. $HP_0^{(n+t)/4} \subset HP^n$

such that the tangent space to the totally geodesic submanifold

1. $S^{n-(n+t)}(q)$,  
2. $CP^{n-(n+t)/2}(q)$, or  
3. $HP^{n-(n+t)/4}(q)$,

which contains the point $q \in M$ and is orthogonal to (1) $S_0^{n+t}$, (2) $CP_0^{(n+t)/2}$, or (3) $HP_0^{(n+t)/4}$, respectively, has zero intersection with the tangent space $T_qM$.

The image $\tilde{M}$ of $M$ under the above mapping $q \in M \rightarrow \tilde{q} \in S_0^{n+t}$, $CP_0^{(n+t)/2}$, and $HP_0^{(n+t)/4}$, resp., is a smooth submanifold of the same rank; as in Lemma 4.7, the second quadratic forms of $M$ and $\tilde{M}$ are proportional.

The estimates of $\dim L$ for a ruled submanifold and tests for $h = 0$ for a parabolic submanifold in CROSS can be obtained in terms of the dimension of $\tilde{M}$ or the codimension of $M$. But a key role is played by the number $t$ from the definition of the unique projectivity and the following lemma.

**Lemma 4.8** [34]. For submanifolds $\tilde{M}$ and $M = \pi^{-1}(\tilde{M})$, where $\pi$ are the Hopf fibrations with an $s$-dimensional fiber, the inequality $r(M) \leq r(\tilde{M}) + 2s$ holds.

The standard totally geodesic foliations on CROSS are the generalized Hopf fibrations (1.12). For a submanifold $\tilde{M}$ in the base of the Hopf fibration, the inverse image $\pi^{-1}(\tilde{M})$ is a ruled submanifold in a sphere or in $CP^{2n+1}$ with equidistant 1-, 3-, 7-, or 2-dimensional rulings (fibers).

**Definition 4.12.** Consider the following classes of ruled submanifolds in $S^m$ and $CP^m$ with complex structure $J$:

1. $M^{n+t} \subset S^m$ with the ruling $L = S^n$,  
2. $M^{n+t} \subset CP^m$ with a ruling $L = RP^n$ that satisfies $J(TL) \perp TM$,  
3. $M^{n+t} \subset CP^m$ with a ruling $L = CP^{n/2}$ that satisfies $J(TM) = TM$.

We can also consider some classes of ruled submanifolds in $HP^m$ and $CP^2$. In [236, 242], the following classes of ruled submanifolds in CROSS are introduced:

- $RS_1$ that are uniquely projectable,  
- $RS_2$ with positive $K_{mix}$.

They generalize the well-known class $RS_0$ of ruled developable (or strongly parabolic) submanifolds and are connected with the following classes of parabolic submanifolds:

- $PS_1$ that are uniquely projectable,  
- $PS_2$ with positive sectional curvature.

**Theorem 4.27** [242]. Assume that a complete submanifold $M$ in $S^N$ or $CP^N$ belongs to $RS_2$ (resp., $PS_2$). Then any (complete) ruling $L \subset M$ has a neighborhood that belongs to $RS_1$ (resp., $PS_1$). The dimension of a ruling of such a submanifold is estimated by

$$\dim L \leq \begin{cases} \rho(\text{codim } L) - 1, \\ 2 \text{ for the Kählerian case.} \end{cases}$$

For a submanifold $M$ from $RS_1$ or $PS_1$, the above inequality can be proved by considering the projection of $M$ onto a totally geodesic subspace of the same dimension, and then, by using Theorem 3.1. Moreover,
for a ruled submanifold in Theorem 4.27, the structural tensor of the foliation \( \{L\} \) has no eigenvectors \([242]\) (compare with Lemma 3.6 and Lemma 3.8).

**Corollary 4.9** [238]. Let \( M^n \) be a complete submanifold in \( S^m(1) \), or \( CP^m(1) \) with \( J(TM) = TM \), and let \( \text{Ric}_M \) not exceed \( n - 2 \) or \( n/4 \), respectively. If

\[
r(M) \leq \begin{cases} 
    n - \nu(n) & \text{for } S^m(1), \\
    n - 1 & \text{for } CP^m(1),
\end{cases}
\]

then \( M^n \) is totally geodesic.

In the above statement, \( \nu(n) \) is defined as in Theorem 3.1.

The simplest ruled submanifold in a Euclidean space is a cylinder \( f(M_1 \times \mathbb{R}^n) \subset \mathbb{R}^N \). There are no cylinders in CROSS; in these spaces, the Segre embedding serves as a standard example of a ruled submanifold that is a metric product.

**Definition 4.13.** The Segre embedding \( M^m+n = f(S^m \times S^n) \subset S^{m+n} \) is given by \( f: (u_1, \ldots, u_m, v_1, \ldots, v_n) \rightarrow (u_1v_1, \ldots, u_nv_j, \ldots, u_nv_n) \), where \( \{u_i\} \) and \( \{v_j\} \) are the coordinates of points on the unit spheres \( S^m \) and \( S^n \).

The complex Segre embedding \( M^{m+n} = f(CP^m \times CP^n) \subset CP^{m+n} \) can be similarly defined.

The rulings of the Segre embedding can be chosen in two canonical ways. The simplest Segre embedding \( M^2 = f(S^1 \times S^1) \subset S^3 \) (Clifford torus) is a flat surface.

**Remark 4.1.** Any smooth spherical fibration (in the topological sense) \( S^k \subset E \overset{\pi}{\rightarrow} B \) with compact base \( B \) admits a smooth ruled embedding into a sphere \( S^N \) with large \( N \), i.e., an embedding whose fibers are large \( k \)-dimensional spheres [217]; such embeddings are called large-sphere fibrations. There are infinitely many topologically nonequivalent smooth 3-sphere fibrations of \( S^7 \) (whose bases are not diffeomorphic to \( S^4 \)). These fibrations can be realized as ruled submanifolds in \( S^{N+7} \) with large \( N \). However, for \( N = 0 \) any smooth fibration of \( S^7 \) by large 3-spheres is topologically equivalent to the Hopf fibration [105], i.e., the topological structure of a compact ruled submanifold depends on its codimension.

If \( M \subset S^N \) is foliated by large spheres, then there is a hierarchy of three questions stated here in an increasing order of difficulty [217]: (1) given two foliations of this kind, are they topologically equivalent? (2) if they are topologically equivalent, is it possible to deform one into the other via a one-parameter family of such foliations? (3) what is the homotopy type of the space of all such foliations?

These problems are completely solved for large-circle foliations of \( S^3 \) (see Sec. 2.6). A more complicated case where \( M = S^p \times S^q \) is embedded into \( S^{p+q+1} \) is considered in [217] in the standard way, where some special cases are also studied.

The curvature tensor of the ambient Riemannian manifold (CROSS) along ruled submanifolds from Definition 4.12 has the same simple form

\[
\bar{R}(x, y)z = 0, \quad x \in TL, \ y \in TM, \ z \perp (x \wedge y), \quad ||J_z, x \in TL, \ y \in TM, \ z \perp (x \wedge y \wedge Jx \wedge Jy)\text{ for the Kählerian case}
\]

\[
\bar{R}(y, x)x = kx^2y, \ x \in TL, \ y \in TL^\perp.
\]

We will also use the following inequality:

\[
(A_\xi x)^2 + (A_\xi u)^2 \leq k \quad (x, u \in TL, \ x \perp u, \ ||x|| = ||u|| = ||\xi|| = 1).
\]

Let \( x_1, \ldots, x_n, y_1, \ldots, y_n \) be an orthonormal basis in \( T_mM \) such that \( x_i \in T_mL, \ y_i \in T_mL^\perp \). Let \( A_\xi \) denote the symmetric matrix of the second quadratic form for a normal vector \( \xi \in T_mM^\perp, \ ||\xi|| = 1 \), in this basis. If, for example, \( A_\xi x_i = \sqrt{k}y_j \) for some numbers \( i, j \) (an extremal situation), then the matrix \( A_\xi \) satisfying the condition (4.21) has only two nonzero elements; its rank is equal to two.
Note that from (4.21) and (4.20), by virtue of the Gauss equation (2.3a), we obtain \( K_{\text{mix}} \geq 0 \). Moreover, (6.33) follows from the “compact” inequality
\[
||A_\xi||^2 := \text{tr}(A_\xi^2) \leq 2k, \quad |\xi| = 1.
\] (4.22)

We define the nondecreasing integer sequence \( \bar{p}(n) = \max\{\rho(t): t \leq n\} \). From the definition of \( \rho(n) \) (see Sec. 1.1), we obtain the estimates
\[
2[\log_2 n] \leq \bar{p}(n) \leq 2[\log_2 n] + 2 \quad ([x] denotes the integer part of x).
\]

The following synthetic concept is effective for local (i.e., in a neighborhood of a generator) research of ruled and parabolic submanifolds.

**Definition 4.14.** A smooth submanifold \( M \subset \tilde{M} \) is \( t \)-uniquely projectable along a generator \( L \subset M \) if \( t \) is a minimum integer such that for some point \( m \in L \), there exists a subspace \( V \subset T_m M^\bot \) of dimension \( \text{codim } M - t \) that remains transversal to \( M \) under the \( \nabla \)-translation along any path in \( L \). For \( t = 0 \), such \( M \) is uniquely projectable along \( L \).

For a submanifold in a space form \( \tilde{M}(k) \) or CROSS, Definition 4.14 generalizes Definition 4.11.

A key result of this section is the following theorem.

**Theorem 4.28.** Let \( M^{n+n} \) be a ruled (Kähler) submanifold in a Riemannian (Kähler) space \( \tilde{M} \) with rulings \( \{L^\nu\} \), and let conditions (4.19), (4.20), and (4.21) with \( k > 0 \) hold. If \( \nu \geq \bar{p}(n) \) (or \( \nu > 2 \) for the Kählerian case), then \( M^{n+n} \) is locally isometric to the product \( L^\nu \times M^2_k \) and codim \( M \geq \nu n \). Moreover, if \( \tilde{M} = S^N \) or \( CP^N \) and codim \( M = \nu n \), then \( M \) is congruent to the domain of the Segre submanifold.

**Corollary 4.10.** A ruled submanifold \( M^{n+n} \subset \tilde{M} \) satisfying properties (4.19) and (4.20) with \( K_{\text{mix}} > 0 \) along a ruling \( L^\nu \) is uniquely projectable. In particular, we have \( \nu < \rho(n) \) (or \( \nu = 2 \) for the Kählerian case).

Condition (4.21) can be replaced (without affecting Theorem 4.28) by the integral inequality for the second quadratic form as follows:

along any leaf geodesic \( \gamma: [0, \pi/(2\sqrt{k})] \to M \), for any \( \nabla^\bot \)-parallel unit vector field \( \xi(t) \in T_\gamma M^\bot \) we have
\[
\int_\gamma ||A_\xi(t)||dt \leq \pi\sqrt{k}.
\]

**Idea of the Proof of Theorem 4.28.** For simplicity, we assume that \( k = 1 \). For the totally geodesic foliation \( \{L\} \) (on the ruled submanifold \( M \)), we define the structural tensor \( B: TL^\bot \times TL^\bot \to TL \). In view of the given inequality for the dimension, for every point of \( M \), there exist unit vectors \( x \in TL \), \( y \in TL^\bot \) and real \( \lambda \leq 0 \) satisfying the property \( B(x, y) = \lambda y \). The \( L \)-parallel Jacobi field \( y(t) \), \( y(0) = y \), along the geodesic \( \gamma(t) \), \( \gamma'(0) = x \), can be written as \( y(t) = (\cos(t) + \lambda \sin(t))\hat{y} + (\sin(t))\hat{v} \), where \( \hat{y} \) and \( \hat{v} \) are obtained by using the \( \nabla \)-parallel transport of the vectors \( y \) and \( v = h(x, y) \neq 0 \) along \( \gamma \). Note that the normal vector \( v \) under the \( \nabla^\bot \)-parallel transport along the path \( \gamma(t) \), \( 0 \leq t \leq t_0 \), where \( t_0 = \cot^{-1}(-\lambda) \leq \pi/2 \), turns in \( \tilde{M} \) through the angle \( \pi/2 \), although the velocity of rotation is less than 1. In view of the extremal situation, we find that the normal vector \( v \) turns uniformly in some plane along \( \gamma(t) \), \( 0 \leq t \leq \pi/2 \), through the angle \( \pi/2 \) and \( \lambda = 0 \) holds. Hence we deduce that we can “chop off” the vector field \( Z_1: L \to TL^\bot \) containing the unit vector \( ||A_\xi: \nabla_u Z_1 = R(Z_1, u)u = 0, w \perp Z_1 \Rightarrow B(u, w) \perp Z_1. \) We will repeat this process \( n \) times and obtain the splitting of \( M \) along \( TL \) and \( TL^\bot \). Using Codazzi equations, we find the metric decomposition of \( M \) with codim \( M \geq \nu n \).

Next we give similar decomposition and rigidity results for parabolic submanifolds in CROSS. The curvature tensor of a submanifold \( M \) in \( S^N(k) \) or a Kähler submanifold \( M \) in \( CP^N(k) \) has the following simple form:
\[
\hat{R}(x, y)z \begin{cases} 
= 0, & x, y \in TM, \ z \perp (x \wedge y), \\
||Jz, x, y \in TM, \ z \perp (x \wedge y \wedge Jx \wedge Jy) \end{cases} \text{ for the Kählerian case,}
\]

(4.23)
\[ R(y, x)x = kx^2y, \quad (x, y \in TM \text{ for the Kählerian case } y \perp (x \land Jx)). \] (4.24)

We will use the following inequality (cf. (4.21)):

\[ A^{2}_{x}x + A^{2}_{u}u \leq k \quad (x, u \in TM, \ x \perp u, \ |x| = |u| = |\xi| = 1). \] (4.25)

Note that from (4.25) and (4.24) in a real case, by virtue of the Gauss equation (2.3 a), we obtain \( K_M \geq 0 \). Moreover, (4.25) follows from the inequality (cf. (4.22))

\[ ||A_\xi||^2 \leq 2k, \quad |\xi| = 1. \] (4.26)

**Definition 4.15.** For a submanifold \( M \subset \tilde{M} \) and any \( s \)-dimensional subspace \( U_s \subset T_qM \), let \( h(U_s) = Ph: T_qM \times T_qM \to U_s \) be a bilinear form, where \( h \) is the second fundamental form of \( M \), and let \( P: T_qM^\perp \to U_s \) be an orthoprojector. The \( s \)-null index of \( M \), where \( s \leq \text{codim} \, M \), is defined by the formula \( \mu_s(M) = \min\{|N(U_s): U_s \subset T_qM, \ q \in M\} \), where \( N(U_s) \) is the null space of the bilinear form \( h(U_s) \).

Note that this definition is different from that given in [54]. The following inequalities hold, where \( p = \text{codim} \, M \):

\[ \mu(M) = \mu_p(M) \leq \ldots \leq \mu_2(M) \leq \mu_1(M) = \dim M - r(M). \] (4.27)

**Lemma 4.9.** The Segre submanifold \( M^{n+\nu} = f(S^n \times S^n) \subset S^{n} \) satisfies the conditions \( r(M) = 2 \min\{\nu, n\} \) and \( \mu_1(M) = \mu_2(M) + 1 \).

Thus, the classical Segre embedding is a parabolic submanifold when \( \nu \neq n \).

**Theorem 4.29.** Let \( M^n \subset \tilde{M}^{n+p} \) be a complete submanifold satisfying properties (4.23a), (4.24), \( 0 < r(M) \leq n - \beta(n) \), and one of the following inequalities:

(a) (4.25), \quad (b) \( K_M \geq 0 \), \quad (c) \( \frac{1}{k} \text{Ric}_M \geq n - 2 \),

where \( k > 0 \) and \( \mu_1(M) \leq \mu_2(M) + 1 \) for (b) and (c). Then \( M \) is locally isometric to the product (a), (b) \( S^{n-r(M)/2} \times M_2^{r(M)/2} \), (c) \( S^{n-1} \times M_2 \) with \( r(M) = 2 \), respectively, and \( p \geq \frac{1}{2}r(M)(n - r(M)/2) \). Moreover, if \( \tilde{M} = S^{n}(k) \) and \( p = \frac{1}{2}r(M)(n - r(M)/2) \), then the submanifold \( M^n \) is congruent to the Segre embedding.

The proof of Theorem 4.29 is based on the relationship between \( K_M > 0 \) and the synthetic concept of a \( t \)-uniquely projectable submanifold. The only difference with the proof of Theorem 4.28 is that together with every vector field \( Z_i, i \leq r(M)/2 \), we also split off the field \( W_i \) in which the mixed sectional curvature is \( k \).

The decomposition of Kähler parabolic submanifolds can be deduced from weaker assumptions on the rank \( r(M) \) and on the curvature.

**Theorem 4.30.** Let \( M^n \subset \tilde{M}^{n+p} \) be a complete Kählerian submanifold satisfying properties (4.23b), (4.24), \( 0 < r(M) < n \), and one of the following inequalities: (a) \( \frac{1}{k} \text{Ric}_M \geq n \), (b) \( |h(x, y)|/(|x| \cdot |y|) \leq \sqrt{k} \), (c) the \( (bi) \)secional curvature of \( M \) is nonnegative. Then \( M \) is locally isometric to the product (a) \( CP^{n-1} \times M_2^{r(M)/2} \) with \( r(M) = 4 \), (b) \( CP^{n-r(M)/2} \times M_2^{r(M)/2} \), respectively, and \( p \geq \frac{1}{2}r(M)(n - r(M)/2) \). If, in addition, \( \tilde{M} = CP^{m}(k) \) and \( p = \frac{1}{2}r(M)(n - r(M)/2) \), then \( M^n \) is congruent to the complex Segre submanifold.

**Corollary 4.11.** Let \( M^n \subset \tilde{M} \) be a complete submanifold satisfying the properties \( K_M > 0 \), (4.23), and (4.24), and let \( \nu(M) \geq \nu(n) \) (or \( \nu > 0 \) for the Kählerian case). Then \( M \) is totally geodesic.

**4.4.2. Ruled and parabolic submanifolds in a Riemannian space of positive curvature.** For a ruled submanifold \( M \) in a Riemannian space \( \tilde{M} \) with nonconstant mixed sectional curvature along the rulings, we can assume that

\[ k_2 \geq R(x, y) \geq k_1 > 0, \quad x \in TL, \ y \in TL^\perp. \] (4.28)

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We will also use the following condition that generalizes (4.21):

\[(A_x x)^2 + (A_x u)^2 \leq k(1 - 5(1 - k_1/k_2)(a(L_0)^2/k + 1))^2\]

\[(x, u \in TL, x \perp u, |x| = |u| = |\xi| = 1). \quad (4.29)\]

Combining the area-of-L-parallel-fields method for foliations (see Theorem 3.6) with the synthetic method of Theorem 4.28 for ruled submanifolds in space forms, we obtain the Segre-type decomposition of submanifolds in a Riemannian space with positive sectional curvature.

**Theorem 4.31.** Let \(M^{n+\nu} \subset M\) be a ruled submanifold with a ruling \(\{L^v\}\), and let conditions (4.19), (4.28), and (4.29) with \(k = (k_2 + k_1)/2\) hold. If \(\nu \geq \rho(n)\) \((\nu > 2\) for the Kählerian case), then \(k_1 = k_2\) and \(M^{n+\nu}\) is locally isometric to the product \(L^v \times M^n\), and \(\text{codim } M \geq \nu n\).

**Corollary 4.12.** Let \(M^{n+\nu} \subset M\) be a ruled submanifold with a ruling \(\{L^v\}\), and let there exist a point \(m \in M\) such that along any geodesic \(\gamma: [0, \pi/\sqrt{k}] \to L_0, \gamma(0) = m\), the following conditions hold: (4.19), (4.28), and

\[|h(\gamma', y)|/|y| < \sqrt{k}(1 - 5(1 - k_1/k_2)(a(L_0)^2/k + 1)), \quad y \in T_\gamma L^v,\]

with \(k = (k_2 + k_1)/2\). Then \(\nu < \rho(n)\) or \(\nu = 2\) and \(n\) is divisible by 4 for the Kählerian case.

Combining the method of 3-dimensional volume of L-parallel fields (see Theorem 3.8) with the synthetic method of Theorem 4.28, we can also obtain a Segre-type decomposition of ruled submanifolds in a Riemannian space similar to Theorem 4.31. However, at present, we cannot obtain a more precise curvature pinching.

Next we give similar decomposition and rigidity results for parabolic submanifolds in a Riemannian space of positive sectional curvature. For the nonconstant sectional curvature of \(M\) along a submanifold \(M\)

\[k_2 \geq \overline{K}(x, y) \geq k_1 > 0, \quad x, y \in TM,\]

below we will use the following condition generalizing (4.25) or (4.26):

\[||A_\xi|| \leq \sqrt{2k}(1 - 5(1 - k_1/k_2)(a(\xi_0)^2/k + 1)), \quad |\xi| = 1, \quad (4.32)\]

or the slightly weaker inequality

\[|h(x, y)/(|x| \cdot |y|) \leq \sqrt{k}(1 - 5(1 - k_1/k_2)(a(\xi_0)^2/k + 1)), \quad x, y \in TM. \quad (4.33)\]

The following real \(a(\xi)\) is related to the turbulence from Sec. 3.5.

**Definition 4.16.** Let \(M \subset M\) be a complete submanifold with property (4.23), and let \(k > 0\) be a constant. For the unit normal vector \(\xi \in T_m M^v\) with the condition \(r(\xi) = r(M) < \dim M\), we consider \(\Gamma(m, \xi)\), the sheaf of geodesics \(\gamma: [0, \pi/\sqrt{k}] \to M\) with \(\gamma(0) = m\) and the initial directions \(\gamma'(0) \in \ker A_\xi\), and define a real

\[a(\xi) = \sup \frac{|(\nabla_\gamma A_{\xi_0})' y| + k}{|A_{\xi_0} x|},\]

where \(y, z \perp \ker A_{\xi_0}, y \perp z, |y| = |z| = 1, \xi(0) = \xi, \nabla_\gamma^2 \xi(t) = 0,\) and \(\gamma \in \Gamma(m, \xi)\).

We can see that \(a(\xi)\) is an analog of the turbulence \(a(L)\) for some foliation by generators on a domain of \(M\). In view of Corollary 4.2, \(a(L)\) is well defined.

Combining the method of variations of Theorem 3.6 for foliations with the synthetic method of Theorems 4.29 and 4.30 for parabolic submanifolds in CROSS, we arrive at the following theorem.

**Theorem 4.32.** Let \(M^n \subset M^{n+p}\) be a complete submanifold satisfying conditions (4.23), (4.31), and (4.32) or (4.33) with \(\mu_1(M) \leq \mu_2(M) + 1\) (in the Kählerian case without inequalities for \(\mu_3(M)\)), where \(k = (k_1 + k_2)/2\) and \(\xi\) is any unit normal vector having \(r(\xi) = r(M)\). If

\[0 < r(M) \leq \left\{ \begin{array}{ll} n - \rho(n), & \text{for the Kählerian case,} \end{array} \right. \]

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then \( k_2 = k_1 \), \( M \) is locally isometric to the product \( L^{n-r(M)/2} \times M_2^{r(M)/2} \) and \( p \geq \frac{1}{2} r(M)(n - r(M)/2) \).

If (4.33) in Theorem 4.32 is given with strict inequality, then we obtain tests of totally geodesic submanifolds.

**Corollary 4.13.** Let \( M^n \subset \tilde{M} \) be a complete submanifold satisfying conditions (4.23), (4.31), and (4.33) with strict inequality (\( k \) and \( \xi \) are as in Theorem 4.32), and let

\[
r(M) \leq \begin{cases} n - \nu(n), & \text{for the Kählerian case.} \\
n - 1 & \text{for the Kählerian case.}
\end{cases}
\]

Then \( M \) is totally geodesic.

### 4.4.3. Pseudo-Riemannian isometric immersions.

The universal covering of a complete pseudo-Riemannian manifold \( M^n_p \) (of dimension \( n \) and with a metric of signature \( (p, n - p) \)) of constant curvature \( k \) can be represented by (a hypersurface in the first and second cases)

\[
\tilde{M}_p^n = \begin{cases} \sum_{i=1}^{p+1} x_i^2 - \sum_{j=1}^{q} x_{p+1+j}^2 = \sqrt{k} & \text{in } \mathbb{R}^{p+1}_{p+1}, \text{ if } p \neq 1, k > 0, \\
\sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{q+1} x_{p+j}^2 = -\sqrt{-k} & \text{in } \mathbb{R}^{p+1}_{q+p}, \text{ if } q \neq 1, k < 0, \\
a \text{pseudo-Euclidean space } \mathbb{R}^{n+1} & \text{if } k = 0.
\end{cases}
\]

Hence the case \( k > 0 \) is dual to the case \( k < 0 \).

In [4], the relative nullity foliation of an indefinite Riemannian manifold isometrically immersed into an indefinite space form is studied. This foliation is totally geodesic and gives rise to a Riccati-type matrix ODE along a geodesic in a leaf. This differential equation can be applied in several cases for estimating the index of relative nullity for geodesically complete Lorentz submanifolds in the Lorentz sphere. In [5], pseudo-Riemannian Kählerian strongly parabolic embeddings are studied. In [140], products of complex submanifolds of indefinite complex space forms are studied, and the criteria for a submanifold corresponding to the Segre embedding are given.

The decomposition results (similar to Corollary 4.11 and Theorem 4.29) are valid for submanifolds with rulings of a pseudo-Riemannian manifold, which generalize the results given in [4]. We will give them in their simplest forms (see [246]). Let \( M_p^n(S_q^m(1)) \) be a pseudo-Riemannian manifold (a sphere with radius 1) of dimension \( m \) with metric of signature \( (p, m - p) \).

**Theorem 4.33.** Let \( M_p^{n+\nu} \subset S_q^m(1) \) be a ruled submanifold with geodesically complete rulings and the following condition on its second quadratic forms:

\[
(A\xi x)^2 \geq 0, \quad (A\xi x)^2 + (A\xi u)^2 \leq 1 \quad (x, u \in TL, \xi^2 = x^2 = u^2 = 1, (x, u) = 0).
\]

If \( \nu \geq p + \rho(n) \), then \( M_p^{n+\nu} \) is locally isometric to the product \( S_q^m \times B_0^n \), and \( m \geq n + \nu + \nu p, q \geq np \) holds; if, in addition, \( m = n + \nu + \nu p \) and \( q = np \), then the submanifold \( M_p^{n+\nu} \) is congruent to a domain of the pseudo-Riemannian Segre embedding \( S_q^n \times S_0^m \subset S_{q+p}^{n+m} \).

**Theorem 4.34.** Let \( M_p^n \subset S_q^{n+p}(1) \) be a geodesically complete submanifold satisfying one of the following conditions:

(a) \( (A\xi x)^2 \geq 0, \quad (A\xi x)^2 + (A\xi y)^2 \leq 1 \quad (\xi^2 = x^2 = y^2 = 1, (x, y) = 0) \),

(b) \( 0 \leq K(x, y) \leq 1 \),

(c) \( K(x, y) \leq 1, \quad \text{Ric}(x) \geq n - 2 \), for (b) (c): \( \mu_1(M) \leq \mu_2(M) + 1 \),

where \( x^2 = y^2 = 1, (x, y) = 0 \), and the restriction of the metric on the plane \( \ker A_{\xi} (\xi^2 = 1, r(\xi) = r(M)) \) has signature \((p, n - r(M) - p)\). If \( r(M) \leq n - p - \rho(n - p) \), then \( M_p^n \) is locally isometric to the product \( S_p^{n-r(M)/2} \times B_0^{r(M)/2}, \quad S_p^{n+1} \times B_0^1, \quad r(M) = 2 \).
and $p \geq \frac{1}{2} r(M)(n-r(M)/2)$, $q \geq \frac{1}{2} pr(M)$. If, moreover, $p = \frac{1}{2} r(M)(n-r(M)/2)$ and $q = \frac{1}{2} pr(M)$, then the submanifold $M$ is congruent to the pseudo-Riemannian Segre embedding.

The existence of a ruled structure on a parabolic submanifold $M^n$ with a definite metric in a pseudo-Riemannian space form is proved in [35]. Note that a pseudo-Riemannian manifold whose sectional curvature preserves its sign actually has a constant curvature (see [35]). The indefinite isometric immersions of constant curvature were studied by many other authors.

**Theorem 4.35** [35]. (a) A pseudo-Riemannian manifold $M^n_p$ of constant negative sectional curvature cannot be locally isometrically embedded into $\mathbb{R}^{2n-2}_p$, where $n-p = 2n-2-p'$. (b) A complete pseudo-Riemannian manifold $M^n_p$ of constant negative sectional curvature cannot be isometrically embedded into $\mathbb{R}^{2n-1}_p$, where $n-p = 2n-1-p' \neq 0, 1, 3, 7$.

**Example 4.3** [35]. The local isometric immersion of $M^n_p$ of constant negative curvature into $\mathbb{R}^{2n-1}_p$, where $n-p = 2n-2-p'$, is given by formulas

\[
\begin{align*}
x_{2i-1} &= \left\{ \begin{array}{ll}
  a_i \exp(u_i) \cos(u_i) & \text{if } 1 \leq i \leq p-1, \\
  a_i \exp(u_i) \cosh(u_i) & \text{if } p \leq i \leq n-1,
\end{array} \right.
\\
x_{2i} &= \left\{ \begin{array}{ll}
  a_i \exp(u_i) \sin(u_i) & \text{if } 1 \leq i \leq p-1, \\
  a_i \exp(u_i) \sinh(u_i) & \text{if } p \leq i \leq n-1,
\end{array} \right.
\\
x_{2i-1} &= \int_0^u \sqrt{1 - \exp(2t)} \, dt,
\end{align*}
\]

where $x_i$ are coordinates in $\mathbb{R}^{2n-1}_p$, and $\sum_i a_i^2 = 1$.

The proof of Theorem 4.35 is based on the Otsuki lemma (see Sec. 4.2.2). Obviously, Theorem 4.35 is also valid for $n-p = 0, 1, 3, 7$ [135]; this is connected with the fact that the parallelizable spheres have dimensions $1, 3,$ and $7$.

### 4.5. Ruled Submanifolds with Conditions on the Mean Curvature

Some authors studied ruled submanifolds in space forms with additional conditions on its mean-curvature vector. The well-known theorem by Catalan states that the only ruled minimal surfaces in $\mathbb{R}^3$ are the plane and the helicoid (see [170] for the case of $S^3$ and [55] for hyperbolic space $H^3$). These classical results can be generalized in the following three ways.

**1.** The classification of minimal ruled submanifolds $M^{n+1}$ with $n = 1$ in space forms was obtained independently in [175], [282], and [16] (for hypersurfaces in $\mathbb{R}^N$ see [27]). It is shown in these works that the minimal ruled submanifolds are generalized helicoids. This extends the Catalan theorem in one direction.

**Definition 4.17** [16]. Let $c = c(t)$ be a curve in a space form $	ilde{M}(k)$ of constant curvature and with the Frenet frame $e_1, \ldots, e_m$. We set $\nu = [\frac{n}{2}]$ and define the map $f: \mathbb{R}^{r+1} \to \tilde{M}(k)$ by $f(s, t_1, \ldots, t_\nu) = \exp \sum_{j=1}^\nu t_j e_{2j}(s)$, which is called (in any parametrization) the helicoid associated with the curve $c$. 

**Lemma 4.10** [16]. The helicoid $f$ associated with a curve $c: \mathbb{R} \to \tilde{M}(k)$ describes a minimal immersion, wherever it is regular. The helicoid $f$ admits a one-parameter subgroup $A(s)$ of rigid motions of $\tilde{M}(k)$ such that $A(s)f(t, t_1, \ldots, t_\nu) = f(s + t, t_1, \ldots, t_\nu)$, $s, t, t_1, \ldots, t_\nu \in \mathbb{R}$.

However, a ruled minimal immersion invariant for a one-parameter subgroup $A(s)$ of rigid motions of $\tilde{M}(k)$ is not necessarily a helicoid.
Definition 4.18 [16] Let $A(s)$ be a one-parameter subgroup of rigid motions of $\tilde{M}(k)$, and let $S$ be a $\nu$-dimensional complete totally geodesic submanifold of $\tilde{M}(k)$ orthogonal to the orbits of $A(s)$. Then the map $f: \mathbb{R} \times S \rightarrow \tilde{M}(k)$ defined by $f(s,p) = A(s)p$ describes a ruled immersion, whenever it is regular. These maps (minimal whenever they are regular) are called generalized helicoids.

Note that a generalized helicoid in a sphere or a hyperbolic space is a restriction of a certain submanifold in Euclidean space. The classification of generalized helicoids in $\mathbb{R}^{N+1}$ up to the motion of the ambient space is as follows [16]:

$$f(s, t_1, \ldots, t_\nu) = \sum_{i=1}^{r} t_i e_i(s) + \sum_{i=1}^{\nu-r} t_{\nu+i} V_{2r+i} + sbV_{\nu+r+1},$$

where the vectors $V_1, \ldots, V_{N+1}$ constitute some special orthonormal basis of $\mathbb{R}^{N+1}$, the fields $e_i(s)$ are given by $e_i(s) = \cos a_i s \ V_{2i-1} + \sin a_i s \ V_{2i}$ ($1 \leq i \leq \nu$), and $a_1, \ldots, a_\nu, b; s, t_1, \ldots, t_\nu$ are real numbers and variables. A similar result is valid for $\mathbb{H}^{N+1}$.

Theorem 4.36 [16]. Let $M^\nu(k)$ be a minimal ruled submanifold of $\tilde{M}^\nu(k)$. Then there exist a generalized helicoid $f: \mathbb{R} \times \tilde{M}^\nu(k) \rightarrow \tilde{M}^\nu(k)$ and an open set $U \subset \mathbb{R} \times \tilde{M}^\nu(k)$ such that $f$ restricted to $U$ parametrizes $M$.

Consider a generalization of helicoids in which, instead of “screw lines,” “nicely curved submanifolds” in space forms are used.

The notion of a (locally) symmetric submanifold $M$ of the Euclidean space $\mathbb{R}^n$ is given in [88] as a submanifold (locally) mapped into itself for each $x \in M$ under the reflection of $\mathbb{R}^n$ with respect to the (affine) normal space to $M$ at $x$. Locally symmetric submanifolds of an arbitrary Riemannian space are studied in [277].

Definition 4.19. A submanifold $M \subset \tilde{M}(c)$ is (locally) symmetric if, for every $x \in M$, $M$ is (locally) mapped into itself by the geodesic reflection of $\tilde{M}(c)$ with respect to the subspace $F_x = \exp_x(T_xM^+ \setminus M)$. This $F_x$ is called the submanifold of symmetry.

In [163], the $k$-symmetric submanifolds of $\mathbb{R}^n \ (k \geq 2)$ are defined by using suitable isometries of $\mathbb{R}^n$. The authors of [163] observed that a 2-symmetric submanifold $M$ is invariant under the reflections of $\mathbb{R}^n$ with respect to subspaces of the normal spaces of $M$; therefore, 2-symmetric submanifolds appear as a generalization of symmetric submanifolds.

Definition 4.20. A submanifold $M \subset \tilde{M}(c)$ is 2-symmetric if, for every $x \in M$, there exists an involutive isometry $\sigma_x$ of $\tilde{M}(c)$ such that, locally, $M$ is mapped by $\sigma_x$ into itself and $x$ is a fixed point of $\sigma_x$ isolated on $M$.

It is easy to verify that a totally geodesic submanifold of a symmetric submanifold is a 2-symmetric submanifold, but it is not a symmetric submanifold in general.

The vector space $\frac{1}{N_x} M$ generated by values of the second fundamental form $h = h_0$ at a fixed point $x \in M$ is called the first normal space of $M$ at $x$. Then the $k$th normal space $\frac{k}{N_x} M$ at $x \in M$ is the vector subspace of $T_x\tilde{M}$ generated by values of the $(k-1)th$ fundamental form $h^{k-1}$ of $M$:

$$h^{k-1}: T_xM \times \frac{k-1}{N_x} M \rightarrow (T_xM \oplus \frac{1}{N_x} M \oplus \ldots \oplus \frac{k-1}{N_x} M)^{\perp}.$$ Note that for $k \geq l$, we have $h^{k} = 0$; for $k \leq l$, we have $\frac{k}{N_x} M \neq 0$, $l$ is the maximal value of $k$ such that $\frac{k}{N_x} M \neq 0$ and the vector spaces $\frac{k}{N_x} M$ are mutually orthogonal. For the submanifold $M$, we set $V_x = \oplus_{i=0}^{q} \frac{k}{N_x} M$, where $q = \lfloor\frac{k}{2}\rfloor$. 1768
Theorem 4.37. If $M$ is a 2-symmetric submanifold in $\tilde{M}(c)$, then
\[ \nabla^k h = 0 \text{ for each } k. \] (4.34)

Definition 4.21. A submanifold $M \subset \tilde{M}(c)$ is nicely curved if $\dim N_x M$ does not depend on $x \in M$ for all $k \in \mathbb{N}$.

Recall that a submanifold $M$ of the space form $\tilde{M}(c)$ is said to be essential if it is not contained in any proper totally geodesic submanifold of $\tilde{M}(c)$.

Lemma 4.11. If $M$ is a connected nicely curved submanifold of a real space form $\tilde{M}(c)$, then $M$ is an essential submanifold of a totally geodesic submanifold of $\tilde{M}(c)$ whose dimension is equal to $\dim(\oplus_{i=0}^k N_x M)$.

In particular, the normal space $T_x M^\perp$ of $M$ at $x$ is given by $T_x M^\perp = \oplus_{i=1}^k N_x M$.

Definition 4.22. A submanifold $\tilde{M}$ of $\tilde{M}(c)$, which is a tubular neighborhood of a nicely curved $M$ satisfying condition (4.34) in the set \{exp$_x v$, $x \in M$, $v \in V_z$\}, is foliated by totally geodesic submanifolds of $\tilde{M}(c)$ and is called the multihelicoid.

If $\dim M = 1$, then condition (4.34) implies that $M$ is just a curve of constant curvature, and $\tilde{M}$ becomes a helicoid associated with a curve in the sense of Definition 4.17.

Theorem 4.38 [182]. Any multihelicoid $\tilde{M}$ in $\tilde{M}(c)$ over a nicely curved submanifold $M$ satisfying condition (4.34) is minimal.

For a nicely curved submanifold $M$ in $\tilde{M}(c)$, we can consider the map $\nu: M \to G(p, m - p)$ into the Grassmannian manifold, where $p = \dim(\oplus_{i=0}^k N_x M)$, given by the formula
\[ \nu(x) = \oplus_{i=0}^{2i+1} N_x M \quad (x \in M, q = \left[\frac{r-1}{2}\right]). \] (4.35)

Recall that in any Grassmannian manifold $G(p, q)$, there exist flat totally geodesic submanifolds in each dimension $m \leq \min(p, q - p)$. If $H$ is a flat totally geodesic submanifold of $G(p, q)$, then the map $\exp_a: T_a H \to H, a \in H$, is a totally geodesic (local) isometry (see [184]). If we identify $\mathbb{R}^m$ with $T_a H$, then $\exp_a$ yields a totally geodesic map of $\mathbb{R}^m$ into $G(p, q)$.

Recall that the map $f: M \to N$ between two manifolds is totally geodesic if the covariant derivative $\nabla(df)$ of the differential $df$ is zero (see [78] and Example 2.1).

Theorem 4.39 [184]. Let $M$ be a complete connected nicely curved submanifold of dimension $m$ in $\mathbb{R}^m$, and let $\nu: M \to G(p, m - p)$ be a totally geodesic map defined by (4.35). Then
(i) if $\dim \ker d\nu = 0$, then $\nu(M)$ is a complete totally geodesic submanifold of $G(p, m - p)$,
(ii) if $\dim \ker d\nu \neq 0$, then $d\nu$ defines a foliation on $M$ whose leaves are Euclidean spaces of dimension $r$; moreover, the leaf space $B$ is a complete totally geodesic submanifold of $G(p, m - p)$, the map $\nu$ factors into a Riemannian submersion with totally geodesic leaves, $\pi: M \to B$, followed by a totally geodesic immersion $j: B \to G(p, m - p)$ ($B$ is a totally geodesic submanifold of $G(p, m - p)$ and its connection coincides with the one induced by $j$), and the fiber space $M \to B$ has a flat connection with totally geodesic horizontal fibers,
(iii) $M$ has nonnegative curvature and is locally symmetric.

For a submanifold $M \subset \tilde{M}$ with a ruling $L \subset M$, we can consider a geodesic reflection of $\tilde{M}$ with respect to $L$ that maps a neighborhood of $L$ in $M$ back into $\tilde{M}$.

Definition 4.23 [182, 183]. A ruled submanifold $M^r \subset \mathbb{R}^m$ is symmetric if the geodesic reflection of $\tilde{M}$ with respect to every ruling $L$ maps some neighborhood of $L$ in $M$ back into $M$ and maps the rulings into rulings.
Since a geodesic reflection of a space form $\tilde{M}(c)$ with respect to a totally geodesic submanifold is an isometry (rigid motion), a sufficient condition for a ruled submanifold $M \subset \tilde{M}(c)$ to be symmetric is that $M$ be locally mapped into itself by a geodesic reflection with respect to each of its rulings.

Any multihelicoid $M$ associated with a nicely curved 2-symmetric submanifold in $\tilde{M}(c)$ is also a symmetric ruled submanifold in the sense of Definition 4.23 [183]. The following theorem partially answers the question of how symmetric ruled submanifolds in space forms can be classified.

**Theorem 4.40** [183]. A symmetric ruled submanifold $M^{v+n}$ in a space form $\tilde{M}(c)$ is a minimal submanifold.

On the other hand, minimal submanifolds of a Euclidean space can be regarded as a special case of submanifolds of finite type, which were introduced by B.-Y. Chen in 1984 (for a recent survey, see [62]).

**Definition 4.24** [62]. A submanifold $M \subset \mathbb{R}^N$ is of finite type if every coordinate of its position vector field $f$ can be written as a finite sum of eigenfunctions of the Laplacian $\Delta$ of $M$, i.e., $f = f_0 + f_1 + \ldots + f_m$, where $f_0$ is a constant vector and $\Delta f_i = \lambda_i f_i$, $1 \leq i \leq m$. If, in particular, all eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ are pairwise different, then $M$ is of $m$-type.

Note that every minimal submanifold of a Euclidean space is of 1-type (with $\lambda_1 = 0$), since $\Delta f = 0$. Also, we note that a plane curve of finite type is a part of a circle or a straight line. In [71], the Catalan theorem was extended to ruled submanifolds $M^{v+n}$ of a finite type with $n = 1$ in $\mathbb{R}^N$.

**Theorem 4.41** [71]. A ruled submanifold $M^{v+1}$ in $\mathbb{R}^N$ is of finite type if and only if $M^{v+1}$ is a part of a cylinder on a curve of finite type or a part of a generalized helicoid. In particular, a ruled submanifold $M^{v+1}$ in $\mathbb{R}^{v+2}$ is of finite type if and only if $M^{v+1}$ is a part of a hyperplane, a circular cylinder, a helicoid $\Gamma^2$ in $\mathbb{R}^3$, a cone $K^3$ with vertex $p$ on a minimal ruled surface in a sphere $S^3$ centered at $p$, or a cylinder over $\Gamma^2$ or $K^3$.

Finite-type ruled submanifolds modeled on the basis of spherical submanifolds are studied in [89]. The test for a finite-type Gauss map for ruled submanifolds is given in [12].

**Theorem 4.42** [12]. The only ruled submanifolds $M^{v+1}$ in the Euclidean space $\mathbb{R}^{v+p}$ with finite type Gauss map are cylinders over curves of finite type and $(v+1)$-dimensional Euclidean spaces.

(3) The only ruled submanifolds $M^{v+1}$ of $\mathbb{R}^N$ of constant nonzero mean curvature are generalized cylinders, i.e., products of the form $\mathbb{R}^v \times c(t) \subset \mathbb{R}^v \times \mathbb{R}^{N-v}$, where $c(t) \subset \mathbb{R}^{N-v}$ is a curve of constant first curvature [16]. For $k < 0$, there are no ruled submanifolds $M^{v+1}$ of $\tilde{M}(k)$ of constant nonzero mean curvature [16]. If $k > 0$, the situation is quite different.

**Theorem 4.43** [15]. (a) If $M^{v+1}$ is a ruled submanifold in $\tilde{M}^N(k)$, $k > 0$, of constant nonzero mean curvature, then $N \geq 2v+1$ and $M$ is locally isometric to the product $\mathbb{R} \times N^{v+1}(k)$. (b) If $f: M^{v+1} \to S^{2v+1}(k)$ is a ruled submanifold of a mean curvature of nonzero constant length, then $v + 1$ is even. Given $a > 0$, for any complete smooth curve $c$ in the symmetric space $O(2p)/U(p)$, there exists an isometric ruled immersion $f_{c,a}: \mathbb{R} \times S^{2p-1} \to S^{4p-1}$ of a mean curvature of constant length which is well defined up to a rigid motion of $S^{4p-1}$. Moreover, if $c$ and $c_1$ coincide in a rigid motion of $O(2p)/U(p)$, then $f_{c,a}$ and $f_{c_1,a}$ are the same up to a rigid motion of $S^{4p-1}$.

Thus, we have a complete classification of ruled submanifolds $M^{v+1}$ of codimension $v$ in space forms with mean curvature of constant length. If, in Theorem 4.43, the mean-curvature vector is parallel or the normal bundle is trivial, then $v = 1$ and $M$ is the product of two circles in $S^3 \subset S^N$.

### 4.6. Submanifolds with Spherical Generators

Let us consider submanifolds with generators that are totally umbilic in an ambient space.

**Definition 4.25** [223]. A 1-form $\lambda$ on $TM^\perp$ is called the principal curvature of a submanifold $M \subset \tilde{M}$ if $T_m(\lambda) = \{x \in T_m M : A_\xi x = \lambda(\xi) x, \xi \in T_m M^\perp\}$ is at least a 1-dimensional subspace for all $m \in M$.
Let $G(\lambda) \subset M$ be an open regularity domain, where $\dim T_m(\lambda)$ takes its minimal value $\mu(\lambda, M)$. The principal-curvature function $\lambda$ is smooth on $G(\lambda)$.

**Theorem 4.44** [223]. Assume that the principal curvature function $\lambda$ of a (complete) submanifold $M \subset \tilde{M}(k)$ with $\mu(\lambda, M) > 1$ is given. Then the subspaces $T_m(\lambda)$ form an integrable distribution on $G(\lambda)$ whose leaves $\{L(\lambda)\}$ are (complete) totally umbilic submanifolds (spherical generators) in $\tilde{M}$, and, therefore, in $M$.

There exist isometric immersions without principal curvature, for instance, the immersion of the Veronese surface into $S^4$. The principal curvatures for hypersurfaces in a conformally flat Riemannian space are studied in [139]. For recent studies of completeness of curvature surfaces of submanifolds in complex space forms and pseudo-Riemannian spaces, see [32].

**Corollary 4.14** [223, 203]. Let $M(k) \subset \tilde{M}(\tilde{k})$ be an isometric immersion of space forms under the conditions $k > \tilde{k}$ and $\dim M < \dim \tilde{M} \leq 2 \dim M - 2$. Then, at every point $m \in M$, there exists exactly one principal curvature $\lambda_m$ with $\dim T(\lambda_m) \geq 2$; moreover, $\dim T(\lambda_m) \geq 2 \dim M - \dim \tilde{M} + 1 \geq 3$ (for the case $\dim M + 2 = \dim \tilde{M}$, see [132]). All statements of Theorem 4.44 can be applied to this principal-curvature function.

Conformally flat submanifolds of small codimension possess a similar property. Conformally flat hypersurfaces are naturally connected with other classes (defined extrinsically) such as special quasiumbilic hypersurfaces, loci of $n$-spheres, and canal hypersurfaces [60]. The local structure of conformally flat hypersurfaces was discovered in [57]: they are generically foliated by codimension-1 spheres.

**Definition 4.26** [192]. A submanifold $M^m$ of $\mathbb{R}^{m+p}$, $p \leq m$, is generically foliated by spheres if there exists a dense open set $U \subset M$ such that $U = \bigcup_{i=0}^{p} U_i$, where $U_i$ (for every $i$) is an open set of $M$ that is foliated by $(m - p + i)$-dimensional umbilic submanifolds of $\mathbb{R}^{m+p}$. We denote this class of submanifolds by $\mathcal{F}S^p_m$. The points of $U$ are generic points. A submanifold $M$ is strongly generically foliated by spheres if $M \in \mathcal{F}S^p_m$ and the restriction of $TM^\perp$ to each leaf $S$ of $U$ is parallel in $TS^\perp$. We denote this class of submanifolds by $\mathcal{S}T^p_m$.

This definition implies, in particular, that the leaves in $U_i$ are open sets of standard spheres $S^{m+p+i}$. Let $\mathcal{C}F^p_m$ be the class of conformally flat submanifolds of $\mathbb{R}^{m+p}$. In view of Theorem 4.44, we have the following theorem.

**Theorem 4.45** (local) [192]. If $1 \leq p \leq m - 3$, then $\mathcal{C}F^p_m \subset \mathcal{F}S^p_m$.

**Theorem 4.46** (global) [192]. Let $M^m$ be an orientable conformally flat submanifold of $\mathbb{R}^{m+p}$ ($p \leq m - 3$). If $M$ is $k$-regular (i.e., the leaves have dimension $k$ everywhere and form a regular foliation on $M$, see Theorem 4.45), then $M$ is a sphere bundle (the fibers are spheres) over an $(m - k)$-dimensional manifold.

For the classification of hypersurfaces in Theorem 4.46, see [53].

**Example 4.4** [192]. Consider a standard immersion of the sphere $S^{m-2}(1)$ into $\mathbb{R}^{m-1}$ and any isometric immersion of the compact surface $\mathbb{H}^2(-1)$ of curvature $-1$ into $\mathbb{R}^{17}$ (by the Nash theorem). The product $S^{m-2}(1) \times \mathbb{H}^2(-1)$ is then isometrically immersed into $\mathbb{R}^{m+16}$. If $m \geq 19$, we obtain the situation of Theorem 4.46.

The following theorem generalizes the classical investigations by Riemann [231] (a minimal surface in $\mathbb{R}^3$ that is foliated by circles in parallel planes must be either a piece of a catenoid or the example now called the "Riemann staircase") and Enneper [79] (if a minimal surface in $\mathbb{R}^3$ is foliated by circles or by their arcs, then the planes containing these curves must be parallel).

**Theorem 4.47** [143]. If $M^{n+1}$ is a complete and nonplanar minimal hypersurface in $\mathbb{R}^{n+2}$ ($n \geq 2$) and an open subset of $M$ is foliated by pieces of $\nu$-dimensional spheres, then $M$ is a higher-dimensional catenoid (a hypersurface of revolution).
REFERENCES


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214. B. Papantoniou, “Contact Riemannian manifolds satisfying $R(\xi, x) \ast R = 0$ and $\xi \in (k, \mu)$-nullity distribution,” *Yokohama Math. J.*, **40**, 149–161 (1993).


