FINITE-DIMENSIONAL PERTURBATIONS OF SELF-ADJOINT OPERATORS

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Abstract.

We study finite-dimensional perturbations $A+\gamma B$ of a self-adjoint operator A acting in a Hilbert space $\mathfrak{H}$. We obtain asymptotic estimates of eigenvalues of the operator $A+\gamma B$ in a gap of the spectrum of the operator A as $\gamma \to 0$, and asymptotic estimates of their number in that gap. The results are formulated in terms of new notions of characteristic branches of A with respect to a finite-dimensional subspace of $\mathfrak{H}$ on a gap of the spectrum $\sigma(A)$ and asymptotic multiplicities of endpoints of that gap with respect to this subspace. It turns out that if A has simple spectrum then under some mild conditions these asymptotic multiplicities are not bigger than one. We apply our results to the operator $(Af)(t)=tf(t)$ on $L^2([0,1],\rho_c)$, where ρ_c is the Cantor measure, and obtain the precise description of the asymptotic behavior of the eigenvalues of $A+\gamma B$ in the gaps of $\sigma(A)=\mathfrak{C}(=$ the Cantor set).

1. Introduction

An extensive literature exists on the perturbations of self-adjoint operators. In many cases ones studies perturbations of the form:

$$A(\gamma)=A+\gamma B, \tag{1.1}$$

where A, B are self-adjoint operators, acting on a Hilbert space $\mathfrak{H}$, γ is a real or complex parameter. If λ_0 is an isolated eigenvalue of A, then it is possible to find branches of eigenvalues $\lambda(\gamma)$ and eigenvectors $X(\gamma)$ of the operator $A(\gamma)$ in the form of Taylor expansions:

$$\lambda(\gamma)=\lambda_0+\lambda_2\gamma+\lambda_2\gamma^2+\dots,$$

$$X(\gamma)=X_0+X_1\gamma+X_2\gamma^2+\dots.$$

Such expansions have been obtained first by E. Schrödinger [1] in connection with problems of Quantum Mechanics. These methods have been developed by many mathematicians

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also for the case when the spectrum of the operator A is not discrete (see [2], [3], [4]). Other approaches to perturbation theory of linear operators are connected with some assumptions concerning the perturbing operator B. H. Weyl [5] proved that if the operator B is compact, then the continuous spectrum of $A(\gamma)$ is the same as that of A. It means that only the discrete part of the spectrum of the operator A can vary under such perturbations. This fact will be basic in our investigations. In the papers [6], [7] and [8], the invariance of the absolutely continuous spectrum of A under finite rank and trace class perturbations is proved.

In this paper we study the discrete part of the spectrum of the operator $A + \gamma B$. We restrict ourselves only to finite rank perturbing operators B, and study in detail the asymptotic behavior of the eigenvalues of the operator $A + \gamma B$ in a gap of the spectrum $\sigma(A)$ of A as $\gamma \to 0$. It turns out that even in the case of finite rank perturbations this behavior can be rather complicated. It is determined by the asymptotic behavior of the diagonal block of the resolvent $R_\lambda(A)$ of A, corresponding to the range of B near the endpoints of the gap. We obtain asymptotic estimates of the eigenvalues in terms of so called *characteristic branches* of A with respect to finite-dimensional subspace of $\mathfrak{H}$ on a gap of the spectrum $\sigma(A)$ (Definitions 3.1, 3.3). A new notion of *asymptotic multiplicities of endpoints* of that gap with respect to this subspace is introduced (Definition 3.2). It is shown that the asymptotic multiplicity is responsible for the number of the eigenvalues of $A(\gamma)$ which are born in the gap from this endpoint when γ varies from 0 to some nonzero small value. The full information concerning the number and asymptotic behavior of the eigenvalues (estimates from above and from below) is obtained in the case of a non-negative perturbing operator B (Theorem 4.1). The picture is more complicated in the case of indefinite B: in this case we obtain only estimates of the multiplicities of the eigenvalues near the endpoints of a gap of the spectrum $\sigma(A)$ and some information concerning a localization of these eigenvalues (Theorems 5.1, 5.2, Corollary 5.1 and Proposition 5.1).

We study also the following interesting question: how is the asymptotic multiplicity of an endpoint of the spectrum of A connected with the multiplicity of an endpoint of its spectrum? It is known that in the case where the spectrum of A is discrete, the number of eigenvalues which are born from an eigenvalue of A under a small perturbation is not bigger than the multiplicity of this eigenvalue. This situation suggests the conjecture that the asymptotic multiplicities of the endpoints of the gaps of the spectrum of A are no more than the multiplicity of the spectrum of A. However, it turns out this is not true. We construct an example (Example 6.1), in which the operator A has a simple spectrum and B is a self-adjoint non-positive operator of rank two, but the asymptotic multiplicity of the right endpoint of a gap of the spectrum of A with respect to the range of B is equal to two. By Theorem 4.1 two eigenvalues are born in the gap from this endpoint as γ varies from 0 to a small positive value γ_0. In this example the operator A is represented in the canonical form of the multiplication operator by the independent variable in the space

$L_2(\mathbf{R},\rho)$, and the functions belonging to the range of the operator B oscillate sharply near the endpoint of the gap. It turns out that if A is the multiplication operator and functions from the range of B satisfy some Besov-type condition near the endpoint of the gap, then the asymptotic multiplicity of this endpoint with respect to the range of B is no more than one, i.e. no more than one eigenvalue can be born from the endpoint when γ varies from 0 to a small γ_0 (Theorem 6.1). With the physical point of view in Example 6.1, we observe a resonance phenomenon: the sharply oscillated perturbation excitates not only the endpoint of the gap, but also the points of the spectrum of A accumulating to it from outside of the gap, therefore an additional eigenvalue appears. We believe that it is possible to generalize Theorem 6.1 to the case when $\sigma(A)$ has an arbitrary finite multiplicity m; indeed, it is sufficient to overcome some technical difficulties. In this case, under certain conditions, the asymptotic multiplicity of the endpoint with respect to the range of the operator B will be no more than the multiplicity m.

We apply our results to finite dimensional perturbations of the operator A of multiplication by the independent variable in the space $L_2([0,1],\rho_c)$, where ρ_c is the Cantor measure. This is a generic example of perturbations of operators with singular continuous spectra. We obtain estimates of the rate of convergence of the eigenvalues in the gaps of the spectrum of A to the endpoints of the gaps (Theorems 7.1, 7.3). In these estimates the number $\alpha = \log_3 2$ (which is the Hausdorff dimension of the Cantor set) plays an important role. It is predicted that in the general case of a self-adjoint operator A with singular continuous spectrum $\sigma(A)$, the Hausdorff dimension of $\sigma(A)$ plays an important role in the asymptotic estimates of eigenvalues of $A+\gamma B$ as $\gamma \to 0$.

The paper is divided into eight sections. After this Introduction, in Section 2 we reduce the problem of the description of the discrete spectrum of $A+\gamma B$ in the gap of $\sigma(A)$ to the study of some operator pencils in finite-dimensional spaces. In Section 3 we introduce the notions of the characteristic branches of the operator A with respect to a subspace on a gap of $\sigma(A)$, and of the asymptotic multiplicities of endpoints of the gap with respect to this subspace. We establish also some properties connected with these notions. In Section 4 we obtain the asymptotic estimates (as $\gamma \to 0$) of the eigenvalues of $A(\gamma)$ in the gap of $\sigma(A)$ in the case of the non-negative or non-positive perturbing operator B. In Section 5 we study the case of an indefinite perturbing operator B. Section 6 is devoted to the case where the spectrum of A is simple. In Section 7 we apply our results to finite-dimensional perturbations of the multiplication operator in $L_2([0,1],\rho_c)$, where ρ_c is the Cantor measure. The last section is an Appendix, in which we establish some arithmetical properties of the Cantor set which are used in Section 7.

We shall assume in our paper that A and B are bounded, self-adjoint operators on a Hilbert space $\mathfrak{H}$. We remark that it is possible to generalize our results of to the case of an unbounded self-adjoint operator A.

We shall use the following notations:

$\mathbf{N}$ - the set of all natural numbers;

$\mathbf{R}$ - the field of all real numbers;

$\mathbf{C}$ - the field of all complex numbers;

$\mathfrak{H}$ - a complex Hilbert space;

$A|_{\mathfrak{G}}$ - the restriction of a linear operator A to a subspace $\mathfrak{G} \subset \mathfrak{H}$;

$P_{\mathfrak{G}}$ - the orthogonal projection onto a subspace $\mathfrak{G} \subseteq \mathfrak{H}$;

$\sigma(A)$ - the spectrum of a linear operator A which acts in $\mathfrak{H}$:

$\sigma(\{\Gamma(\lambda)\})$ - the spectrum of a pencil $\{\Gamma(\lambda)\}_{\lambda \in E}$ $(E \subset \mathbf{C})$ of linear operators that act in a Hilbert space $\mathfrak{G}$:

$J_\sigma(A)$ - a gap (finite or infinite) in the spectrum of a self-adjoint operator A;

$R_\lambda(A) = (A - \lambda I)^{-1}$ - the resolvent of a linear operator A;

$\mathfrak{H}(B)$ - the closure of the range of a linear operator B which acts in $\mathfrak{H}$;

$E_B(\Delta)$ - the spectral projection of a self-adjoint operator B, corresponding to a measurable subset $\Delta \subseteq \mathbf{R}$;

$\mathfrak{H}^-(B)$, $\mathfrak{H}^+(B)$ - the invariant subspaces corresponding to the positive and negative spectrum of a self-adjoint operator B, i.e.:

$$\mathfrak{H}^+(B) = E_B((0,\infty))(\mathfrak{H}), \quad \mathfrak{H}^-(B) = E_B((-\infty,0))(\mathfrak{H}).$$

Notice that we will use for brevity the same notation I for the identity operators acting in the space $\mathfrak{H}$ as well as in any subspace of it; this does not lead to a confusion.

2. A reduction of the problem to a finite-dimensional case

2.1° Let A, B be bounded self-adjoint operators acting on a Hilbert space $\mathfrak{H}$ and assume that B has finite rank N, i.e.

$$\dim(\mathfrak{H}(B)) = N < \infty. \tag{2.1}$$

Furthermore, everywhere in this paper we assume that in (1.1) the parameter γ is real.

The following statement is essentially known, but we have not found an explicit proof of it in the literature. In our proof we use the methods of M. G. Krein, which he applied in the theory of self-adjoint extensions of self-adjoint operators ([9], [10]).

Proposition 2.1. *If the condition (2.1) is satisfied, then each gap $J_\sigma(A)$ in the spectrum of A contains no more than N eigenvalues of the operator $A(\gamma)$ (1.1) (taking into account their multiplicities), and each of them (if they exist) tends to one of the endpoints of the gap when $\gamma \to 0$.*

Proof. Assume that the gap $J_\sigma(A) = (a, b)$ is bounded. Denote by $E(\gamma)$ the invariant subspace of the operator $A(\gamma)$, corresponding to the part of its spectrum $\sigma(A(\gamma)) \cap (a, b)$.

Assume on the contrary $\dim(E(\gamma)) > N$. Then by virtue of (2.1) there exists a vector $e \neq 0$ such that

$$e \in E(\gamma) \cap (\mathfrak{H}(B))^{\perp}.$$

It follows that

$$\|(A+\gamma B)e - ce\| < d\|e\|, \tag{2.2}$$

where $c = \frac{a+b}{2}$, $d = \frac{b-a}{2}$. Furthermore, $Be = 0$ because

$$\ker(B) = \mathfrak{H}(B)^{\perp}. \tag{2.3}$$

We therefore obtain from (2.2) that

$$\|Ae - ce\| < d\|e\|.$$

On the other hand, since (a, b) is a gap of $\sigma(a)$, one has:

$$\forall x \in \mathfrak{H} : \quad \|Ax - cx\| \geq d\|x\|.$$

This contradiction proves that $\dim(E(\gamma)) \leq N$, i.e. we have proved the first statement of the proposition.

Let us proceed with the proof of the second statement. It is known that the eigenvalues of the operator $A(\gamma)$ lying in the gap (a, b) depend continuously on γ ([2]). Therefore it is enough to prove that

$$\forall \epsilon \in \left(0, \frac{b-a}{2}\right) \ \exists \delta > 0 \ \ \forall \gamma \in (-\delta, \delta) : \quad \sigma(A(\gamma)) \cap (a+\epsilon, b-\epsilon) = \emptyset.$$

Assume, on the contrary, that this statement is false. Then there exist a number $\epsilon_0 > 0$ and a sequence $\{\gamma_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \gamma_n = 0$ and for any n there exists an eigenvalue $\lambda(\gamma_n)$ of the operator $A(\gamma_n)$ such that $\lambda(\gamma_n) \in (a+\epsilon_0, b-\epsilon_0)$ and

$$\lim_{n\to\infty} \lambda(\gamma_n) = \lambda_0 \in [a+\epsilon_0, b-\epsilon_0].$$

Then, by the above mentioned continuous dependence of eigenvalues on γ, the number λ_0 is an eigenvalue of the operator A lying in (a, b). The latter contradicts the fact that (a, b) is a gap of $\sigma(A)$.

The case of an unbounded gap (a, b) can be studied analogously, taking into account that $\sigma(A+\gamma B) \subseteq (-\|A\| - |\gamma|\|B\|, \|A\| + |\gamma|\|B\|)$. ■

2.2° We turn now to the description of eigenvalues of $A(\gamma)$ in a gap $J_\sigma(A)$ in terms of a pencil of operators that act in the N-dimensional subspace $\mathfrak{H}(B)$. This pencil is:

$$\{\Gamma_B(\lambda)\}_{\lambda \in J_\sigma(A)}, \tag{2.4}$$

where

$$\Gamma_B(\lambda) = P_{\mathfrak{H}(B)} R_\lambda(A) B|_{\mathfrak{H}(B)}. \tag{2.5}$$

Lemma 2.1. *Let $J_\sigma(A) = (a, b)$. Then*

$$\sigma(A(\gamma)) \cap (a, b) = \sigma(\{I + \gamma \Gamma_B(\lambda)\}) \tag{2.6}$$

and the multiplicity of each eigenvalue $\lambda \in (a, b)$ of the operator $A(\gamma)$ coincides with $\dim(\ker(I + \gamma \Gamma_B(\lambda)))$.

Proof. Consider in $\mathfrak{H}$ the pencil of operators $\{I + \gamma R_\lambda(A)B\}_{\lambda \in (a,b)}$. The representation

$$I + \gamma R_\lambda(A)B = R_\lambda(A)(A(\gamma) - \lambda I) \ \ (\lambda \in (a, b))$$

implies that

$$\sigma(A(\gamma)) \cap (a, b) = \sigma(\{I + \gamma R_\lambda(A)B\}) \tag{2.7}$$

and moreover

$$\forall \lambda \in (a, b) : \ \ \dim(\ker(A(\gamma) - \lambda I))) = \dim(\ker(I + \gamma R_\lambda(A)B)). \tag{2.8}$$

Consider the orthogonal splitting:

$$\mathfrak{H} = \mathfrak{H}(B) \bigoplus \mathfrak{H}(B)^\perp \tag{2.9}$$

and the block matrix representation of the operator $I + \gamma R_\lambda(A)B$ with respect to it:

$$I + \gamma R_\lambda(A)B = \{r_{i,j}(\lambda, \gamma)\}_{i,j=1}^2. \tag{2.10}$$

Taking into account (2.3), we obtain:

$$r_{1,1}(\lambda, \gamma) = (I + \gamma \Gamma_B(\lambda))P_{\mathfrak{H}(B)}, \tag{2.11}$$

$$r_{1,2}(\lambda, \gamma) = 0, \tag{2.12}$$

$$r_{2,1}(\lambda, \gamma) = \gamma(I - P_{\mathfrak{H}(B)})R_\lambda(A)BP_{\mathfrak{H}(B)},$$

and

$$r_{2,2}(\lambda, \gamma) = I - P_{\mathfrak{H}(B)}. \tag{2.13}$$

By (2.12) the block matrix (2.10) is triangular. Then (2.11), (2.13) imply that

$$\sigma(\{I + \gamma R_\lambda(A)B\}) = \sigma(\{I + \gamma \Gamma_B(\lambda)\}) \tag{2.14}$$

and

$$\forall\lambda \in (a,b) \quad \dim(\ker(I+\gamma R_\lambda(A)B)) = \dim(\ker(I+\gamma\Gamma_B(\lambda))). \tag{2.15}$$

From (2.7), (2,8), (2.11), (2.12), (2.13) the assertions of our lemma follows. ■

We shall consider a modification of the operator pencil (2.4) connected with the following orthogonal splitting of the subspace $\mathfrak{H}(B)$:

$$\mathfrak{H}(B) = \mathfrak{H}^+(B) \bigoplus \mathfrak{H}^-(B), \tag{2.16}$$

where the spaces $\mathfrak{H}^+(B), \mathfrak{H}^-(B)$ were defined in the introduction. This splitting is valid by virtue of (2.3). We shall consider also the following positive self-adjoint operators which act in the summands of the above splitting:

$$\hat{B}_+ = B|_{\mathfrak{H}^+(B)}, \quad \hat{B}_- = -B|_{\mathfrak{H}^-(B)}. \tag{2.17}$$

Let $J_\sigma(A) = (a,b)$. Consider the pencil of operators $\{\hat{\Gamma}_B(\lambda)\}_{\lambda\in(a,b)}$ which act in the subspace $\mathfrak{H}(B)$, where

$$\hat{\Gamma}_B(\lambda) = |B|^{\frac{1}{2}} R_\lambda(A) B^{[\frac{1}{2}]}|_{\mathfrak{H}(B)}, \tag{2.18}$$

and $B^{[\frac{1}{2}]}$ is defined via the splitting (2.16) by

$$B^{[\frac{1}{2}]} = \mathrm{diag}(\hat{B}_+^{\frac{1}{2}}, -\hat{B}_-^{\frac{1}{2}}) \tag{2.19}$$

(see (2.17)). It is clear that

$$|B|^{\frac{1}{2}} = \mathrm{diag}(\hat{B}_+^{\frac{1}{2}}, \hat{B}_-^{\frac{1}{2}}). \tag{2.20}$$

The relations (2.18), (2.19), (2.20) imply that the operator $\hat{\Gamma}_B(\lambda)$ has the following block matrix representation with respect to the splitting (2.16):

$$\hat{\Gamma}_B(\lambda) = \{\hat{\gamma}_B^{i,j}(\lambda)\}_{i,j=+,-} \tag{2.21}$$

with the elements:

$$\hat{\gamma}_B^{+,+}(\lambda) = \hat{B}_+^{\frac{1}{2}} P_{\mathfrak{H}^+(B)} R_\lambda(A) \hat{B}_+^{\frac{1}{2}}, \tag{2.22}$$

$$\hat{\gamma}_B^{-,-}(\lambda) = -\hat{B}_-^{\frac{1}{2}} P_{\mathfrak{H}^-(B)} R_\lambda(A) \hat{B}_-^{\frac{1}{2}}, \tag{2.23}$$

$$\hat{\gamma}_B^{+,-}(\lambda) = -\hat{B}_+^{\frac{1}{2}} P_{\mathfrak{H}^+(B)} R_\lambda(A) \hat{B}_-^{\frac{1}{2}},$$

$$\hat{\gamma}_B^{-,+}(\lambda) = (\hat{\gamma}_B^{+,-}(\lambda))^\star. \tag{2.24}$$

Lemma 2.2. *Let $J_\sigma(A) = (a, b)$ and let $\lambda \in (a, b)$. Then the operator*

$$T = |B|^{\frac{1}{2}}_{|\mathfrak{H}(B)}$$

is continuously invertible, and realizes the similarity between the operators $\hat{\Gamma}_B(\lambda)$ and $\Gamma_B(\lambda)$:

$$\Gamma_B(\lambda) = T^{-1}\hat{\Gamma}_B(\lambda)T. \tag{2.25}$$

Hence, for all $\gamma \in \mathbf{R}$ and $\lambda \in (a, b)$

$$\sigma(I + \gamma\Gamma_B(\lambda)) = \sigma(I + \gamma\hat{\Gamma}_B(\lambda))$$

and

$$\dim(\ker(I + \gamma\Gamma_B(\lambda)) = \dim(\ker(I + \gamma\hat{\Gamma}_B(\lambda))).$$

Proof. T is continuously invertible by virtue of (2.1) and (2.3). The representation (2.25) follows from (2.5), (2.18), and the equality $B = B^{[\frac{1}{2}]}|B|^{\frac{1}{2}}$. ■

3. The characteristic branches and the asymptotic multiplicities of the endpoints of a gap

3.1° In this section we shall introduce a new notion of *multiplicity for the endpoints of a gap* $J_\sigma(A)$, which is a generalization of the usual notion of the multiplicity for eigenvalues. For this purpose we shall define the notion of so *characteristic branches* which will play an important role in the sequel. We shall assume throughout this section that B is non-negative:

$$B \geq 0 \tag{3.1}$$

and has a finite rank N.

Consider a gap $J_\sigma(A)$ and the pencil of self-adjoint operators that act in the space $\mathfrak{H}(B)$:

$$\tilde{\Gamma}_B(\lambda) = B^{\frac{1}{2}}R_\lambda(A)B^{\frac{1}{2}}|_{\mathfrak{H}(B)} \tag{3.2}$$

with $\lambda \in J_\sigma(A)$. We shall need the following lemma:

Lemma 3.1. *The operator function $\tilde{\Gamma}_B(\lambda)$ increases on $J_\sigma(A)$, i.e.*

$$\tilde{\Gamma}_B(\lambda_1) > \tilde{\Gamma}_B(\lambda_2) \text{ whenever } \lambda_1, \lambda_2 \in J_\sigma(A), \text{ and } \lambda_1 > \lambda_2.$$

Moreover, there exist N branches of eigenvalues

$$\{\mu_k^B(\lambda)\}_{k=1,2,\ldots,N} \tag{3.3}$$

of the operators $\tilde{\Gamma}_B(\lambda)$ which satisfy the following conditions:

(a) for any fixed $\lambda \in J_\sigma(A)$, the sequence (3.3) is non-decreasing and exhausts all the eigenvalues of the operator $\tilde{\Gamma}_B(\lambda)$ (taking into account their multiplicities):

$$\mu_1^B(\lambda) \le \mu_2^B(\lambda) \le \cdots \le \mu_N^B(\lambda);$$

(b) the functions $\{\mu_k^B(\lambda)\}$ $(k = 1, 2, \ldots, N)$ are continuous and increasing in λ.

Proof. The formula

$$\frac{d}{d\lambda}\tilde{\Gamma}_B(\lambda) = B^{\frac{1}{2}} R_\lambda^2(A) B^{\frac{1}{2}}|_{\mathfrak{H}(B)}$$

and the equality (2.3) imply that $\frac{d}{d\lambda}\tilde{\Gamma}_B(\lambda)$ is self-adjoint and positive: $\frac{d}{d\lambda}\tilde{\Gamma}_B(\lambda) > 0$. Hence the family of quadratic forms

$$\tilde{\Gamma}_B(\lambda)[u] = (\tilde{\Gamma}_B(\lambda)u, u)$$

is increasing with respect to λ on the interval $J_\sigma(A)$. This establishes the first statement of the lemma. The family of operators (3.2) act in a common finite-dimensional subspace $\mathfrak{H}(B)$ of $\mathfrak{H}$ and depend continuously (in the operator norm) on $\lambda \in J_\sigma(A)$. Therefore the family of their spectral projections depends continuously in norm on λ. This yields the continuity of the $\{\mu_k^B(\lambda)\}$ $(k = 1, 2, \ldots, N)$ in the parameter λ. The mini-max characterization of the eigenvalues of self-adjoint operators imply that the branches of eigenvalues (3.3), arranged in non-decreasing order as in (a), are strictly increasing in the parameter λ. ■

Corollary 3.1. *Assume that $J_\sigma(A) = (a, b)$. Then the following limits (finite or infinite) exist:*

$$\mu_{a,k}^B = \lim_{\lambda \to a+0} \mu_k^B(\lambda), \quad \mu_{b,k}^B = \lim_{\lambda \to b-0} \mu_k^B(\lambda), \quad (k = 1, 2 \ldots, N) \tag{3.4}$$

and

$$\mu_{a,1}^B \le \mu_{a,2}^B \le \cdots \le \mu_{a,N}^B, \quad \mu_{b,1}^B \le \mu_{b,2}^B \le \cdots \le \mu_{b,N}^B.$$

Let us introduce some definitions.

Definition 3.1. Let $J_\sigma(A) = (a, b)$. We call the branches (3.3) of eigenvalues of the family of operators (3.2), ordered as in Lemma 3.1, the *characteristic branches of the operator A in the gap $J_\sigma(A)$ with respect to the operator B*. If $\mathfrak{G}$ is a finite-dimensional subspace of $\mathfrak{H}$ and $B = P_{\mathfrak{G}}$, then we call these branches the *characteristic branches of the operator A in the gap $J_\sigma(A)$ with respect to the subspace $\mathfrak{G}$*.

Definition 3.2. Let $J_\sigma(A) = (a, b)$. Let $0 \le l(a), l(b) \le N$ be defined via the conditions:

$$-\infty = \mu_{a,1}^B = \mu_{a,2}^B = \cdots = \mu_{a,l(a)}^B < \mu_{a,l(a)+1}^B \le \mu_{a,l(a)+2}^B \le \cdots \le \mu_{a,N}^B, \tag{3.5}$$

and

$$\mu^B_{b,1} \le \cdots \le \mu^B_{b,N-l(b)} < +\infty = \mu^B_{b,N-l(b)+1} = \mu^B_{b,N-l(b)+2} = \cdots = \mu^B_{b,N} \tag{3.6}$$

(see (3.4)). We call the numbers $l(a)$, $l(b)$ the *asymptotic multiplicities of the endpoints* a, b *of* $J_\sigma(A)$ *with respect to the operator* B, and denote them by

$$l(a) = M(a, A, B), \quad l(b) = M(b, A, B).$$

If $\mathfrak{G}$ is an N-dimensional subspace of $\mathfrak{H}$ and B coincides with the orthogonal projection $P_{\mathfrak{G}}$, then we call the numbers $M(a, A, B)$, $M(b, A, B)$ the *asymptotic multiplicities of the endpoints* a, b *of* $J_\sigma(A)$ *with respect to the subspace* $\mathfrak{G}$, and denote them by $M_{\mathfrak{G}}(a, A)$ and $M_{\mathfrak{G}}(b, A)$, respectively.

Definition 3.3. Let $J_\sigma(A) = (a, b)$. The subsets of the branches (3.3):

$$\{\mu^B_k(\lambda)\}_{k=1,2,\ldots,l(a)} \ \ (l(a) = M(a, A, B)),$$

and

$$\{\mu^B_k(\lambda)\}_{k=l(b)+1,l(b)+2,\ldots,N} \ \ (l(b) = M(b, A, B))$$

are called the *main characteristic branches of the operator* A *with respect to the operator* B *near the endpoints* a and b *respectively.* In case $P = P_{\mathfrak{G}}$, where $\mathfrak{G}$ is an N-dimensional subspace of $\mathfrak{H}$, we call the branches

$$\{\mu^P_k(\lambda)\}_{k=1,2,\ldots,l(a)} \ \ (l(a) = M_{\mathfrak{G}}(a, A)),$$

and

$$\{\mu^P_k(\lambda)\}_{k=l(b)+1,l(b)+2,\ldots,N} (l(b) = M_{\mathfrak{G}}(b, A))$$

the *main characteristic branches of the operator* A *with respect to the subspace* $\mathfrak{G}$ *near the endpoints* a *and* b *respectively.*

Remark 3.1. The numbers $M_{\mathfrak{G}}(a, A)$, $M_{\mathfrak{G}}(b, A)$ and the corresponding main characteristic branches characterize the asymptotic behavior near the endpoints a, b of $J_\sigma(A)$ of the diagonal block

$$\tilde{\Gamma}_P(\lambda) = P_{\mathfrak{G}} R_\lambda(A)|_{\mathfrak{G}} \tag{3.7}$$

of the resolvent $R_\lambda(A)$ corresponding to the subspace $\mathfrak{G} \subset \mathfrak{H}$.

Remark 3.2. In the previous consideration the gap $J_\sigma(A) = (a, b)$ may be infinite. Since the operator A is bounded, there are two such possibilities: either $J_\sigma(A) = (-\infty, \min \sigma(A))$ or $J_\sigma(A) = (\max \sigma(A)), +\infty)$. It is well known that $\|R_\lambda(A)\| \le (|\lambda| - \|A\|)^{-1}$ for $|\lambda| > \|A\|$, and in particular, $\lim_{|\lambda|\to+\infty} \|R_\lambda(A)\| = 0$. Therefore the above considerations imply that for any nonnegative finite rank operator B

$$M(-\infty, A, B) = M(+\infty, A, B) = 0.$$

The following example justifies the term "asymptotic multiplicity".

Example 3.1. Assume that A is a self-adjoint operator that acts on an N-dimensional Hilbert space $\mathfrak{H} = \mathbf{C}^N$. Then $\sigma(A)$ consists of N eigenvalues (with multiplicity counted) $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$. Assume that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_m < \lambda_{m+1}, \tag{3.8}$$

i.e. the multiplicity of the eigenvalue λ_1 is m. Consider the gap $J_\sigma(A) = (\lambda_1, \lambda_{m+1})$ in $\sigma(A)$, and let $\mathfrak{G} = \mathfrak{H}$. The eigenvalues of the resolvent $R_\lambda(A)$ have the form:

$$\mu_k(\lambda) = \frac{1}{\lambda_k - \lambda}, \quad (k = 1, 2, \ldots, N).$$

We see from (3.8) that

$$\lim_{\lambda\to\lambda_1+0} \mu_k(\lambda) \begin{cases} = -\infty & \text{for } \ 1 \leq k \leq m, \\ > -\infty & \text{for } \ m+1 \leq k \leq N. \end{cases}$$

It follows that $M_{\mathfrak{G}}(\lambda_1, A) = m$, i.e. in this case the asymptotic multiplicity coincides with the usual multiplicity of the eigenvalue λ_1. The same is true for all other gaps and their endpoints.

3.2° We shall establish now some invariant of the asymptotic multiplicity. Let us denote

$$L = \|\tilde{B}^{-1}\|^{-1}, \quad K = \|B\|, \tag{3.9}$$

where $\tilde{B} := B|_{\mathfrak{H}(B)}$. Let $\{\mu_k(\lambda)\}$ be the characteristic branches of A with respect to the subspace $\mathfrak{H}(B)$, i.e.

$$\mu_k(\lambda) = \mu_k^P(\lambda), \quad (k = 1, 2, \ldots, N), \tag{3.10}$$

where

$$P = P_{\mathfrak{H}(B)}. \tag{3.11}$$

Theorem 3.1. *Let $J_\sigma(A) = (a,b)$. Then*

$$M(a,A,B) = M_{\mathfrak{H}(B)}(a,A), \quad M(b,A,B) = M_{\mathfrak{H}(B)}(b,A). \tag{3.12}$$

Furthermore, the characteristic branches of A with respect to B admit the following estimates for any $\lambda \in (a,b)$:

$$L\mu_k(\lambda) \le \mu_k^B(\lambda) \le K\mu_k(\lambda), \;\; (k = 1,2,\ldots,N). \tag{3.13}$$

Proof. Observe that by virtue of (2.1) and (2.3) the operator $\tilde{B} = B|_{\mathfrak{H}(B)}$ is continuously invertible. We also have the representation:

$$\tilde{\Gamma}_B(\lambda) = \tilde{B}^{\frac{1}{2}}\tilde{\Gamma}_P(\lambda)\tilde{B}^{\frac{1}{2}},$$

Denote by Λ_k the set of the all k-dimensional subspaces of the space $\mathfrak{H}(B)$. Then by the mini-max characterization of eigenvalues of self-adjoint operators, we obtain for the operator $\tilde{\Gamma}_B(\lambda)$ the estimate:

$$\begin{aligned}
\mu_k^B(\lambda) &= \min_{L\in\Lambda_k}\left(\max_{x\in L,\|x\|=1}\left((\tilde{\Gamma}_P(\lambda)\tilde{B}^{\frac{1}{2}}x, \tilde{B}^{\frac{1}{2}}x)\right)\right) \\
&= \min_{L\in\Lambda_k}\left(\max_{x\in L,\|x\|=1}\left((\tilde{\Gamma}_P(\lambda)\frac{\tilde{B}^{\frac{1}{2}}x}{\|\tilde{B}^{\frac{1}{2}}x\|}, \frac{\tilde{B}^{\frac{1}{2}}x}{\|\tilde{B}^{\frac{1}{2}}x\|})\|\tilde{B}^{\frac{1}{2}}x\|^2\right)\right) \\
&\le \|\tilde{B}\| \min_{L\in\Lambda_k}\left(\max_{y\in L,\|y\|=1}(\tilde{\Gamma}_P(\lambda)y,y)\right) = \|B\|\;\mu_k(\lambda).
\end{aligned}$$

This establishes the right inequality in (3.13). The left inequality can be proved analogously using the invertibility of the operator $\tilde{B}$. Finally, it is easy to see the inequalities (3.13) imply the equalities (3.12). ■

3.3° Let $\mathfrak{G}$ be a subspace of $\mathfrak{H}$ with $\dim(\mathfrak{G}) = N < \infty$. Choose in $\mathfrak{G}$ an orthonormal basis

$$g_1, g_2, \ldots, g_N$$

and consider the matrix representation of the operator function $\tilde{\Gamma}_P(\lambda)$ (3.7) with respect to this basis:

$$\tilde{\Gamma}(\lambda) = \{\tilde{\gamma}_{i,j}(\lambda)\}_{i,j=1}^N \tag{3.14}$$

with

$$\tilde{\gamma}_{i,j}(\lambda) = (R_\lambda(A)g_j, g_i). \tag{3.15}$$

In terms of the matrix representation (3.14), (3.15) we can formulate a criterion for the equalities

$$M_{\mathfrak{G}}(a,A) = 0, \tag{3.16}$$

$$M_{\mathfrak{G}}(b,A) = 0. \tag{3.17}$$

Proposition 3.1. *Let $J_\sigma(A) = (a, b)$. The relation (3.16) (or, (3.17)) is valid if and only if the following relations are fulfilled:*

$$\lim_{\lambda \to a+0} \tilde{\gamma}_{i,i}(\lambda) > -\infty \qquad \textit{for } i = 1, 2, \dots, N$$

$$(\textit{respectively, } \lim_{\lambda \to b-0} \tilde{\gamma}_{i,i}(\lambda) < +\infty) \quad \textit{for } i = 1, 2, \dots, N).$$

Proof. The statement follows from the definition of the multiplicities $M_{\mathfrak{G}}(a, A)$, $M_{\mathfrak{G}}(b, A)$ and the formula:

$$\sum_{i=1}^{N} \tilde{\gamma}_{i,i}(\lambda) = \sum_{i=1}^{N} \mu_i(\lambda) = \operatorname{trace}(\tilde{\Gamma}(\lambda)),$$

where $\mu_i(\lambda) = \mu_i^P(\lambda)$, $P = P_{\mathfrak{G}}$, and from the monotonicity of the functions $\tilde{\gamma}_{i,i}(\lambda)$, $\mu_i(\lambda)$ on the gap (a, b). ■

4. Estimates of eigenvalues in a gap for semi-definite perturbations

We turn now to the problem on perturbations stated in Section 2. We shall assume in this section that the perturbing operator B is semi-definite, i.e. one of two conditions is satisfied: either

$$B \geq 0 \tag{4.1,}$$

or

$$B \leq 0 \tag{4.2}$$

As above, we assume that B has finite rank N. It turns out that in this case one can find the number of eigenvalues of the operator $A(\gamma) = A + \gamma B$ in a gap $J_\sigma(A)$ for γ small enough, and estimate from above and from below the rate of their convergence to the endpoints of the gap as $\gamma \to 0$. We formulate these results in terms of the asymptotic multiplicity and the main characteristic branches (see Definitions 3.1 - 3.3).

We shall assume that $\gamma > 0$ and restrict ourselves to the case (4.1). The other cases (where either $\gamma < 0$ or where (4.2) holds) can be considered analogously.

Theorem 4.1. *Assume that (4.1) holds and let (a, b) be a gap of $\sigma(A)$. If $a > -\infty$, then for a small enough $\gamma > 0$ the number of eigenvalues $\lambda_k(\gamma)$ of the operator $A(\gamma)$ in (a, b), counting their multiplicities, is equal to*

$$l(a) = M_{\mathfrak{H}(B)}(a, A). \tag{4.3}$$

Thus, one has

$$a < \lambda_{l(a)}(\gamma) \leq \lambda_{l(a)-1}(\gamma) \leq \cdots \leq \lambda_2(\gamma) \leq \lambda_1(\gamma) < b, \tag{4.4}$$

with multiplicity counted. Furthermore, all the eigenvalues $\lambda_k(\gamma)$ converge monotonically to the left endpoint a of the gap $J_\sigma(A)$ when $\gamma \downarrow 0$, and the following estimates of the rate of their convergence are valid:

$$\forall \lambda \in (a,b): \quad \mu_k^{-1}(-\frac{1}{K\gamma}) \leq \lambda_k(\gamma) \leq \mu_k^{-1}(-\frac{1}{L\gamma}) \tag{4.5}$$

(k = 1, 2, ... $l(a)$). Here μ_k^{-1} are the inverses of the functions $\mu_k(\lambda)$, the main characteristic branches of the operator A with respect to the subspace $\mathfrak{H}(B)$ near the endpoint a, and the constants K, L are defined by (3.9).

Moreover, if $m := min\{\lambda; \lambda \in \sigma(A)\}$, then the left infinite gap $(-\infty, m)$ of $\sigma(a)$ contains no eigenvalues of the operator $A(\gamma)$ for a small enough $\gamma > 0$.

Proof. Let us observe first that in our case

$$\hat{\Gamma}_B(\lambda) = \tilde{\Gamma}_B(\lambda)$$

(see (2.18), (3.2)). Using Lemmas 2.1 and 2.2 we see that the set $\sigma(A(\gamma)) \cap (a,b)$ coincides with the spectrum $\sigma(\Phi(\gamma))$ of the pencil of the operators

$$\Phi(\gamma) = \{I + \gamma\tilde{\Gamma}_B(\lambda)\}_{\lambda \in (a,b)},$$

which act in the N-dimensional space $\mathfrak{H}(B)$. Furthermore, the multiplicity of eigenvalue $\lambda \in \sigma(A(\gamma)) \cap (a,b)$ coincides with $\dim(\ker(\Phi(\gamma)))$.

Consider the characteristic branches of A on (a,b) with respect to B:

$$\mu_1^B(\lambda) \leq \mu_2^B(\lambda) \leq \cdots \leq \mu_N^B(\lambda). \tag{4.6}$$

We see that $\sigma(\Phi(\gamma))$ coincides with the union of the sets of solutions of the equations:

$$\mu_k^B(\lambda) = -\frac{1}{\gamma} \tag{4.6k}$$

(k = 1, 2, ... , N). Since the functions $\mu_k^B(\lambda)$ increase, each of these equations has at most one solution.

In view of Theorem 3.1,

$$M(a,A,B) = l(a)$$

(see (4.3)). Therefore

$$\lim_{\lambda \to a+0} \mu_k(\lambda) \begin{cases} = -\infty & \text{for } 1 \leq k \leq l(a), \\ > -\infty & \text{for } l(a)+1 \leq k \leq N \end{cases}$$

(see (3.10), (3.11)). From this we conclude that for a small enough $\gamma > 0$ the equation (4.6k) has a unique solution $\lambda_k(\gamma)$ in (a,b) for every $k = 1, 2 \ldots, l(a)$, and it has no solutions in this gap for $k = l(a)+1, l(a)+2, \ldots, N$. So, for a small enough $\gamma > 0$ the set $\sigma(\Phi(\gamma))$ consists of $l(a)$ numbers, which are ordered according to (4.4) (see (4.6)). We see from the equations (4.6k) that each function $\lambda_k(\gamma)$ is increasing, because each $\mu_k^B(\lambda)$ has the same property. Also, the number of repetitions of $\lambda_k(\gamma)$ in the sequence (4.4) coincides with the number of repetitions of μ_k in the sequence (4.6) for $\lambda = \lambda_k(\gamma)$. The last number coincides with the multiplicity of the eigenvalue μ_k of the operator $\tilde{\Gamma}_B(\lambda)$, which is equal to

$$\dim(\ker(I + \gamma\tilde{\Gamma}_B(\lambda))).$$

So, for a small enough $\gamma > 0$, the sequence of eigenvalues (4.4) exhausts the set $\sigma(A(\gamma)) \cap (a, b)$, and these eigenvalues repeat themselves according to their multiplicities.

Taking into account the monotonicity of the functions $\mu_k(\lambda)$, we obtain the estimates (4.5) from the equations (4.6k) and (3.13). These estimates imply the convergence of $\lambda_k(\gamma)$ to the endpoint a as $\gamma \downarrow 0$. This convergence is monotone because the functions $\lambda_k(\gamma)$ are monotone.

The last assertion of the theorem follows from Remark 3.2. ■

Remark 4.1. If the condition (4.2) is fulfilled, the formulation of Theorem 4.1 is the same as above, except that the right endpoint b of the gap (a, b) of $\sigma(A)$ appears instead of the left endpoint a, and the right infinite gap $(M, +\infty)$ (where $M := \max\{\lambda; \lambda \in \sigma(A)\}$) contains no eigenvalues of $A(\gamma)$ for $\gamma > 0$ small enough.

5. Multiplicities and localization of eigenvalues for indefinite perturbations

5.1°. It turns out that in terms of the characteristic branches and the asymptotic multiplicities it is possible to obtain information concerning multiplicities and localization of eigenvalues in a gap of $\sigma(A)$ also for indefinite perturbations.

In this section we assume that B is a self-adjoint operator of finite rank N, but we do not assume the definiteness of B, i.e. $\sigma(B)$ can contain both positive and negative numbers. We assume without loss of generality that $\gamma > 0$.

Theorem 5.1. *Consider a gap (a, b) of the spectrum $\sigma(A)$. Then for any $\lambda_0 \in (a, b)$ there exists a number $\gamma(\lambda_0) > 0$, such that for any $\gamma \in (0, \gamma(\lambda_0))$ the multiplicity of each eigenvalue of the operator $A(\gamma)$ which belongs to $(a, \lambda_0]$ (or, to $[\lambda_0, b)$) is at most $M_{\mathfrak{H}^+(B)}(a, A)$ (respectively, $M_{\mathfrak{H}^-(B)}(b, A)$).*

Proof. By Lemmas 2.1 and 2.2, the statement of the theorem is equivalent to the following statement:

There exists a $\gamma(\lambda_0) > 0$ so that for all $\gamma \in (0, \gamma(\lambda_0))$ we have

$$\forall \lambda \in (a, \lambda_0] : \quad \dim(\hat{\mathfrak{G}}(\lambda, \gamma)) \leq j(a), \tag{5.1}$$

and

$$\forall\lambda \in [\lambda_0, b): \quad \dim(\hat{\mathfrak{G}}(\lambda,\gamma)) \le j(b). \tag{5.2}$$

Here

$$\hat{\mathfrak{G}}(\lambda,\gamma) = \ker(\Gamma(\lambda,\gamma)), \quad \Gamma(\lambda,\gamma) = I + \gamma\hat{\Gamma}_B(\lambda) \tag{5.3}$$

(see (2.18)), and

$$j(a) = M_{\mathfrak{H}^+(B)}(a, A), \quad j(b) = M_{\mathfrak{H}^-(B)}(b, A). \tag{5.4}$$

In order to prove this statement we shall derive some estimates. The block matrix representation (2.21), the equalities (2.22) and (2.23), and the relations (2.24) and (3.2) imply:

$$\mathrm{Re}(\hat{\Gamma}_B(\lambda)) = \mathrm{diag}\{\tilde{\Gamma}_{B_+}(\lambda), -\tilde{\Gamma}_{B_-}(\lambda)\}, \tag{5.5}$$

where

$$B_+ = P_{\mathfrak{H}^+(B)}B, \quad B_- = P_{\mathfrak{H}^-(B)}B. \tag{5.6}$$

Then one has:

$$\mathrm{Re}((\Gamma(\lambda,\gamma)u, u)) = (u,u) + \gamma[(\tilde{\Gamma}_{B_+}(\lambda)u_+, u_+) - (\tilde{\Gamma}_{B_-}(\lambda)u_-, u_-)], \tag{5.7}$$

for all $u \in \mathfrak{H}(B)$, where

$$u_+ = P_{\mathfrak{H}^+(B)}u, \quad u_- = P_{\mathfrak{H}^-(B)}u. \tag{5.8}$$

Let $\{\nu_k(\lambda)\}_{k=1}^N$ be the characteristic branches of the operator A on the gap (a, b) with respect to the operator B_+. Then by Theorem 3.1 the number of the main characteristic branches near the endpoint a coincides with $j(a)$ (see (5.4)). In other words:

$$\lim_{\lambda\to a+0} \nu_k(\lambda) \begin{cases} = -\infty & \text{for } \ k = 1, 2, \ldots, j(a), \\ > -\infty & \text{for } \ k = j(a)+1, j(a)+2, \ldots, N. \end{cases} \tag{5.9}$$

For any fixed $\lambda \in (a, b)$ consider the eigenvectors $\{e_k(\lambda)\}_{k=j(a)+1}^N$ corresponding to the eigenvalues $\{\nu_k(\lambda)\}_{k=j(a)+1}^N$ and the subspace

$$\mathfrak{F}(\lambda) = \mathrm{span}\{e_k(\lambda)\}_{k=j(a)+1}^N. \tag{5.10}$$

Then

$$\forall u_+ \in \mathfrak{F}(\lambda): \quad (\tilde{\Gamma}_{B_+}(\lambda)u_+, u_+) \ \ge \inf_{j(a)+1\le k\le N} \nu_k(\lambda)\|u_+\|^2.$$

Using the last estimate and (5.9), we obtain:

$$\inf_{\lambda\in[a,\lambda_0)} \inf_{u_+\in\mathfrak{F}(\lambda)} \frac{(\tilde\Gamma_{B_+}(\lambda)u_+,u_+)}{\|u_+\|^2} > -\infty. \tag{5.11}$$

By Lemma 3.1, the operator function $\tilde\Gamma_{B_-}(\lambda)$ is increasing on (a,b), therefore one has:

$$\sup_{\lambda\in[a,\lambda_0)} \left(\sup_{u_-\in\mathfrak{H}^-(B)} \frac{(\tilde\Gamma_{B_-}(\lambda)u_-,u_-)}{\|u_-\|^2} \right) <$$

$$\sup_{u_-\in\mathfrak{H}^-(B)} \left(\frac{(\tilde\Gamma_{B_-}(\lambda_0)u_-,u_-)}{\|u_-\|^2} \right) < +\infty. \tag{5.12}$$

We put:

$$\tilde{\mathfrak{F}}(\lambda) = \mathfrak{F}(\lambda) \bigoplus \mathfrak{H}^-(B). \tag{5.13}$$

Then we conclude from (5.7), (5.8), (5.11) and (5.12): there exists $\gamma(\lambda_0) > 0$ such that for every $\gamma \in (0,\gamma(\lambda_0))$:

$$\inf_{\lambda\in[a,\lambda_0)} \left(\inf_{u\in\tilde{\mathfrak{F}}(\lambda)} \frac{\mathrm{Re}((\Gamma(\lambda,\gamma)u,u))}{\|u\|^2} \right) \geq \frac{1}{2}. \tag{5.14}$$

Taking into account the inequality

$$\mathrm{Re}((\Gamma(\lambda,\gamma)u,u)) \leq \|\Gamma(\lambda,\gamma)u\|\|u\|,$$

we obtain from (5.14):

$$\inf_{\lambda\in[a,\lambda_0)} \left(\inf_{u\in\tilde{\mathfrak{F}}(\lambda)} \frac{\|\Gamma(\lambda,\gamma)u\|}{\|u\|} \right) \geq \frac{1}{2} \tag{5.15}$$

for any $\gamma \in (0,\gamma(\lambda_0))$.

Let us proceed to the estimation of $\dim(\hat{\mathfrak{G}}(\lambda,\gamma))$. We shall prove the inequality (5.1) with $\gamma(\lambda_0)$ chosen in (5.14). Assume, on the contrary, that

$$\exists\gamma_1 \in (0,\gamma(\lambda_0)),\ \ \exists\lambda_1 \in (a,\lambda_0],\quad \dim(\hat{\mathfrak{G}}(\lambda_1,\gamma_1)) > j(a). \tag{5.16}$$

On the other hand, the definitions (5.10) and (5.13) of the subspaces $\mathfrak{F}(\lambda)$, $\tilde{\mathfrak{F}}(\lambda)$ imply that

$$\dim(\mathfrak{H}(B)/\tilde{\mathfrak{F}}(\lambda)) = j(a).$$

In view of the assumption (5.16) we obtain

$$\hat{\mathfrak{G}}(\lambda_1,\gamma_1) \cap \tilde{\mathfrak{F}}(\lambda_1) \neq \{0\}.$$

The last relation and definition (5.3) of the subspaces $\hat{\mathfrak{G}}(\lambda,\gamma)$ contradict the relation (5.15). So the inequality (5.1) is valid.

The inequality (5.2) can be proved analogously. ■

Corollary 5.1. *Assume that at least one of the following conditions is fulfilled:*

$$M_{\mathfrak{H}^+(B)}(a, A) = 0, \tag{5.17}$$

or

$$M_{\mathfrak{H}^-(B)}(b, A) = 0. \tag{5.18}$$

Then in the case (5.17), all the eigenvalues of the operator $A(\gamma)$ lying in the gap (a, b) (if they exist) can tend only to the right endpoint b when $\gamma \to 0+$, and in the case (5.18) they can tend only to the left endpoint a. If both (5.17) and (5.18) hold simultaneously, then for a γ small enough there are no eigenvalues of $A(\gamma)$ in the gap (a, b).

The last corollary, Proposition 3.1 and Remark 3.2 imply the following statement:

Proposition 5.1. *Let (a, b) be a gap of the spectrum $\sigma(A)$ and let*

$$\{g_i\}_{i=1}^{m_+}, \quad \{g_i\}_{i=m_++1}^{N}$$

be orthonormal bases in the spaces $\mathfrak{H}^+(B)$, $\mathfrak{H}^-(B)$ respectively. Assume that at least one of the following conditions is fulfilled:

(i) either $a = -\infty$, or

$$\lim_{\lambda \to a+0} (R_\lambda(A)g_i, g_i) > -\infty \ \text{ for } i = 1, 2, \ldots, m_+; \tag{5.19}$$

(ii) either $b = +\infty$, or

$$\lim_{\lambda \to b-0} (R_\lambda(A)g_i, g_i) < +\infty \ \text{ for } i = m_+ + 1, m_+ + 2, \ldots, N. \tag{5.20}$$

Then all the conclusions of Corollary 5.1 are valid with references to (i), (ii) instead of (5.17), (5.18).

5.2° We turn now to a statement on the localization of eigenvalues. For this purpose consider the following family of subsets of a gap (a, b) of $\sigma(A)$:

$$\Pi(\gamma) = \{\lambda \in (a, b) : \nu(\lambda) \geq \frac{1}{C_B \gamma}\}, \tag{5.21}$$

where

$$C_B = \max(\|B_+\|, \|B_-\|), \tag{5.22}$$

$$\nu(\lambda) = \sum_{j=1}^{m_+} |\nu_j^+(\lambda)| + \sum_{j=1}^{m_-} |\nu_j^-(\lambda)| \tag{5.23}$$

and

$$\{\nu_j^+(\lambda)\}_{j=1}^{m_+}, \quad \{\nu_j^-(\lambda)\}_{j=1}^{m_-}$$

are the characteristic branches of the operator A in the gap (a, b) with respect to the subspaces $\mathfrak{H}^+(B)$, respectively $\mathfrak{H}^-(B)$, and $m_+ = \dim(\mathfrak{H}^+(B))$, $m_- = \dim(\mathfrak{H}^-(B))$.

Theorem 5.2. *Let (a, b) be a gap of the spectrum $\sigma(A)$ and let $\sigma(\gamma) = \sigma(A(\gamma)) \cap (a, b)$. Then*

$$\sigma(\gamma) \subseteq \Pi(\gamma). \tag{5.24}$$

Proof. By Lemmas 2.1 and 2.2, the set $\sigma(\gamma)$ is the union of the solutions of the following equations on the interval (a, b):

$$\tau_j^B(\lambda) = -\frac{1}{\gamma} \tag{5.25j}$$

$(j = 1, 2, \ldots, N)$, where

$$\{\tau_1^B(\lambda), \tau_2^B(\lambda), \ldots, \tau_N^B(\lambda)\} = \sigma(\hat{\Gamma}_B(\lambda))$$

(see (2.18)).

Assume that $\lambda(\gamma) \in \sigma(\gamma)$. This means that $\lambda(\gamma) \in (a, b)$ and there exists a k such that $\lambda(\gamma)$ satisfies the equation (5.25k). By a well known property of singular numbers of operators (see [11]), we have:

$$\sum_{j=1}^{N} |\operatorname{Re}(\tau_j^B(\lambda))| \leq \sum_{j=1}^{N} s_j(\operatorname{Re}(\hat{\Gamma}_B(\lambda))).$$

In particular, this inequality implies:

$$|\operatorname{Re}(\tau_k^B(\lambda(\gamma)))| = \frac{1}{\gamma} \leq \sum_{j=1}^{N} s_j(\operatorname{Re}(\hat{\Gamma}_B(\lambda(\gamma)))). \tag{5.26}$$

Using equality (5.5), we obtain from (5.26):

$$\frac{1}{\gamma} \leq \sum_{j=1}^{m_+} |\nu_j^{B_+}(\lambda(\gamma))| + \sum_{j=1}^{m_-} |\nu_j^{B_-}(\lambda(\gamma))|, \tag{5.27}$$

where

$$\{\nu_j^{B_+}(\lambda)\}_{j=1}^{m_+} \quad \text{and } \{\nu_j^{B_-}(\lambda)\}_{j=1}^{m_-}$$

are the characteristic branches of the operator A in the gap (a, b) with respect to the operators B_+ and B_- respectively. Also, by a well known property of eigenvalues of operators (see [11]),

$$|\nu_i^{B_+}(\lambda)| \leq \|B_+\| |\nu_i^+(\lambda)|, \quad |\nu_j^{B_-}(\lambda)| \leq \|B_-\| |\nu_j^-(\lambda)| \tag{5.28}$$

$(i = 1, 2, \ldots, m_+, \quad j = 1, 2, \ldots, m_-)$. From the inequalities (5.27) and (5.28) we obtain that $\lambda(\gamma)$ satisfies the inequality

$$\nu(\lambda(\gamma)) \geq \frac{1}{C_B \gamma},$$

in which the constant C_B and the function $\nu(\lambda)$ are defined by (5.22), (5.23) respectively. This means that $\lambda(\gamma) \in \Pi(\gamma)$ (see (5.21). So the inclusion (5.24) is valid. ■

6. The case of simple spectrum

6.1° Let us recall that $\sigma(A)$ is called *simple*, if the linear subspace

$$\operatorname{span}(\{E_A((a,b))e\}_{a,b\in\mathbf{R},\ a<b})$$

is dense in the space $\mathfrak{H}$ for some vector $e \in \mathfrak{H}$. Each such vector is called a *cyclic vector* for the operator A. It is known that in this case the operator A has the canonical form of a multiplication operator. More precisely, any self-adjoint operator A with simple spectrum is unitarily equivalent to the following operator:

$$(\tilde{A}y)(t) = ty(t), \tag{6.1}$$

which acts in the space $L_2(\mathbf{R}, \rho)$, where the measure ρ is constructed from the spectral measure E_A of A and a cyclic vector e via $\rho(\Delta) = (E_A(\Delta)e, e)$.

We turn now to the example mentioned in the introduction, which shows that the asymptotic multiplicity of an endpoint of a gap of $\sigma(A)$ with respect to a subspace can be bigger than the multiplicity of $\sigma(A)$.

Example 6.1. Define a monotone function ρ on $\mathbf{R}$ in the following manner:

$$\rho(t) = \begin{cases} 0 \ \text{ for } \ t < 0, \\ t \ \text{ for } \ t \in [0,1), \\ 1 \ \text{ for } \ t \geq 1. \end{cases}$$

For simplicity we shall denote by ρ also the Lebesgue-Stieltjes measure generated by the function ρ. Let $\tilde{A}$ be the operator defined by (6.1) on $\mathfrak{H} = L_2(\mathbf{R}, \rho)$. Consider the sequence $\{a_k\}_{k=1}^{\infty}$ defined via $a_{2k+1} = \frac{1}{2^k}$ and $a_{2k} = \frac{3}{2^{k+1}}$. Thus,

$$a_1 = 1, \qquad \text{and } \ a_{2k-1} - a_{2k} = a_{2k} - a_{2k+1} = \Delta_k, \tag{6.2}$$

where

$$\Delta_k = \frac{1}{2^{k+1}}. \tag{6.3}$$

Let $g_1, g_2 \in \mathfrak{H}$ be defined the

$$g_1(t) \equiv 1$$

and

$$g_2(t) = \begin{cases} 1 \ \text{ for } \ a_{2k} < t \leq a_{2k-1}, \\ -1 \ \text{ for } \ a_{2k+1} < t \leq a_{2k}, \\ 0 \ \text{ for } \ t \in \mathbf{R} \setminus (0,1]. \end{cases} \tag{6.4}$$

Then g_1, g_2 are normalized and orthogonal in $\mathfrak{H}$. Let

$$\mathfrak{G} = \text{span}\{g_1, g_2\}.$$

We claim that

$$M_{\mathfrak{G}}(0, \tilde{A}) = 2. \tag{6.5}$$

Using Remark 4.1 we see that two eigenvalues of the operator $\tilde{A} - \gamma P_{\mathfrak{G}}$ are created in the gap $(-\infty, 0)$ from the endpoint 0 as γ varies from 0 to a small value $\gamma_0 > 0$.

In order to prove (6.5), we shall estimate the elements of the matrix $\tilde{\Gamma}(\lambda)$ ((3.14), (3.15)). Assume that $\lambda < 0$. Using (6.2) and (6.4) one has:

$$\tilde{\gamma}_{1,1}(\lambda) = \tilde{\gamma}_{2,2}(\lambda) = \int_0^1 \frac{dt}{t-\lambda} = \ln(\frac{1}{|\lambda|}) + \ln(1-\lambda), \tag{6.6}$$

and

$$\tilde{\gamma}_{1,2}(\lambda) = \tilde{\gamma}_{2,1}(\lambda) = \int_{\mathbf{R}} \frac{g_2(t)dt}{t-\lambda}$$

$$= \sum_{k=1}^{\infty}(-\int_{a_{2k}}^{a_{2k}+\Delta_k} \frac{dt}{t-\lambda} + \int_{a_{2k}-\Delta_k}^{a_{2k}} \frac{dt}{t-\lambda})$$

$$= \sum_{k=1}^{\infty} \ln(1 - \frac{\Delta_k^2}{(a_{2k}-\lambda)^2}). \tag{6.7}$$

Using the fact that $a_{2k} = \frac{3}{2^{k+1}}$, the estimate

$$\frac{\Delta_k^2}{(a_{2k}-\lambda)^2} \leq \frac{1}{9} \ \ (\lambda < 0, \ \ k = 1, 2, \ldots),$$

and the inequality:

$$|\ln(1-x)| \leq \frac{9}{8}x \quad \forall x \in [0, \frac{1}{9}],$$

we can estimate the series (6.7) in the following manner:

$$|\tilde{\gamma}_{1,2}(\lambda)| \ \leq \ \frac{9}{8} \sum_{k=1}^{\infty} \frac{1}{(3+2|\lambda|2^k)^2} \quad (\lambda < 0).$$

Estimating the last series by means of the integral, we obtain for $\lambda < 0$

$$|\gamma_{1,2}(\lambda) \ \leq \ \frac{9}{8} \int_0^{\infty} \frac{dt}{(3+2|\lambda|2^t)^2}.$$

After the change of the variable $\tau = 2^t|\lambda|$ in the last integral, one has for $-1 < \lambda < 0$

$$|\tilde{\gamma}_{1,2}(\lambda)| \le \frac{9}{8\ln 2}\int_{|\lambda|}^{\infty}\frac{d\tau}{(3+2\tau)^2\tau}$$

$$\le \frac{9}{8\ln 2}\left(\frac{1}{9}\int_{|\lambda|}^{1}\frac{d\tau}{\tau} + \frac{1}{4}\int_{1}^{\infty}\frac{d\tau}{\tau^3}\right) = \frac{1}{8\ln 2}(\ln(\frac{1}{|\lambda|}) + \frac{9}{8}\Big). \tag{6.8}$$

It is easy to find the eigenvalues of the matrix $\tilde{\Gamma}(\lambda)$:

$$\mu_1(\lambda) = \tilde{\gamma}_{1,1}(\lambda) + \tilde{\gamma}_{1,2}(\lambda), \quad \mu_2(\lambda) = \tilde{\gamma}_{1,1}(\lambda) - \tilde{\gamma}_{1,2}(\lambda).$$

Then from the equality (6.6) and from the estimate (6.8) we obtain:

$$\lim_{\lambda\to -0}\mu_k(\lambda) = +\infty \quad \text{for} \quad k = 1, 2,$$

i.e. the desired equality (6.5) is valid.

6.2° In the above example the basic function g_2 oscillates sharply near the endpoint 0 of the gap $(-\infty, 0)$ of $\sigma(\tilde{A})$. In what follows we shall show that if the self-adjoint operator A with simple spectrum is represented in the canonical form of a multiplication operator $\tilde{A}$ (6.1) in the space $L_2(\mathbf{R}, \rho)$, and if all the functions from a finite-dimensional subspace $\mathfrak{G}$ of $L_2(\mathbf{R}, \rho)$ satisfy some Besov type condition near the endpoint b (or a) of $J_\sigma(A) = (a, b)$, then $M_{\mathfrak{G}}(b, A) \le 1$ (respectively, $M_{\mathfrak{G}}(a, A) \le 1$).

Definition 6.1. Let ρ be a non-decreasing function on $\mathbf{R}$ with $-\infty < \rho(-\infty) < \rho(\infty) < \infty$, let $f \in L_2(\mathbf{R}, \rho)$, and let $c \in \mathbf{R}$. We say that f satisfies *condition* $B(c)$ (*local Besov condition at the point* c) if the following condition holds: Either $\rho(c_+) > \rho(c_-)$ and

$$\frac{f(t) - f(c)}{t - c} \in \mathfrak{H}. \tag{6.9}$$

Or, $\rho(c_+) = \rho(c_-)$ and there is a choice of the value $f(c)$ so that (6.9) holds.

Theorem 6.1. *Let $\tilde{A}$ be the operator (6.1) in the space $\mathfrak{H} = L_2(\mathbf{R}, \rho)$, and let (a, b) be a gap in $\sigma(\tilde{A})$. Assume that $b < +\infty$, and let $\mathfrak{G} \subseteq \mathfrak{H}$ be an N-dimensional subspace so that every function $g \in \mathfrak{G}$ satisfies condition $B(b)$. Then*

$$M_{\mathfrak{G}}(b, \tilde{A}) \le 1. \tag{6.10}$$

Similarly, if $a > -\infty$ and every function $g \in \mathfrak{G}$ satisfies condition $B(a)$, then $M_{\mathfrak{G}}(a, \tilde{A}) \le 1$.

Proof. We shall restrict ourselves to the case of the right endpoint b, because the proof for the left endpoint a is analogous. Let $\tilde{\Gamma}(\lambda)$ ((3.14), (3.15)) be the matrix representation of the operator $\tilde{\Gamma}_P(\lambda)$ (3.7) with respect to an orthonormal basis $\{g_1(t), g_2(t), \dots, g_N(t)\}$ of the subspace $\mathfrak{G}$. Represent this matrix in the form:

$$\tilde{\Gamma}(\lambda) = \tilde{\Gamma}_1(\lambda) + \tilde{\Gamma}_2(\lambda), \tag{6.11}$$

where

$$\tilde{\Gamma}_1(\lambda) = \{\tilde{\gamma}_{i,j}^{(1)}(\lambda)\}_{i,j=1}^N, \tag{6.12}$$

$$\tilde{\gamma}_{i,j}^{(1)}(\lambda) = s(\lambda) g_i(b) \bar{g}_j(b), \tag{6.13}$$

$$s(\lambda) = \int_{\mathbf{R}} \frac{d\rho(t)}{t-\lambda}, \tag{6.14}$$

and

$$\tilde{\Gamma}_2(\lambda) = \{\tilde{\gamma}_{i,j}^{(2)}(\lambda)\}_{i,j=1}^N, \tag{6.15}$$

$$\tilde{\gamma}_{i,j}^{(2)}(\lambda) = \int_{\mathbf{R}} \frac{\big(g_i(t)\bar{g}_j(t) - g_i(b)\bar{g}_j(b)\big)}{t-\lambda}\, d\rho(t)$$

$$= \int_{\mathbf{R}} \Big(g_i(t) \frac{\bar{g}_j(t) - \bar{g}_j(b)}{t-\lambda} + \frac{g_i(t) - g_i(b)}{t-\lambda} \bar{g}_j(b) \Big)\, d\rho(t). \tag{6.16}$$

It is clear from (6.12), (6.13), that the matrix $\tilde{\Gamma}_1(\lambda)$ has rank 1, and that the only non-zero eigenvalue is

$$\tilde{\mu}_N(\lambda) = \sum_{i=1}^N |g_i(b)|^2 s(\lambda)$$

(see (6.14)). Hence the resolvent of the matrix $\tilde{\Gamma}_1(\lambda)$ has the form:

$$R_\mu(\tilde{\Gamma}_1(\lambda)) f = \frac{(\tilde{e}_N(\lambda), f)\tilde{e}_N(\lambda)}{\tilde{\mu}_N(\lambda) - \mu} - \frac{1}{\mu} \sum_{k=1}^{N-1} (\tilde{e}_k(\lambda), f)\tilde{e}_k(\lambda), \quad f \in \mathfrak{H}, \tag{6.17}$$

were $\{\tilde{e}_k(\lambda)\}_{k=1}^N$ are the normalized eigenvectors of the matrix $\tilde{\Gamma}_1(\lambda)$. Let us fix $\lambda_0 \in (a,b)$. We obtain from (6.15), (6.16), and condition $B(b)$ for the functions $\{g_j\}_{j=1}^N$:

$$\tilde{G}(\lambda_0) = \sup_{\lambda \in [\lambda_0, b)} \|\tilde{\Gamma}_2(\lambda)\| < \infty. \tag{6.18}$$

Consider the cases:

$$(i) \quad \lim_{\lambda \to b-0} \tilde{\mu}_N(\lambda) < +\infty,$$

$$(ii) \quad \lim_{\lambda \to b-0} \tilde{\mu}_N(\lambda) = +\infty.$$

We see from (6.17) that in case (i) there exists $R > 0$ such that for every $\lambda \in [\lambda_0, b)$:

$$\sup_{\mu \in C_R} \|R_\mu(\tilde{\Gamma}_1(\lambda))\| < (\tilde{G}(\lambda_0))^{-1}, \tag{6.19}$$

where

$$C_R = \{\mu \in \mathbf{C} : |\mu| = R\}. \tag{6.20}$$

In case (ii) there exists $R > 0$ and $\lambda_1 \in [\lambda_0, b)$ such that (6.19) is valid for any $\lambda \in [\lambda_1, b)$. Taking into account (6.19), one has in both cases (i) and (ii) that there exist $R > 0$ and $\lambda_1 \in (a, b)$ such that for every $\lambda \in [\lambda_1, b)$:

$$\sup_{\mu \in C_R} \|R_\mu(\tilde{\Gamma}_1(\lambda))\tilde{\Gamma}_2(\lambda)\| < 1. \tag{6.21}$$

Let us join the matrices $\tilde{\Gamma}(\lambda)$ (6.11) and $\tilde{\Gamma}_1(\lambda)$ by the homotopy:

$$\tilde{\Gamma}(\lambda, s) = \tilde{\Gamma}_1(\lambda) + s\tilde{\Gamma}_2(\lambda) \quad (0 \le s \le 1). \tag{6.22}$$

The inequality (6.21) and the representation

$$\tilde{\Gamma}(\lambda, s) - \mu I = (\tilde{\Gamma}_1(\lambda) - \mu I)(I + sR_\mu(\tilde{\Gamma}_1(\lambda))\tilde{\Gamma}_2(\lambda))$$

imply that for any $\lambda \in [\lambda_1, b)$ and for any $s \in [0, 1]$ the circle C_R (6.20) belongs to the resolvent set of the operator $\tilde{\Gamma}(\lambda, s)$, and the following representation is valid:

$$R_\mu(\tilde{\Gamma}(\lambda, s)) = (I + sR_\mu(\tilde{\Gamma}_1(\lambda))\tilde{\Gamma}_2(\lambda))^{-1} R_\mu(\tilde{\Gamma}_1(\lambda)).$$

We see from this representation that the operator $R_\mu(\tilde{\Gamma}(\lambda, s))$ depends continuously on s in the uniform operator topology, and the family of functions $\{R_\mu(\tilde{\Gamma}(\lambda, \cdot))\}_{\mu \in C_R}$ is equicontinuous. Therefore, the Riesz projector

$$P_s = -\frac{1}{2\pi i} \oint_{C_R} R_\mu(\tilde{\Gamma}(\lambda, s)) d\mu$$

depends continuously on s in the uniform operator topology. Thus

$$\dim(\operatorname{Im}(P_1)) = \dim(\operatorname{Im}(P_0)).$$

But the circle C_R contains in its interior the $(N-1)$-multiple eigenvalue 0 of the matrix $\tilde{\Gamma}_1(\lambda)$. Hence $\dim(\operatorname{Im}(P_0)) \ge N - 1$. So $\dim(\operatorname{Im}(P_1)) \ge N - 1$, i.e for any $\lambda \in [\lambda_1, b)$ the circle C_R contains in its interior at least $N - 1$ eigenvalues $\mu_1(\lambda), \mu_2(\lambda), \ldots, \mu_{N-1}(\lambda)$ of the matrix $\tilde{\Gamma}(\lambda)$ (taking into account their multiplicities). This means that

$$\lim_{\lambda \to b-0} \mu_k(\lambda) < +\infty \quad (k = 1, 2, 3, \ldots, N - 1),$$

i.e. (6.10) is valid. ■

The following result is a corollary of Theorems 6.1 and 4.1.

Proposition 6.1. *Let (a,b) be a gap of $\sigma(\tilde{A})$ and let $B \geq 0$ be an operator of rank $N < \infty$. Assume that if $a > -\infty$ then every function in the subspace $\mathfrak{H}(B)$ satisfies condition $B(a)$. Furthermore, assume that the least characteristic branch $\mu_1(\lambda)$ of the operator $\tilde{A}$ in the gap (a,b) with respect to the subspace $\mathfrak{H}(B)$ satisfies the condition:*

$$\lim_{\lambda \to a+0} \mu_1(\lambda) = -\infty.$$

Then for a small enough $\gamma > 0$ the gap (a,b) contains a unique eigenvalue $\lambda(\gamma)$ of the operator $\tilde{A}(\gamma) = \tilde{A} + \gamma B$, which converges to a monotonically as $\gamma \downarrow 0$. Moreover, the following estimate of the rate of this convergence is valid:

$$\exists \gamma_0 > 0 \;\; \exists K_1, K_2 > 0 \;\; \forall \gamma \in (0, \gamma_0)$$

$$\mu_1^{-1}(-\frac{1}{K_1 \gamma}) \leq \lambda(\gamma) \leq \mu_1^{-1}(-\frac{1}{K_2 \gamma}),$$

where μ_1^{-1} is the inverse of the function $\mu_1(\lambda)$. The left infinite gap $(-\infty, m)$ of $\sigma(\tilde{A})$ (where $m := min\{\lambda; \lambda \in \sigma(A)\}$) contains no eigenvalues of $A(\gamma)$ for a small enough $\gamma > 0$.

7 The case of the Cantor spectral measure

7.1° In this section we denote by $\mathfrak{C}$ the classical (2-3) Cantor set on the segment $[0,1]$. Let $\tilde{A}$ be the multiplication operator (6.1) which acts in the Hilbert space

$$\mathfrak{H} = L_2(\mathbf{R}, \rho_c) \tag{7.1}$$

with the *Cantor measure* ρ_c associated with the extension of the Cantor function

$$\rho_c(t) = \begin{cases} 0 \;\text{ for }\; t < 0, \\ C(t) \;\text{ for }\; 0 \leq t \leq 1, \\ 1 \;\text{ for }\; t > 1, \end{cases} \tag{7.2}$$

where $C(t)$ is the classical (2-3) Cantor function. Then $\mathfrak{C} = \operatorname{supp}(\rho_c)$. We shall study the finite-dimensional perturbations of the operator $\tilde{A}$ on the basis of the results in the previous sections.

We need in the sequel some arithmetical notions, connected with the Cantor set.

Let $g \geq 2$ be an integer. Recall that the *g-representation* of a number $x \in [0,1)$ is the expansion of x on the base g: $x = \sum_{j \in \mathbf{N}} x_j g^{-j}$, where $x_j \in \{0, 1, \ldots g-1\}$. The x_j's are called *the coefficients of x in the g-representation.* In case x admits both finite and infinite representations we choose the finite one. In this way every $x \in [0,1)$ admits a *unique* g-representation.

Definition 7.1. We call the mapping that associates to each real number $p \in [0,1)$ in the 2-representation the real number $q \in [0,1]$ in the 3-representation with the same coefficients *the (2-3)-mapping*. We denote it by $q = P_{2,3}(p)$. Thus

$$P_{2,3}(\sum_{j=1}^{\infty} p_j 2^{-j}) = \sum_{j=1}^{\infty} p_j 3^{-j}.$$

Remark 7.1. According to the definition of the Cantor set $\mathfrak{C}$:

$$\mathfrak{C} = \mathrm{cl}(2P_{2,3}([0,1))).$$

Consider the set $\mathfrak{B}$ consisting of the point 0 and the right endpoints of connected components of $[0,1] \setminus \mathfrak{C}$. They are the numbers, whose 3-representations are finite and contain only the coefficients 0 and 2. One has

$$\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n, \tag{7.3}$$

where

$$\mathfrak{B}_n = \{c \in \mathfrak{B} : \mathrm{len}(c) \leq n\}, \tag{7.4}$$

and $\mathrm{len}(c)$ is the length of the 3-representation of the number c. Each set of $\mathfrak{B}_n$ contains exactly 2^n numbers:

$$\mathfrak{B}_n = \{b_k^{(n)}\}_{k=0}^{2^n-1}, \tag{7.5}$$

where $b_0^{(n)} = 0$ and $b_{k_1}^{(n)} < b_{k_2}^{(n)}$, if $k_1 < k_2$. Similarly let $\mathfrak{A}$ be the set consisting of the point 1 and the left endpoints of connected components of $[0,1] \setminus \mathfrak{C}$. In view of the symmetry of the Cantor set $\mathfrak{C}$ with respect to the point $t = \frac{1}{2}$ one has:

$$\mathfrak{A} = \{c : \quad c = 1 - b, \quad b \in \mathfrak{B}\}. \tag{7.6}$$

Denote also

$$\mathfrak{A}_n = \{c \in \mathfrak{A} : \quad c = 1 - b, \quad b \in \mathfrak{B}_n\}.$$

Thus

$$\mathfrak{A}_n = \{a_k^{(n)}\}_{k=0}^{2^n-1}, \tag{7.7}$$

where

$$a_k^{(n)} = 1 - b_{2^n-k}^{(n)} \tag{7.8}$$

(see 7.5). The intervals

$$(a_k^{(n)}, b_k^{(n)}) \quad (n \in \mathbf{N}, \quad k = 1, 2, \ldots, 2^n - 1)$$

are removed from $[0, 1]$ after n steps of the construction of the Cantor set. Thus one has:

$$\mathbf{R} \setminus \mathfrak{C} = (-\infty, 0) \cup (\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^n-1} (a_k^{(n)}, b_k^{(n)})) \cup (1, +\infty). \tag{7.9}$$

7.2° In what follows we shall need some estimates on the Cauchy transform of the measure ρ_c:

$$\Phi(\lambda) = \int_{\mathbf{R}} \frac{d\rho_c(t)}{t - \lambda}. \tag{7.10}$$

Denote by $\mathfrak{R}$ the set of numbers from [0,1), whose 2-representation is finite. The following essentially known result will play an important role in the sequel. Since we did not find a proof in the literature, we enclose a proof in the Appendix (Theorems A.1, A.2):

Proposition 7.1. *Let*

$$\alpha = \log_3 2. \tag{7.11}$$

Then for every $r_1, r_2 \in \mathfrak{R}$, with $r_1 > r_2$

$$(r_1 - r_2)^{\frac{1}{\alpha}} \leq 2^{\frac{1}{\alpha}} \, (2P_{2,3}(r_1) - 2P_{2,3}(r_2)). \tag{7.12}$$

An inverse estimate to (7.12) is valid in a right (or, left) semi-neighborhoods of the right (respectively, left) endpoints of connected components of the set $\mathbf{R} \setminus \mathfrak{C}$. More explicitly, if (a, b) is a bounded connected component of $\mathbf{R} \setminus \mathfrak{C}$, then the following relations hold:

$$\{b\} + 2P_{2,3}([0, (b-a)^\alpha) \cap \mathfrak{R}) = \mathfrak{B} \cap [b, b + (b-a)),$$
$$\{a\} - 2P_{2,3}([0, (b-a)^\alpha) \cap \mathfrak{R}) = \mathfrak{A} \cap (a - (b-a)), a],$$

and the following inequalities are valid:

$$\forall c \in \mathfrak{B} \cap [b, b + (b-a)) : \quad (c - b)^\alpha \ \leq \ 2^\alpha (r - r_b), \tag{7.13}$$

$$\forall l \in \mathfrak{A} \cap (a - (b-a), a] : \quad (a - l)^\alpha \leq 2^\alpha (q - q_a), \tag{7.14}$$

where

$$r = P_{2,3}^{-1}(\frac{c}{2}), \quad r_b = P_{2,3}^{-1}(\frac{b}{2}),$$
$$q = P_{2,3}^{-1}(\frac{1-l}{2}), \quad q_a = P_{2,3}^{-1}(\frac{1-a}{2}).$$

If $(a, b) = (-\infty, 0)$ then

$$\forall c \in \mathfrak{B} : \quad c^\alpha \ \leq \ 2^\alpha r, \tag{7.15}$$

and if $(a, b) = (1, +\infty)$ then

$$\forall l \in \mathfrak{A} : \quad (1 - l)^\alpha \ \leq \ 2^\alpha q. \tag{7.16}$$

We shall need the following lemma.

Lemma 7.1. *Let $f \in C[0,1]$. Then for any interval $[c_1, c_2] \subseteq [0,1]$ the following relation is valid:*

$$\int_{c_1}^{c_2} f(t) d\rho_c(t) = \lim_{n\to\infty} \frac{1}{2^n} \sum_{c \in \mathfrak{B}_n(c_1,c_2)} f(c) = \lim_{n\to\infty} \frac{1}{2^n} \sum_{c \in \mathfrak{A}_n(c_1,c_2)} f(c), \tag{7.17}$$

where

$$\mathfrak{A}_n(c_1,c_2) = \mathfrak{A}_n \cap [c_1,c_2], \quad \mathfrak{B}_n(c_1,c_2) = \mathfrak{B}_n \cap [c_1,c_2] \tag{7.18}$$

(see (7.5), (7.7), (7.8))).

Proof. Consider the following sequence of monotone non-decreasing functions:

$$\sigma_n(t) = \begin{cases} 0 & \text{for } t \le 0, \\ \frac{k}{2^n} & \text{for } t \in (b_{k-1}^{(n)}, b_k^{(n)}], \quad k = 1, 2, \ldots, 2^n - 1, \\ 1 & \text{for } t > b_{2^n-1}^n. \end{cases} \tag{7.19}$$

It is clear that

$$\forall t \in (0,1]: \quad \lim_{n\to\infty} \sigma_n(t) = \rho_c(t).$$

Therefore by Helly's theorem:

$$\int_{c_1}^{c_2} f(t) d\rho_c(t) = \lim_{n\to\infty} \int_{\mathbf{R}} f(t) d\sigma_n(t).$$

Using (7.19), we see that the first equality of (7.17) is valid. The second equality of (7.17) is proved analogously. ■

The following proposition establishes some inequalities between the integrals with respect to the Cantor and the Lebesgue measures. As before, $\alpha = \log_3 2$

Proposition 7.2. *Let (a,b) be a connected component of $\mathbf{R} \setminus \mathfrak{C}$. Assume that f is a non-negative monotone continuous function, so that either*

(1) $b < +\infty$, and f is defined on $[b, \infty)$,

or

(2) $a > -\infty$, and f is defined on $(-\infty, a]$.

Then the following estimates are valid:

(i) In case (1), if f is non-increasing then

$$\forall d \in [b, 1): \int_b^d f(t)\, d\rho_c(t) \le \int_0^{2(d-b)^\alpha} f((\frac{u}{2})^{\frac{1}{\alpha}} + b)\, du. \tag{7.20}$$

Moreover, if $a > -\infty$ then

$$\forall d \in [b, b + (b-a)): \int_0^{(\frac{d-b}{2})^\alpha} f(2u^{\frac{1}{\alpha}} + b)\, du \le \int_b^d f(t)\, d\rho_c(t). \tag{7.21}$$

(ii) In the case (1), if f is non-decreasing and $a > -\infty$, then for any $d \in [b, b+(b-a))$

$$(7.22) \qquad \int_0^{(\frac{d-b}{2})^\alpha} f((\frac{u}{2})^{\frac{1}{\alpha}} + b)\, du \;\le\; \int_b^d f(t)\, d\rho_c(t) \;\le\; \int_0^{2(d-b)^\alpha} f(2u^{\frac{1}{\alpha}} + b)\, du.$$

(iii) In the case (2), if f is non-increasing and $b < +\infty$, then for any $d \in (a-(b-a), a]$

$$(7.23) \qquad \int_0^{(\frac{a-d}{2})^\alpha} f(a - (\frac{u}{2})^{\frac{1}{\alpha}})\, du \;\le\; \int_d^a f(t)\, d\rho_c(t) \;\le\; \int_0^{2(a-d)^\alpha} f(a - 2u^{\frac{1}{\alpha}})\, du.$$

(iv) In the case (2), if f is non-decreasing then

$$(7.24) \qquad \forall d \in (0, a]: \; \int_d^a f(t) d\rho_c(t) \;\le\; \int_0^{2(a-d)^\alpha} f(a - (\frac{u}{2})^{\frac{1}{\alpha}})\, du.$$

Moreover, if $b < +\infty$ then

$$(7.25) \qquad \forall d \in (a-(b-a), a]: \; \int_0^{(\frac{a-d}{2})^\alpha} f(a - 2u^{\frac{1}{\alpha}})\, du \;\le\; \int_d^a f(t)\, d\rho_c(t).$$

(v) If in the cases (i) and (ii) $a = -\infty$ and $b = 0$, then the corresponding estimates are valid for all $d \in [0,1)$. If in the cases (iii) and (iv) $a = 1$ and $b = +\infty$, then the corresponding estimates are valid for all $d \in (0,1]$.

Proof. We prove the estimates (7.20), (7.21) in the case (i). Let $\mathfrak{R}_n$ be the set of all numbers from the segment $[0,1]$ whose lengths in the 2-representation are not bigger than n. It is clear that $\mathfrak{R} = \bigcup_{n=1}^\infty \mathfrak{R}_n$. According to the definition (7.4) of the set $\mathfrak{B}_n$ we have:

$$(7.26) \qquad \mathfrak{B}_n = 2P_{2,3}(\mathfrak{R}_n)$$

(see Definition 7.1). Denote $r_b = P_{2,3}^{-1}(\frac{b}{2})$. Take $d \in [b, 1)$ and consider the sets:

$$(7.27) \qquad \mathfrak{R}_n(b,d) = \{r \in \mathfrak{R}_n : \;\; 0 \le r - r_b \le 2(d-b)^\alpha\},$$

and

$$(7.28) \qquad \hat{\mathfrak{R}}_n(b,d) = \{r \in \mathfrak{R}_n : \;\; 0 \le r - r_b \le (\frac{d-b}{2})^\alpha\}.$$

The inequalities (7.12) (7.13) imply the inclusions

$$(7.29) \qquad \mathfrak{B}_n(b,d) \subseteq 2P_{2,3}(\mathfrak{R}_n(b,d)),$$

and

$$(7.30) \qquad 2P_{2,3}(\hat{\mathfrak{R}}_n(b,d)) \subseteq \mathfrak{B}_n(b,d) \quad \forall d \in [b, b+(b-a)).$$

(see (7.18)). Using the equality (7.17), the inclusion (7.29), the inequality (7.12) and the fact that the function $f(t)$ is non-increasing and non-negative, one obtains:

$$\int_b^d f(t)\, d\rho_c(t) = \lim_{n\to\infty} \frac{1}{2^n} \sum_{c\in\mathfrak{B}_n(b,d)} f(c) \leq$$

$$\lim_{n\to\infty} \sum_{r\in\mathfrak{R}_n(b,d)} \frac{1}{2^n}\, f(2^{-\frac{1}{\alpha}}(r-r_b)^{\frac{1}{\alpha}}+b).$$

The last sum is the Riemann sum for the integral:

$$\int_0^{2(d-b)^\alpha} f(2^{-\frac{1}{\alpha}}u^{\frac{1}{\alpha}}+b)\, du,$$

i.e. we obtain the inequality (7.20). The inequality (7.21) is obtained in a similar way, using the inclusion (7.30) and the inequality (7.13). The case (ii) is proved by using the estimate (7.13) and the inclusion (7.29) for the right inequality of (7.22), and by using the estimate (7.12) and the inclusion (7.30) for the left inequality of (7.22). The cases (iii), (iv) are obtained from (i), (ii) by using the symmetry of the Cantor set with respect to the point $t=\frac{1}{2}$. In case (v) of infinite interval (a,b) the proof is analogous. ■

We turn now to the estimation of the Cauchy transform $\Phi(\lambda)$ (7.10) of the measure ρ_c.

Proposition 7.3. *Let (a,b) be a connected component of $\mathbf{R}\backslash\mathfrak{C}$. Then there exist constants $K_1,\ K_1$ so that the following estimates are valid:*

$$\forall\lambda\in(\tilde{a},\tilde{b}):\ \frac{K_1}{(dist(\lambda,\mathfrak{C}))^{1-\alpha}} \leq |\Phi(\lambda)| \leq \frac{K_2}{(dist(\lambda,\mathfrak{C}))^{1-\alpha}}, \tag{7.31}$$

where α is defined by (7.11) and

$$(\tilde{a},\tilde{b}) = \begin{cases} (a,b), & \textit{if } a>-\infty \textit{ and } b<+\infty, \\ (-1,0), & \textit{if } a=-\infty \textit{ and } b=0, \\ (1,2), & \textit{if } a=1 \textit{ and } b=+\infty. \end{cases}$$

Proof. Assume that the interval (a,b) is bounded. We prove the estimates (7.31) for $\lambda\in I_b$, where $I_b=[\frac{a+b}{2},b)$. One has:

$$\Phi(\lambda)=\Phi_1(\lambda)+\Phi_2(\lambda),$$

where

$$\Phi_1(\lambda)=\int_b^1 \frac{d\rho_c(t)}{t-\lambda},\quad \Phi_2(\lambda)=\int_0^a \frac{d\rho_c(t)}{t-\lambda}.$$

It is clear that the function $\Phi_2(\lambda)$ is bounded on the semi-interval I_b, therefore it is enough to estimate the function $\Phi_1(\lambda)$. Observe that for any fixed $\lambda \in I_b$, the integrand $\frac{1}{t-\lambda}$ is positive and decreasing on [b,1]. Using the inequality (7.20) with $d = 1$, we obtain for $\lambda \in I_b$:

$$\Phi_1(\lambda) \leq \int_0^{2(1-b)^\alpha} \frac{du}{2^{-\frac{1}{\alpha}} u^{\frac{1}{\alpha}} + b - \lambda}.$$

Using the change of variables $\tau = \frac{u}{(b-\lambda)^\alpha}$ in the last integral, we obtain for $\lambda \in I_b$:

$$\Phi_1(\lambda) \ \leq \ \frac{1}{(b-\lambda)^{1-\alpha}} \int_0^\infty \frac{d\tau}{2^{-\frac{1}{\alpha}} \tau^{\frac{1}{\alpha}} + 1},$$

i.e. the right estimate in (7.31) is valid for $\Phi_1(\lambda)$. Analogously, using the inequality (7.21), we obtain:

$$\Phi_1(\lambda) \ \geq \ \frac{1}{(b-\lambda)^{1-\alpha}} \int_0^\theta \frac{d\tau}{4\tau^{\frac{1}{\alpha}} + 1} \quad (\lambda \in I_b)$$

for some $\theta > 0$, independent of λ. This means that the left estimate in (7.31) is valid for $\Phi_1(\lambda)$. For the left semi-interval $I_a = (a, \frac{a+b}{2}]$ the proof is analogous. In this case the function $\Phi_1(\lambda)$ is bounded and we must estimate the function $|\Phi_2(\lambda)|$ near the endpoint a using the estimates (7.24), (7.25). The cases of the unbounded intervals (a, b) are treated analogously using the conclusion (v) of Proposition 7.2. ■

7.3^o We turn now to the study of finite-dimensional perturbations of the operator $\tilde{A}$ (6.1) in the space $\mathfrak{H}$ (7.1). Assume that the perturbing operator B has the form:

$$By = \sum_{j=1}^{N} b_j (y, g_j) g_j, \tag{7.32}$$

where

$$\{g_i\}_{i=1}^N \tag{7.33}$$

is an orthonormal system in the space $\mathfrak{H}$. Using the results of the Sections 4, 5 we shall study the asymptotic behavior (as as $\gamma \to 0+$) of the eigenvalues of the operator

$$\tilde{A}(\gamma) = \tilde{A} + \gamma B \tag{7.34}$$

in the gaps of $\sigma(\tilde{A})$ (i.e. in the connected components of $\mathbf{R} \setminus \mathfrak{C}$).

In what follows we shall need the following lemma.

Lemma 7.2. *Let (a, b) be a connected component of $\mathbf{R}\backslash\mathfrak{C}$. Let $h \in \mathfrak{H}$, $\nu > 0$ and $\beta > \nu - \frac{\alpha}{2}$ (where $\alpha = \log_3 2$).*

(i) If $b < +\infty$ and $|h(t)| \leq L|t-b|^{\beta}$ for some $L > 0$ and all $t \in [b, 1]$, then the function $f_b(t) = \frac{h(t)}{|t-b|^{\nu}}$ belongs to $L_2([b, 1], \rho_c)$.

(ii) If $a > -\infty$ and $|h(t)| \leq L|t-a|^{\beta}$ for some $L > 0$ and all $t \in [0, b]$, then the function $f_a(t) = \frac{h(t)}{|t-a|^{\nu}}$ belongs to $L_2([0, a], \rho_c)$.

Proof. Using the right inequality (7.22) (see Proposition 7.2) we obtain:

$$\int_b^{\theta} \frac{|h(t)|^2}{(t-b)^{2\nu}}\,dt \;\leq\; L^2 \int_b^{\theta} (t-b)^{2(\beta-\nu)}\,d\rho_c(t) \;\leq\; L^2 4^{-(\beta-\nu)/\alpha} \int_0^{2(\theta-b)^{\alpha}} u^{2(\beta-\nu)/\alpha}\,du,$$

where $\theta := b + \min\{b-a, 1\}$. The integral on the right hand side converges, because according to the condition of the lemma, $\frac{2(\beta-\nu)}{\alpha} > -1$. This establishes (i). The assertion (ii) can be proved in the same way, using the right inequality (7.23). ■

In the case of a semi-definite perturbing operator B we obtain the following result:

Theorem 7.1. *Let (a, b) be a gap of $\sigma(\tilde{A})$. Assume that the perturbing operator B (7.32) and the orthonormal system (7.33) satisfy the conditions:*

(i) $b_j > 0$ for $j = 1, 2, \ldots, N$;

(ii) there exists a number $\beta \in (1-\frac{\alpha}{2}, +\infty)$ such that $g_j(t) \in Lip_{\beta}[0, 1]$ for $j = 1, 2, \ldots, N$ (where $\alpha = \log_3 2$);

(iii) for any point t belonging to the Cantor set $\mathfrak{C}$, $g_1(t) \neq 0$.

If $a > -\infty$, then for a small enough $\gamma > 0$ the gap (a, b) contains a unique eigenvalue $\lambda(\gamma)$ of the operator $\tilde{A}(\gamma)$ (7.34), which converges monotonically to a as $\gamma \downarrow 0$. Moreover, the following estimates of the rate of this convergence are valid:

$$\exists\gamma_0 > 0,\ \ \exists L_1, L_2 > 0\ \ \forall\gamma \in (0, \gamma_0):$$

$$L_1\gamma^{\frac{1}{1-\alpha}} \;\leq\; \lambda(\gamma) - a \;\leq\; L_2\gamma^{\frac{1}{1-\alpha}}.$$

The gap $(-\infty, 0)$ contains no eigenvalues of the operator $\tilde{A}(\gamma)$ for a small enough $\gamma > 0$.

Proof. Assume that $a > -\infty$. Observe that by the condition (i) $B \geq 0$. The condition (ii) and Lemma 7.2 with $\nu = 1$ imply that any function $g \in \mathfrak{H}(B)$ is continuous on the segment $[0, 1]$ and satisfies condition $B(a)$ (see Definition 6.1). Furthermore, the functions $g_i(t)$ $(i = 1, 2, \ldots, N)$ are bounded on $[0, 1]$ and by the condition (iii), $m_g = \min_{t\in\mathfrak{C}} |g_1(t)| > 0$. It follows that for the elements $\tilde{\gamma}_{i,j}(\lambda)$ of the matrix representation $\tilde{\Gamma}(\lambda)$ (3.14) of the operator $\tilde{\Gamma}_P(\lambda)$ (see (3.14), (3.15) and (3.7)) the following estimates are valid:

$$\exists C > 0,\ \ \forall\lambda \in (a, b):\ \ |\tilde{\gamma}_{i,j}(\lambda)| \;\leq\; C\int_0^1 \frac{d\rho_c(t)}{|t-\lambda|} \quad (i, j = 1, 2. \ldots, N), \tag{7.35}$$

$$(7.36)\qquad \tilde{\gamma}_{1,1}(\lambda) = \int_0^1 |g_1(t)|^2 \frac{d\rho_c(t)}{t-\lambda} \ \le\ -m_g^2 \int_0^a \frac{d\rho_c(t)}{\lambda - t} + C \int_b^1 \frac{d\rho_c(t)}{t-\lambda}.$$

Observe that by mini-max characterization of the eigenvalues of the self-adjoint operator $\tilde{\Gamma}_P(\lambda)$, applied to the least eigenvalue $\mu_1(\lambda)$ of this operator, the following estimate is valid:

$$\mu_1(\lambda) \le \tilde{\gamma}_{1,1}(\lambda), \quad \forall \lambda \in (a,b).$$

Furthermore, it is known that

$$\mu_1(\lambda) \ge -\|\tilde{\Gamma}_P(\lambda)\|.$$

Using the last inequality, the estimates (7.35)and (7.36), and Proposition 7.3, we obtain:

$$\exists C_1, C_2 > 0, \ \ \exists \lambda_0 \in (a,b), \quad \forall \lambda \in (a, \lambda_0):$$

$$-\frac{C_1}{(\lambda - a)^{1-\alpha}} \ \le\ \mu_1(\lambda) \ \le\ -\frac{C_2}{(\lambda-a)^{1-\alpha}}.$$

The conclusions of our theorem follow from these estimates and from Proposition 6.1 ■

In the case of an indefinite perturbing operator B one has the following result:

Theorem 7.2. *Let (a,b) be a gap of $\sigma(\tilde{A})$ and let B be given by (7.32), where*

$$b_j > 0 \ \ \text{for} \ \ j = 1, 2, \ldots, m_+,$$

$$b_j < 0 \ \ \text{for} \ \ j = m_+ + 1, m_+ + 2, \ldots, N.$$

Assume that

$$(7.37)\qquad \exists \beta \in (\frac{1-\alpha}{2}, +\infty) \ \ \text{so that} \ \ g_j \in Lip_\beta[0,1] \ \ \text{for} \ \ j = 1, 2, \ldots, N$$

(where $\alpha = \log_3 2$). Assume, further that at least one of the following conditions is satisfied:
(i) either $a = -\infty$, or

$$(7.38)\qquad g_j(a) = 0 \ \ \text{for} \ \ 1 \le j \le m_+;$$

(ii) either $b = +\infty$, or

$$(7.39)\qquad g_j(b) = 0 \ \ \text{for} \ \ m_+ + 1 \le j \le N.$$

Then in the case (i) all the eigenvalues of the operator $\tilde{A}(\gamma)$ (7.34) lying in the gap (a,b) (if they exist) can tend only to the right endpoint b as $\gamma \to 0+$, and in the case (ii) they *can tend only to the left endpoint. If both (i) and (ii) hold simultaneously, then for a small enough γ there is no eigenvalues of the operator $\tilde{A}(\gamma)$ in the gap (a, b).*

Proof. Assume that $b < +\infty$. Notice that condition (5.20) of Proposition 5.1 is equivalent to the condition:

$$\frac{|g_j(t)|^2}{t-b} \in L_1([b,1],\rho_c) \;\text{ for }\; m_+ + 1 \le j \le N.$$

The last condition is fulfilled whenever

$$\frac{g_j(t)}{(t-b)^{\frac{1}{2}}} \in L_2([b,1],\rho_c) \;\text{ for }\; m_+ + 1 \le j \le N.$$

The last relations follow from the assumptions (7.37) and (7.39) and from Lemma 7.2 with $\nu = \frac{1}{2}$. Therefore condition (5.20) of Proposition 5.1 is fulfilled. In the same way condition (5.19) of Proposition 5.1 follows from (7.37) and (7.38). Thus the assertions of the theorem follow from Proposition 5.1. ■

Using Theorem 5.2 and Proposition 7.3, we shall obtain the following result on the localization of the eigenvalues:

Theorem 7.3. *Let (a, b) be a gap of $\sigma(\tilde{A})$ and assume that the eigen-basis (7.33) of the perturbing operator B (7.32) satisfies the condition:*

$$g_j \in L_\infty[0,1] \;\textit{ for }\; j = 1, 2, \ldots, N. \tag{7.40}$$

Then, there exists a constant $K > 0$ such that for $\gamma > 0$ and $\alpha = \log_3 2$ the set

$$\tilde{\Pi}_K(\gamma) = \begin{cases} (-K\gamma^{\frac{1}{1-\alpha}}, 0), & \textit{if } a = -\infty \textit{ and } b = 0, \\ (a, a + K\gamma^{\frac{1}{1-\alpha}}) \cup (b - K\gamma^{\frac{1}{1-\alpha}}, b), & \textit{if } a > -\infty \textit{ and } b < +\infty, \\ (1, 1 + K\gamma^{\frac{1}{1-\alpha}}), & \textit{if } a = 1 \textit{ and } b = +\infty \end{cases} \tag{7.41}$$

contains all the points of the spectrum of the operator $\tilde{A}(\gamma)$ lying in the gap (a,b).

Proof. Without loss of generality we can assume that the sets $\{g_j\}_{j=1}^{m_+}$ and $\{g_j\}_{j=m_++1}^{N}$ are bases of the spaces $\mathfrak{H}^+(B)$ and $\mathfrak{H}^-(B)$ respectively. The condition (7.40) implies that the estimate (7.35) is valid for the elements $\tilde{\gamma}_{i,j}(\lambda) = (R_\lambda(A)g_i, g_j)$ of the matrices

$$\tilde{\Gamma}_+(\lambda) = \{\tilde{\gamma}_{i,j}(\lambda)\}_{i,j=1}^{m_+}, \;\text{ and }\; \tilde{\Gamma}_-(\lambda) = \{\tilde{\gamma}_{i,j}(\lambda)\}_{i,j=m_++1}^{N},$$

Then, by Proposition 7.3, the following estimate holds for the function

$$\nu(\lambda) = \operatorname{trace}(|\tilde{\Gamma}_+(\lambda)|) + \operatorname{trace}(|\tilde{\Gamma}_-(\lambda))|$$

(see (5.23)):

$$\exists K_1 > 0, \;\; \forall\lambda \in (a,b): \;\; \nu(\lambda) \;\le\; \frac{K_1}{(\operatorname{dist}(\lambda, \mathfrak{C}))^{1-\alpha}}.$$

This implies that $\Pi(\gamma) \subseteq \tilde{\Pi}_K(\gamma)$ (see (5.21) and (7.41)). This inclusion and Theorem 5.2 yields the conclusion of our theorem. ■

Appendix: Hölder property of the (2-3) mapping

A.1° Our goal in this section is to prove Proposition 7.1, on which the results of section 7 are based. This proof uses some arithmetical arguments. Let us introduce first some notations.

Definition A.1. Let $g \in \mathbf{N}, \quad g \geq 2$ and let $c \in [0,1)$. The coefficients of c in the g-representation are denoted by $\{\sigma_j^{(g)}(c)\}_{j=1}^{\infty}$. Thus

$$c = \sum_{j=1}^{\infty} \sigma_j^{(g)}(c) g^{-j}.$$

The *support of c with respect to the base g* is the set

$$\mathrm{supp}_g(c) = \{j : \quad \sigma_j^{(g)}(c) \neq 0\},$$

The *set of zeros of c with respect to the base g* is

$$\mathrm{nul}_g(c) = \mathbf{N} \setminus \mathrm{supp}_g(c).$$

Definition A.2. Let $g \in N, \quad g \geq 2$ and let $x, y \in [0,1)$ be expanded in their g-representations. We call an integer j a *lending position of x with respect to y*, if the sequence $\{\sigma_i^{(g)}(x)\}_{i=j+1}^{\infty}$ is lexically less than $\{\sigma_i^{(g)}(y)\}_{i=j+1}^{\infty}$.

Remark A.1. Assume that numbers $x, y \in [0,1)$ have finite g-representations and $x \geq y$. It is clear that j is a lending position of x with respect to y if and only if in the standard algorithm of subtraction y from x, using the g-representations, when arriving the $j+1$-th position one has to "borrow" "1" from the j-th position.

Recall that $\mathfrak{R}$ is the set of numbers from $[0,1)$, whose 2-representation is finite.

Lemma A.1. *For any pair of numbers $r_1, r_2 \in \mathfrak{R}$ with $\quad r_1 \geq r_2$*

$$\mathrm{supp}_2(r_1 - r_2) \subseteq \mathrm{supp}_3(2P_{2,3}(r_1) - 2P_{2,3}(r_2)) \tag{A.1}$$

(see Definition A.1).

Proof. Denote

$$a_l = 2P_{2,3}(r_l) \quad (l = 1, 2). \tag{A.2}$$

It is clear that, with respect to the 2-representation, j is a lending position of r_1 with respect to r_2 if and only if, with respect to the 3-representation, j is a lending position of

a_1 with respect to a_2. It is also easy to check that $j \in \mathrm{nul}_3(a_1 - a_2)$ only if one of two conditions holds for the 3-representation of a_1, a_2: either

(i_3) j is not a lending position of a_1 with respect to a_2, and $\sigma_j^{(3)}(a_1) = \sigma_j^{(3)}(a_2)$,

or

(ii_3) j is a lending position of a_1 with respect to a_2, and $\sigma_j^{(3)}(a_1) = 0$. $\sigma_j^{(3)}(a_2) = 2$.

On the other hand, $j \in \mathrm{nul}_2(r_1 - r_2)$ only if one of three conditions holds for the 2-representation of r_1, r_2: either

(i_2) j is not a lending position of r_1 with respect to r_2, and $\sigma_j^{(2)}(r_1) = \sigma_j^{(2)}(r_2)$,

or

(ii_2) j is a lending position of r_1 with respect to r_2, and $\sigma_j^{(2)}(r_1) = 0$, $\sigma_j^{(2)}(r_2) = 1$,

or

(iii_2) j is a lending position of r_1 with respect to r_2, and $\sigma_j^{(2)}(r_1) = 1$, $\sigma_j^{(2)}(r_2) = 0$.

It is clear that (i_2) is equivalent to (i_3) and (ii_2) is equivalent to (ii_3). The above circumstances imply the inclusion $\mathrm{nul}_3(a_1 - a_2) \subseteq \mathrm{nul}_2(r_1 - r_2)$, which is equivalent to (A.1). ■

We turn now to the main results of this section.

Theorem A.1. *Let $r_1, r_2 \in \mathfrak{R}$, with $r_1 > r_2$ and let $a_l = 2P_{2,3}(r_l)$ $(l = 1, 2)$. Then the following estimate is valid:*

$$r_1 - r_2 \ \le \ 2(a_1 - a_2)^\alpha,$$

where $\alpha = \log_3 2$.

Proof. Consider the 2- and the 3-representations of the numbers $r_1 - r_2$ and $a_1 - a_2$ respectively:

$$r_1 - r_2 = \sum_{\nu=1}^{\infty} \sigma_\nu \frac{1}{2^\nu}, \quad a_1 - a_2 = \sum_{\nu=1}^{\infty} \theta_\nu \frac{1}{3^\nu},$$

where $\sigma_\nu \in \{0, 1\}$ and $\theta_\nu \in \{0, 1, 2\}$. Denote by ν_k $(k = 1, 2, \ldots)$ the indices ν for which $\sigma_\nu \neq 0$. Taking into account (A.1) and (A.2) (see Lemma A.1), we obtain:

$$\frac{r_1 - r_2}{(a_1 - a_2)^\alpha} \le \frac{\sum_{k=1}^{\infty} 2^{-\nu_k}}{(\sum_{k=1}^{\infty} 3^{-\nu_k})^\alpha} \le 2^{\nu_1} \sum_{\nu=\nu_1}^{\infty} 2^{-\nu} = 2.$$

The theorem is proven. ■

In what follows we shall use the sets $\mathfrak{A}$ and $\mathfrak{B}$ (see (7.3), (7.4) and (7.6)) of the left and right endpoints of the complementary intervals to the Cantor set $\mathfrak{C}$.

Theorem A.2. *Let (a,b) be a connected component of the set $\mathbf{R}\setminus\mathfrak{C}$. If (a,b) is bounded, then*

$$\{b\}+2P_{2,3}([0,(b-a)^{\alpha})\cap\mathfrak{R})=\mathfrak{B}\cap[b,b+(b-a)), \tag{A.3}$$

$$\{a\}-2P_{2,3}([0,(b-a)^{\alpha})\cap\mathfrak{R})=\mathfrak{A}\cap(a-(b-a)),a], \tag{A.4}$$

where $\alpha=\log_3 2$. Furthermore, the following inequalities are valid:

$$\forall c\in\mathfrak{B}\cap[b,b+(b-a)):\quad (c-b)^{\alpha}\ \le\ 2^{\alpha}(r-r_b), \tag{A.5}$$

$$\forall l\in\mathfrak{A}\cap(a-(b-a),a]:\quad (a-l)^{\alpha}\le 2^{\alpha}(q-q_a), \tag{A.6}$$

where

$$r=P_{2,3}^{-1}(\frac{c}{2}),\ \ r_b=P_{2,3}^{-1}(\frac{b}{2}),$$
$$q=P_{2,3}^{-1}(\frac{1-l}{2}),\ \ q_a=P_{2,3}^{-1}(\frac{1-a}{2}).$$

If $(a,b)=(-\infty,0)$ then

$$\forall c\in\mathfrak{B}:\quad c^{\alpha}\ \le\ 2^{\alpha}r, \tag{A.7}$$

and if $(a,b)=(1,+\infty)$ then

$$\forall l\in\mathfrak{A}:\quad (1-l)^{\alpha}\ \le\ 2^{\alpha}q. \tag{A.8}$$

Proof. If the interval (a,b) is bounded, then there exist $n\in\mathbf{N}$ and $k\in\mathbf{N}\cap[1,2^n-1]$ such that $(a,b)=(a_k^n,b_k^n)$ (see (7.9)). Assume that n is the minimal natural number for which the last property holds. Then $[b,b+(b-a))=[b_k^n,b_k^n+\frac{1}{3^n})$. The number b_k^n has 3-representation of the form:

$$b_k^n=\sum_{j=1}^{n}\sigma_j^{(3)}(b_k^n)3^{-j},$$

where $\sigma_j^{(3)}(b_k^n)\in\{0,2\}$. It is clear that when the number r runs through the set $\{r_b\}+[0,2^{-n})\cap\mathfrak{R}$, the number

$$c=b_k^n+\sum_{j=1}^{\infty}2\sigma_{n+j}^{(2)}(r)3^{-n-j}$$

runs through the set $\mathfrak{B}\cap[b,b+(b-a))$, i.e. we have proved the equality (A.3). It is also clear that $r-r_b=P_{2,3}^{-1}(\frac{c-b}{2})$. Taking into account that $0<\alpha<1$ and the fact that $\sigma_j^{(2)}(r)\in\{0,1\}$, we obtain:

$$(c-b)^{\alpha}\ =\ (\sum_{j=1}^{\infty}2\sigma_{n+j}^{(2)}(r)\,3^{-n-j})^{\alpha}\ \le\ 2^{\alpha}\sum_{j=1}^{\infty}\sigma_{n+j}^{(2)}(r)\,2^{-n-j}\ =\ 2^{\alpha}(r-r_b).$$

This establishes the estimate (A.5). The estimate (A.7) can be proven in the same manner. The relation (A.4) and the estimates (A.6), (A.8) follow from the obtained results in view of the symmetry of the Cantor set with respect to the point $t=\frac{1}{2}$ (see (7.7), (7.8)). ■

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